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HARDY-TYPE INEQUALITIES ON THE REAL LINE

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ABSTRACT. We prove a certain type of inequalities concerning the integral of the Fourier transform of a function integrable on the real line.

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1. INTRODUCTION

Hardy's inequality states that a constant $C > 0$ exists such that

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq C\|f\|_1 \tag{1.1}$$

for all integrable functions f on the circle $\mathbb{T} = [0, 2\pi)$ with $\hat{f}(n) = 0$ for $n < 0$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int}dt, \ \forall n \in \mathbb{Z} \quad \text{and} \quad \|f\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|dt.$$

Questions of how inequality (1.1) can be generalized for all $f \in L^1(\mathbb{T})$ have been raised, and some partial answers were given. Some references on the subject are [3], [4], [5] and [7].

In [4] it was proved that a constant $C > 0$ exists such that

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n} \leq C\|f\|_1^2 + \sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|^2}{n} \tag{1.2}$$

for all $f \in L^1(\mathbb{T})$.

Now, let $f \in L^1(\mathbb{R})$ and let $\|f\|_1$ denote the L^1 norm of f. We shall prove in the next section, that

$$\int_0^{\infty} \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \leq 2\pi\|f\|_1^2 + \int_0^{\infty} \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi \tag{1.3}$$

for all $f \in X$, where X is an infinite dimensional subspace of L^1. Then we interpolate this inequality to get the inequality

$$\int_0^\infty \frac{|\hat{f}(\xi)|^\alpha}{\xi} \leq 2\pi \|f\|_1^\alpha$$

for all $\alpha \geq 2$ when f lies in a certain space.

We emphasize that the main purpose of this article is not only the concrete inequalities that it contains. Rather, we would like to show that the methodology for proving Hardy-type inequalities on the real line is very similar to that for proving such inequalities on the circle.

In other words, it is well known that proving Hardy-type inequalities on the circle depends on the construction of a certain bounded function. This constructed function is, then, used in a standard duality argument to produce the required inequality.

Although the first proof of Hardy's inequality does not depend on such a construction, many proofs were given later depending on the construction of bounded functions whose Fourier coefficients have desired decay properties. We encourage the reader to have a look at [3], [5], [7] and [8] to see how such bounded functions are constructed.

It is a very tough task to construct these bounded functions and usually these functions are constructed through an inductive procedure. We refer the reader to [1] for the most comprehensive discussion of these inductive constructions.

In this article, we prove some Hardy-type inequalities on the real line depending on the construction of a certain bounded function. This bounded function is constructed in a very simple way and no inductive procedure is followed.

We remark that inequality (1.2) was proved first in [6] where the authors gave a quite complicated proof; it uses BMO and the theory of Hankel and Teoplitz operators. Later Koosis [4] gave a simpler proof. In fact, we can imitate the given proof of inequality (1.3), in this article, to prove inequality (1.2) on the circle. This is the only known proof of (1.2) which uses the construction of bounded functions.[1]

2. MAIN RESULTS

We begin by introducing the set

$$X = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^x f(t)dt \in L^1(\mathbb{R}) \right\}.$$

It is clear that X is a subspace of $L^1(\mathbb{R})$. In fact, X is an infinite dimensional space. Indeed, for $\alpha \geq 3$, let

$$f_\alpha(x) = \begin{cases} 0, & x \leq 1; \\ \frac{1}{x^{\alpha+2}} - \frac{\alpha+2}{(\alpha+1)x^{\alpha+3}}, & x > 1. \end{cases}$$

Then $(f_\alpha)_{\alpha \geq 3}$ is a linearly independent set in X. This implies that X is an infinite dimensional subspace of $L^1(\mathbb{R})$.

