Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 1, Article 2, 2006

AN INEQUALITY FOR THE ASYMMETRY OF DISTRIBUTIONS AND A BERRY-ESSEEN THEOREM FOR RANDOM SUMMATION

HENDRIK KLÄVER AND NORBERT SCHMITZ

INSTITUTE OF MATHEMATICAL STATISTICS
UNIVERSITY OF MÜNSTER
EINSTEINSTR. 62
D-48149 MÜNSTER, GERMANY

schmnor@math.uni-muenster.de

Received 16 March, 2004; accepted 15 December, 2005
Communicated by S.S. Dragomir

ABSTRACT. We consider random numbers N_n of independent, identically distributed (i.i.d.) random variables X_i and their sums $\sum_{i=1}^{N_n} X_i$. Whereas Blum, Hanson and Rosenblatt [3] proved a central limit theorem for such sums and Landers and Rogge [8] derived the corresponding approximation order, a Berry-Esseen type result seems to be missing. Using an inequality for the asymmetry of distributions, which seems to be of its own interest, we prove, under the assumption $E|X_i|^{2+\delta} < \infty$ for some $\delta \in (0,1]$ and $N_n/n \to \tau$ (in an appropriate sense), a Berry-Esseen theorem for random summation.

Key words and phrases: Random number of i.i.d. random variables, Central limit theorem for random sums, Asymmetry of distributions, Berry-Esseen theorem for random sums.

2000 *Mathematics Subject Classification.* 60E15, 60F05, 60G40.

1. INTRODUCTION

One of the milestones of probability theory is the famous theorem of *Berry-Esseen* which gives uniform upper bounds for the deviation from the normal distribution in the Central Limit Theorem:

Let $\{X_n, n \geq 1\}$ *be independent random variables such that*

$$EX_n = 0, \quad EX_n^2 =: \sigma_n^2,$$

$$s_n^2 := \sum_{i=1}^{n} \sigma_i^2 > 0,$$

$$\Gamma_n^{2+\delta} := \sum_{i=1}^{n} E|X_i|^{2+\delta} < \infty$$

ISSN (electronic): 1443-5756

056-04

for some $\delta \in (0,1]$ *and* $S_n = \sum_{i=1}^n X_i,\ n \geq 1$. *Then there exists a universal constant* C_δ *such that*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S_n}{s_n} \leq x\right) - \Phi(x)\right| \leq C_\delta \left(\frac{\Gamma_n}{s_n}\right)^{2+\delta},$$

where Φ *denotes the cumulative distribution function of a* $\mathcal{N}(0,1)$*-(normal) distribution* (see e.g. Chow and Teicher [5, p. 299]).

For the special case of identical distributions this leads to:

Let $\{X_n,\ n\geq 1\}$ *be i.i.d. random variables with* $EX_n = 0,\ EX_n^2 =: \sigma^2 > 0,\ E|X_n|^{2+\delta} =: \gamma^{2+\delta} < \infty$ *for some* $\delta \in (0,1]$. *Then there exists a universal constant* c_δ *such that*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) - \Phi(x)\right| \leq \frac{c_\delta}{n^{\delta/2}} \left(\frac{\gamma}{\sigma}\right)^{2+\delta}.$$

Van Beek [2] showed that $C_1 \leq 0.7975$; bounds for other values C_δ are given by Tysiak [12], e.g. $C_{0.8} \leq 0.812;\ C_{0.6} \leq 0.863,\ C_{0.4} \leq 0.950,\ C_{0.2} \leq 1.076.$

On the other hand, there exist also central limit theorems for random summation, e.g. the theorem of *Blum, Hanson and Rosenblatt* [3] which generalizes previous results by Anscombe [1] and Renyi [11]:

Let $\{X_n,\ n\geq 1\}$ *be i.i.d. random variables with* $EX_n = 0$, $\operatorname{Var} X_n = 1,\ S_n := \sum_{i=1}^n X_i$ *and let* $\{N_n, n\geq 1\}$ *be* $\mathbb{N}$*-valued random variables such that* $N_n/n \xrightarrow{P} U$ *where* U *is a positive random variable. Then*

$$P^{S_{N_n}/\sqrt{N_n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Therefore, the obvious question arises whether one can prove also Berry-Esseen type inequalities for random sums. A first result concerning the approximation order is due to *Landers and Rogge* [8] for random variables X_n with $E|X_n|^3 < \infty$; this was generalized by *Callaert* and *Janssen* [4] to the case that $E|X_n|^{2+\delta} < \infty$ for some $\delta \in (0,1]$:

Let $\{X_n,\ n\geq 1\}$ *be i.i.d. random variables with* $E\,X_n = \mu,\ \operatorname{Var} X_n = \sigma^2 > 0$, *and* $E|X_n|^{2+\delta} < \infty$ *for some* $\delta > 0$. *Let* $\{N_n,\ n\geq 1\}$ *be* $\mathbb{N}$*-valued random variables,* $\{\varepsilon_n,\ n\geq 1\}$ *positive real numbers with* $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ *where, for* n *large,* $n^{-\delta} \leq \varepsilon_n$ *if* $\delta \in (0,1]$ *and* $n^{-1} \leq \varepsilon_n$ *if* $\delta \geq 1$. *If there exists a* $\tau > 0$ *such that*

$$P\left(\left|\frac{N_n}{n\tau} - 1\right| > \varepsilon_n\right) = O(\sqrt{\varepsilon_n}),$$

then

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\sum_{i=1}^{N_n}(X_i-\mu)}{\sigma\sqrt{n\tau}} \leq x\right) - \Phi(x)\right| = O(\sqrt{\varepsilon_n})$$

and

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\sum_{i=1}^{N_n}(X_i-\mu)}{\sigma\sqrt{N_n}} \leq x\right) - \Phi(x)\right| = O(\sqrt{\varepsilon_n}).$$

Moreover, there exist several further results on convergence rates for random sums (see e.g. [7] and the papers cited there); as applications, sequential analysis, random walk problems, Monte Carlo methods and Markov chains are mentioned.

However, rates of convergence without any knowledge about the factors are of very limited importance for applications. Hence the aim of this paper is to prove a Berry-Esseen type result for random sums i.e. a uniform approximation with explicit constants. Obviously, due to the dependencies on the moments of the X_n as well as on the asymptotic behaviour of the sequence N_n such a result will necessarily be more complex than the original Berry-Esseen theorem.

For the underlying random variables X_n we make the same assumption as in the special version of the Berry-Esseen theorem:

$$\text{(M)} \qquad : \begin{cases} X_n,\ n \geq 1, \text{ are i.i.d. random variables with } EX_n = 0,\ Var\ X_n = 1 \text{ and} \\ \gamma^{2+\delta} := E|X_n|^{2+\delta} < \infty \text{ for some } \delta \in (0,1]. \end{cases}$$

Similarly as Landers and Rogge [8] or Callaert and Janssen [4] resp. we assume on the random indices

$$\text{(R)} \qquad : \begin{cases} N_n,\ n \geq 1, \text{ are integer-valued random variables and } \zeta_n,\ n \geq 1, \\ \text{real numbers with } \lim_{n\to\infty} \zeta_n = 0 \text{ such that there exist} \\ d, \tau > 0 \text{ with } P(|\frac{N_n}{n\tau} - 1| > \zeta_n) \leq d\sqrt{\zeta_n}. \end{cases}$$

As they are essential for applications, e.g. in sequential analysis, arbitrary dependencies between the indices and the summands are allowed.

2. AN INEQUALITY FOR THE ASYMMETRY OF DISTRIBUTIONS

A main tool for deriving explicit constants for the rate of convergence is an inequality which seems to be of its own interest. For a smooth formulation we use (for the different values of the moment parameter $\delta \in (0,1]$) some (technical) notation: For $\vartheta := 2/(1+\delta)$ and $y \geq 1$ let $g_\delta(y)$ be defined by

$$g_\delta(y) := \begin{cases} 2y^2 - 1 + 2y\sqrt{y^2-1} & \text{if } \delta = 1 \\ \min\Big\{2^\vartheta y^{4\vartheta} - 1 + 2^{\frac{\vartheta+1}{2}} y^{2\vartheta}\sqrt{2^{\vartheta-1}y^{4\vartheta}-1}\,, & \\ \quad 2y^{8\vartheta} - 1 + 2y^{4\vartheta}\sqrt{y^{8\vartheta}-1}\Big\} & \text{if } \delta \in \left[\frac{1}{3}, 1\right) \\ \min\Big\{2^\vartheta y^{(2+2^k)\vartheta} - 1 + 2^{\frac{\vartheta+1}{2}} y^{(1+2^{k-1})\vartheta}\sqrt{2^{\vartheta-1}y^{(2+2^k)\vartheta}-1}, & \\ \quad 2y^{(4+2^{k+1})\vartheta} - 1 + 2y^{(2+2^k)\vartheta}\sqrt{y^{(4+2^{k+1})\vartheta}-1}\Big\} & \text{if } \delta \in \left(\frac{1}{2^k+1}, \frac{1}{2^{k-1}+1}\right],\ k \geq 2 \end{cases}$$

Theorem 2.1. *Let X be a random variable with $EX = 0$, $\operatorname{Var} X = 1$ and $\gamma^{2+\delta} := E|X|^{2+\delta} < \infty$ for some $\delta \in (0,1]$. Then*

$$P(X < 0) \leq g_\delta(\gamma^{2+\delta})P(X > 0) \textit{ and } P(X > 0) \leq g_\delta(\gamma^{2+\delta})P(X < 0).$$

Proof. Let X be a random variable as described above. Let $X_+ = \max\{X, 0\}$, $X_- = \min\{X, 0\}$, $E(X_+) = E(X_-) = \alpha$ and $P(X > 0) = p,\ P(X = 0) = r,\ P(X < 0) = 1 - r - p$. Since $E(X) = 0$, $\operatorname{Var}(X) = 1$ it is obvious that $p, 1 - r - p > 0$. As $\alpha = pE(|X|\,\|X > 0)$, we have $E(|X|\,\|X > 0) = \frac{\alpha}{p}$; analogously $E(|X|\,\|X < 0) = \frac{\alpha}{1-r-p}$.

