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A GENERALIZATION FOR OSTROWSKI'S INEQUALITY IN $\mathbb{R}^2$

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ABSTRACT. We establish a new Ostrowski's inequality in $\mathbb{R}^2$ by using an idea of B.G. Pachpatte.

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A.M. Ostrowski proved the following inequality (see [1, p. 226–227]):

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)||f'||_\infty, \tag{1}$$

for all $x \in [a,b]$, where $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $||f'||_\infty = \sup_{x\in(a,b)}|f'(x)| < \infty$.

Recently, by using a fairly elementary analysis, B.G. Pachpatte [2] established the following inequality of type (1) involving two functions and their derivatives.

Theorem 1. *Let $f,g:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be continuous functions on $[a,b]$ and differentiable on (a,b), whose derivative $f',g':(a,b)\to\mathbb{R}$ are bounded on (a,b), i.e., $||f'||_\infty = \sup_{x\in(a,b)}|f'(x)| < \infty$, $||g'||_\infty = \sup_{x\in(a,b)}|g'(x)|<\infty$. Then*

$$\left|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)dy + f(x)\int_a^b g(y)dy\right]\right|$$
$$\le \frac{1}{2}\{|g(x)|\,||f'||_\infty + |f(x)|\,||g'||_\infty\}\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a),$$

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for all $x \in [a,b]$.

In this paper, by means of B.G. Pachpatte's idea, we prove the following

Theorem 2. *Let* $D = [a,b] \times [a,b]$, $intD = (a,b) \times (a,b)$, $f, g : D \to \mathbb{R}$ *be continuous functions on* D *and differentiable on* $intD$*, whose partial derivatives* $f_x, f_y, g_x, g_y : intD \to \mathbb{R}$ *are bounded on* $intD$*, i.e.,* $||f_x||_\infty = \sup_{(x,y)\in intD} |f_x(x,y)| < \infty$, $||f_y||_\infty = \sup_{(x,y)\in intD} |f_y(x,y)| < \infty$, $||g_x||_\infty = \sup_{(x,y)\in intD} |g_x(x,y)| < \infty$, $||g_y||_\infty = \sup_{(x,y)\in intD} |g_y(x,y)| < \infty$. *Then*

$$\begin{aligned}\Bigg| f(u_1,v_1)g(u_1,v_1) - \frac{1}{2(b-a)^2}\Bigg[g(u_1,v_1)\iint_D f(u_2,v_2)du_2dv_2 \\ + f(u_1,v_1)\iint_D g(u_2,v_2)du_2dv_2\Bigg]\Bigg| \\ \le \frac{1}{2}\left[|g(u_1,v_1)|\,||f_x||_\infty + |f(u_1,v_1)|\,||g_x||_\infty\right]\left[\frac{1}{4} + \frac{\left(u_1 - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a) \\ + \frac{1}{2}\left[|g(u_1,v_1)|\,||f_y||_\infty + |f(u_1,v_1)|\,||g_y||_\infty\right]\left[\frac{1}{4} + \frac{\left(v_1 - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\end{aligned}$$

for all $(u_1, v_1) \in D$.

By taking $g(x,y) = 1$ in Theorem 2, we get the following Ostrowski like inequality in $\mathbb{R}^2$,

Corollary 3.

$$\begin{aligned}\left| f(u_1,v_1) - \frac{1}{(b-a)^2}\iint_D f(u_2,v_2)du_2dv_2\right| \\ \le ||f_x||_\infty\left[\frac{1}{4} + \frac{\left(u_1 - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a) + ||f_y||_\infty\left[\frac{1}{4} + \frac{\left(v_1 - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a).\end{aligned}$$

Proof of Theorem 2. By the mean value theorem, there exist $\xi_1, \eta_1, \xi_2, \eta_2 \in (a,b)$ such that

$$f(u_1,v_1) - f(u_2,v_2) = f_x(\xi_1,v_1)(u_1-u_2) + f_y(u_2,\eta_1)(v_1-v_2), \tag{2}$$

and

$$g(u_1,v_1) - g(u_2,v_2) = g_x(\xi_2,v_1)(u_1-u_2) + g_y(u_2,\eta_2)(v_1-v_2). \tag{3}$$