We remark that if $f \in X$ then $\hat{f}(0) = 0$ and [2]:

$$\left(\int_{-\infty}^x f(t)dt \right)^\wedge (\xi) = \frac{\hat{f}(\xi)}{i\xi}, \quad \xi \neq 0.$$

[1]This is for sure to the best of the author's knowledge. The proof which uses the construction of a bounded function is in an unpublished work of the author. But, the reader of this article will be able to conclude how to prove (1.2) using a duality argument without any difficulties.

Theorem 2.1. *Let $f \in X$, then*

$$\int_0^\infty \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \le 2\pi \|f\|_1^2 + \int_0^\infty \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi. \tag{2.1}$$

Proof. Let $f \in X$ be such that $\hat{f}$ is of compact support, so that the inversion formula holds for f. Let

$$F(x) = \int_{-\infty}^x f(t)dt,$$

and observe that $\|F\|_\infty \le \|f\|_1$ and for real $\xi \neq 0$, $\hat{F}(\xi) = \frac{\hat{f}(\xi)}{i\xi}$. Therefore,

$$\begin{aligned} \|f\|_1^2 &\ge \|F\|_\infty \|f\|_1 \\ &\ge \left| \int_{\mathbb{R}} f(x)\overline{F(x)}dx \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi \overline{F(x)} dx \right|, \end{aligned}$$

where, in the last line, we have used the inversion formula for f. Consequently

$$\begin{aligned} \|f\|^2 &\ge \frac{1}{2\pi} \left| \int_{\mathbb{R}} \hat{f}(\xi) \overline{\int_{\mathbb{R}} F(x) e^{-ix\xi} dx} d\xi \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{F}(\xi)} d\xi \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}\setminus\{0\}} \hat{f}(\xi) \frac{\overline{\hat{f}(\xi)}}{\xi} d\xi \right| \\ &= \frac{1}{2\pi} \left| \int_0^\infty \frac{|\hat{f}(\xi)|^2}{\xi} d\xi - \int_0^\infty \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi \right|, \end{aligned}$$

whence

$$\int_0^\infty \frac{|\hat{f}(\xi)|^2}{\xi} d\xi \le 2\pi \|f\|_1^2 + \int_0^\infty \frac{|\hat{f}(-\xi)|^2}{\xi} d\xi.$$

Thus the inequality holds for all $f \in X$ such that $\hat{f}$ is compactly supported.

Now, let $f \in X$ be arbitrary. Let $g = f * K_\lambda$ where K_λ is the Fejer kernel on $\mathbb{R}$ of order λ. Then, $\hat{g} = \hat{f} \times \hat{K}_\lambda$ is of compact support because $\hat{K}_\lambda(\xi) = 0$ when $|\xi| \ge \lambda$.

We now prove that $G(x) = \int_{-\infty}^x g(y)dy \in L^1(\mathbb{R})$, in order to apply the statement of the theorem on g. Observe that

$$\int_{-\infty}^\infty |G(x)|dx = \int_{-\infty}^\infty \left| \int_{-\infty}^x \int_{-\infty}^\infty f(y-t) K_\lambda(t) dt dy \right| dx. \tag{2.2}$$

Now,

$$\begin{aligned} \int_{-\infty}^x \int_{-\infty}^\infty |f(y-t)K_\lambda(t)| dt dy &\le \int_{-\infty}^\infty \int_{-\infty}^\infty |f(y-t)| K_\lambda(t) dy dt \\ &= \int_{-\infty}^\infty K_\lambda(t) \int_{-\infty}^\infty |f(y-t)| dy dt \\ &= \|f\|_1 \|K_\lambda\|_1 < \infty \end{aligned}$$

because f and K_λ are both integrable on $\mathbb{R}$. Therefore, the Tonelli theorem applies and (2.2) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}|G(x)|dx &= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(y-t)K_\lambda(t)dydt\right|dx \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{x} f(y-t)K_\lambda(t)dy\right|dtdx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_\lambda(t)\left|\int_{-\infty}^{x} f(y-t)dy\right|dtdx \\ &= \int_{-\infty}^{\infty}K_\lambda(t)\int_{-\infty}^{\infty}\left|\int_{-\infty}^{x} f(y-t)dy\right|dxdt. \end{aligned} \tag{2.3}$$

Recall the definition of F in the statement of the theorem and note that

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{x} f(t-y)dy\right|dx &= \int_{-\infty}^{\infty}\left|\int_{-\infty}^{x-t} f(y)dy\right|dx \\ &= \int_{-\infty}^{\infty}|F(x-t)|dx \\ &= \int_{-\infty}^{\infty}|F(x)|dx = \|F\|_1 < \infty\end{aligned}$$

where in the last line we used the assumption that $F \in L^1(\mathbb{R})$.

Whence, (2.3) boils down to saying

$$\int_{-\infty}^{\infty}|G(x)|dx \le \|F\|_1\|K_\lambda\| < \infty.$$

Therefore, the result of the theorem applies for g. That is

$$\int_0^\infty \frac{|\hat{g}(\xi)|^2}{\xi}d\xi \le 2\pi\|g\|_1^2 + \int_0^\infty \frac{|\hat{g}(-\xi)|^2}{\xi}d\xi. \tag{2.4}$$

Recalling that

$$\hat{K}_\lambda(\xi) = \begin{cases} \left(1-\frac{|\xi|}{\lambda}\right), & |\xi| \le \lambda \\ 0, & |\xi| \ge \lambda \end{cases}$$

and that $\|f * K_\lambda\|_1 \le \|f\|_1\|K_\lambda\|_1 = \|f\|_1$, (2.4) reduces to

$$\begin{aligned}\int_0^\infty \frac{|\hat{f}(\xi)|^2|\hat{K}_\lambda(\xi)|^2}{\xi}d\xi &\le 2\pi\|f\|_1^2 + \int_0^\lambda \frac{|\hat{f}(-\xi)|^2(1-\xi/\lambda)^2}{\xi}d\xi \\ &\le 2\pi\|f\|_1^2 + \int_0^\lambda \frac{|\hat{f}(-\xi)|^2}{\xi}d\xi \\ &\le 2\pi\|f\|_1^2 + \int_0^\infty \frac{|\hat{f}(-\xi)|^2}{\xi}d\xi.\end{aligned}$$

Also, it is clear that $|\hat{K}_{\lambda+1}(\xi)| \ge |\hat{K}_\lambda(\xi)|$ for all $\lambda \in \mathbb{R}$ and $\xi \in [0,\infty)$. Hence, the monotone convergence theorem implies

$$\int_0^\infty \frac{|\hat{f}(\xi)|^2}{\xi}d\xi \le 2\pi\|f\|_1^2 + \int_0^\infty \frac{|\hat{f}(-\xi)|^2}{\xi}d\xi,$$

where we have used the fact that $\lim_{\lambda\to\infty}|\hat{K}_\lambda(\xi)| = 1$ for all $\xi \in [0,\infty)$. □

On replacing the function f by $f * f$ the above theorem gives:

Corollary 2.2. *Let $f \in X$, then*

$$\int_0^\infty \frac{|\hat{f}(\xi)|^4}{\xi}d\xi \leq 2\pi\|f\|_1^4 + \int_0^\infty \frac{|\hat{f}(-\xi)|^4}{\xi}d\xi. \tag{2.5}$$

Proof. Observe first that if $\int_{-\infty}^x f(y)dy \in L^1(\mathbb{R})$ then $\int_{-\infty}^x (f*f)(y)dy \in L^1(\mathbb{R})$ and the proof of this conclusion is exactly the same as proving that $G \in L^1(\mathbb{R})$, where G is as in the above theorem, if K_λ is replaced with f. □

Thus, replacing the power 2 by any even power $2m$ makes no difference on (2.1).