Multiplying both sides of (2) and (3) by $g(u_1,v_1)$ and $f(u_1,v_1)$ respectively and adding we get

$$\begin{aligned}2f(u_1,v_1)g(u_1,v_1) - [f(u_2,v_2)g(u_1,v_1) + f(u_1,v_1)g(u_2,v_2)] \\ = g(u_1,v_1)[f_x(\xi_1,v_1)(u_1-u_2) + f_y(u_2,\eta_1)(v_1-v_2)] \\ + f(u_1,v_1)[g_x(\xi_2,v_1)(u_1-u_2) + g_y(u_2,\eta_2)(v_1-v_2)].\end{aligned}$$

Integrate both sides with respect to u_2, v_2 over D. Note that by the proof of the mean value theorem, we know that $f_x(\xi_1,v_1)$, $f_y(u_2,\eta_1)$, $g_x(\xi_2,v_1)$ and $g_y(u_2,\eta_2)$ are Riemann-integrable for $(u_2, v_2) \in D$. Rewriting we get

$$f(u_1,v_1)g(u_1,v_1) - \frac{1}{2(b-a)^2}\left[g(u_1,v_1)\iint_D f(u_2,v_2)du_2dv_2 + f(u_1,v_1)\iint_D g(u_2,v_2)du_2dv_2\right]$$
$$= \frac{1}{2(b-a)^2}g(u_1,v_1)\iint_D [f_x(\xi_1,v_1)(u_1-u_2) + f_y(u_2,\eta_1)(v_1-v_2)]du_2dv_2$$
$$+ \frac{1}{2(b-a)^2}f(u_1,v_1)\iint_D [g_x(\xi_2,v_1)(u_1-u_2) + g_y(u_2,\eta_2)(v_1-v_2)]du_2dv_2.$$

So

$$\left|f(u_1,v_1)g(u_1,v_1) - \frac{1}{2(b-a)^2}\left[g(u_1,v_1)\iint_D f(u_2,v_2)du_2dv_2 + f(u_1,v_1)\iint_D g(u_2,v_2)du_2dv_2\right]\right|$$
$$\le \frac{1}{2(b-a)^2}|g(u_1,v_1)|\left[||f_x||_\infty \iint_D |u_1-u_2|du_2dv_2 + ||f_y||_\infty \iint_D |v_1-v_2|du_2dv_2\right]$$
$$+ \frac{1}{2(b-a)^2}|f(u_1,v_1)|\left[||g_x||_\infty \iint_D |u_1-u_2|du_2dv_2 + ||g_y||_\infty \iint_D |v_1-v_2|du_2dv_2\right].$$

Note that

$$\iint_D |u_1-u_2|du_2dv_2 = (b-a)\left[\frac{(u_1-a)^2+(u_1-b)^2}{2}\right] = \left[\frac{1}{4} + \frac{\left(u_1-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)$$

and

$$\iint_D |v_1-v_2|du_2dv_2 = \left[\frac{1}{4} + \frac{\left(v_1-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a).$$

We obtain that

$$\left|f(u_1,v_1)g(u_1,v_1) - \frac{1}{2(b-a)^2}\left[g(u_1,v_1)\iint_D f(u_2,v_2)du_2dv_2 + f(u_1,v_1)\iint_D g(u_2,v_2)du_2dv_2\right]\right|$$
$$\le \frac{1}{2}[|g(u_1,v_1)|\,||f_x||_\infty + |f(u_1,v_1)|\,||g_x||_\infty]\left[\frac{1}{4} + \frac{\left(u_1-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)$$
$$+ \frac{1}{2}[|g(u_1,v_1)|\,||f_y||_\infty + |f(u_1,v_1)|\,||g_y||_\infty]\left[\frac{1}{4} + \frac{\left(v_1-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a).$$

□

Remark 4. Let $f(x,y) = f(x), g(x,y) = g(x)$ in Theorem 2. Then $f_y = 0, g_y = 0$. We obtain Theorem 1.

Remark 5. Let $f(x,y) = f(x), g(x,y) = 1$ in Theorem 2, we recapture the well known Ostrowski inequality.

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