Now, let $X' = \left\{ f \in X : \hat{f}(\xi) = 0 \text{ when } \xi < 0 \right\}$, then

$$\left(\int_0^\infty \frac{|\hat{f}(\xi)|^{2m}}{\xi}d\xi \right)^{\frac{1}{2m}} \leq \sqrt[2m]{2\pi}\|f\|_1, \quad \forall m \in \mathbb{N}. \tag{2.6}$$

Let $\mathcal{M}$ be the sigma algebra of Lebesgue measurable subsets of $[0,\infty)$ and let μ be the measure given by $d\mu = \frac{d\xi}{\xi}$ where $d\xi$ is the Lebesgue measure. Define a linear mapping $T' : X'^{2m}([0,\infty),\mathcal{M},\mu)$ by

$$T'(f) = \hat{f}.$$

This is a well defined mapping because inequality (2.6) guarantees that $\hat{f} \in L^{2m}([0,\infty),\mathcal{M},\mu)$ when $f \in X'$. Moreover, T' is a continuous linear mapping of norm $\leq \sqrt[2m]{2\pi}$. By the Hahn-Banach theorem, T' extends to a bounded linear mapping $T : L^1 \longrightarrow L^{2m}([0,\infty),\mathcal{M},\mu)$ with norm $\leq \sqrt[2m]{2\pi}$.

Now, by the Riesz-Thorin theorem for interpolating a linear operator [2], T remains continuous as a mapping from L^1 into $L^\alpha([0,\infty),\mathcal{M},\mu)$ for all $\alpha \geq 2$. Thus, we have proved the following result.

Theorem 2.3. *Let $f \in X'$, then for $\alpha \geq 2$, we have*

$$\int_0^\infty \frac{|\hat{f}(\xi)|^\alpha}{\xi}d\xi \leq 2\pi\|f\|_1^\alpha.$$

Remark 1.

(1) If $f \in L^1(\mathbb{T})$ is such that $\hat{f}(n) = 0,\ \forall n < 0$ (that is, $f \in H^1(\mathbb{T})$) then

$$\sum_{n=1}^\infty \frac{|\hat{f}(n)|^m}{n} \leq C\|f\|_1^m.$$

This follows from Hardy's inequality (1.1) when f is replaced by the convolution of f with itself $m \in \mathbb{N}$ times.

A similar interpolation idea as above yields the inequality

$$\sum_{n=1}^\infty \frac{|\hat{f}(n)|^\alpha}{n} \leq C\|f\|_1^\alpha$$

for all $f \in H^1(\mathbb{T})$ when $\alpha \geq 1$.

(2) The above interpolated inequalities can be proved at once using the observation $\|\hat{f}\|_\infty \leq \|f\|_1$ and no interpolation is needed. But we believe that the interpolation idea can be

used to obtain the inequalities

$$\sum_{n=1}^{\infty}\frac{|\hat{f}(n)|^{\alpha}}{n}\leq C\|f\|_1^{\alpha}+\sum_{n=1}^{\infty}\frac{|\hat{f}(-n)|^{\alpha}}{n}\quad \text{(on the circle) and}$$

$$\int_0^{\infty}\frac{|\hat{f}(\xi)|^{\alpha}}{\xi}d\xi\leq 2\pi\|f\|_1^{\alpha}+\int_0^{\infty}\frac{|\hat{f}(-\xi)|^{\alpha}}{\xi}d\xi\quad \text{(on the line)}$$

for $\alpha\geq 2$. The truth of these two inequalities is still an open problem.

These ideas suggest the following question: For $f\in H^1(\mathbb{T})$, is there a constant $C>0$ such that

$$\sum_{n=1}^{\infty}\frac{|\hat{f}(n)|^{\alpha}}{n}\leq C\|f\|_1^{\alpha}$$

when $\alpha>0$? How about for $H^1(\mathbb{R})$? Also, what is the smallest value of $\alpha>0$ such that the above inequality holds?

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