Applying Jensen's inequality to the convex function $f : [0,\infty) \to [0,\infty),\ f(x) = x^{\frac{3+\delta}{2}}$ yields

$$E(|X|^{\frac{3+\delta}{2}}\|X > 0) \geq \left(\frac{\alpha}{p}\right)^{\frac{3+\delta}{2}} \text{ and } E(|X|^{\frac{3+\delta}{2}}\|X < 0) \geq \left(\frac{\alpha}{1-r-p}\right)^{\frac{3+\delta}{2}}.$$

Defining $\beta_z := E|X|^z$ for $z > 0$ we get

$$\begin{aligned} \beta_{\frac{3+\delta}{2}} &= pE\left(|X|^{\frac{3+\delta}{2}}\|X > 0\right) + (1-r-p)E\left(|X|^{\frac{3+\delta}{2}}|X < 0\right) \\ &\geq \alpha^{\frac{3+\delta}{2}} \frac{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}}{(p(1-r-p))^{\frac{1+\delta}{2}}} \end{aligned}$$

and, therefore,

(i) $$\alpha^{\frac{3+\delta}{2}} \le \beta_{\frac{3+\delta}{2}} \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}}$$

Since $\gamma^{2+\delta} < \infty$, we can apply the Cauchy-Schwarz-inequality to $|X|^{1/2}$ and $|X|^{1+\delta/2}$ and obtain

$$\left(E|X|^{\frac{3+\delta}{2}}\right)^2 \le E|X|\, E|X|^{2+\delta} = 2\alpha\gamma^{2+\delta};$$

hence

(ii) $$\alpha \ge \frac{\left(\beta_{\frac{3+\delta}{2}}\right)^2}{2\gamma^{2+\delta}}.$$

Combining (i) and (ii) we obtain

$$\left(\frac{\left(\beta_{\frac{3+\delta}{2}}\right)^2}{2\gamma^{2+\delta}}\right)^{\frac{3+\delta}{2}} \le \beta_{\frac{3+\delta}{2}} \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{p^{\frac{1+\delta}{2}} + (1-r-p)^{\frac{1+\delta}{2}}};$$

hence

$$x^2 \le \frac{1}{4} - \frac{1}{1-r}\left(\frac{1}{2}\right)^{\vartheta+1} \frac{a_1}{a_2}\left(\left(\frac{1}{2}+x\right)^{\frac{1}{\vartheta}} + \left(\frac{1}{2}-x\right)^{\frac{1}{\vartheta}}\right)^{\vartheta}$$

with

$$x = \frac{p}{1-r} - \frac{1}{2}, \quad \vartheta = \frac{2}{1+\delta}, \quad a_1 = \left(\beta_{\frac{3+\delta}{2}}\right)^{\vartheta+2}, \quad a_2 = (\gamma^{2+\delta})^{\vartheta+1}.$$

Obviously $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\vartheta \in [1,2)$.

Since

(iii) $$x^\mu + y^\mu \ge (x+y)^\mu \ \ \forall \mu \in (0,1],\ x, y \ge 0$$

and $0 < 1-r \le 1$ it follows altogether that

(iv) $$|x| \le \frac{1}{2}\sqrt{1 - \left(\frac{1}{2}\right)^{\vartheta-1} \frac{a_1}{a_2}}.$$

For large values of $|x|$ this estimation is rather poor, so we notice, furthermore, that

$$\left(\left(\frac{1}{2}+x\right)^{\frac{1}{\vartheta}} + \left(\frac{1}{2}-x\right)^{\frac{1}{\vartheta}}\right)^{\vartheta} \ge 2^{\vartheta-1}(1-4x^2)^{1/2};$$

hence

(v) $$|x| \le \frac{1}{2}\sqrt{1 - \left(\frac{a_1}{a_2}\right)^2}.$$

From (iv) and (v) it follows that

(vi) $$\frac{p}{1-r-p} \le 2^{\vartheta}\frac{a_2}{a_1} - 1 + 2^{\frac{\vartheta+1}{2}}\sqrt{\frac{a_2}{a_1}\left(2^{\vartheta-1}\frac{a_2}{a_1} - 1\right)}$$

and

(vii) $$\frac{p}{1-r-p} \le 2\left(\frac{a_2}{a_1}\right)^2 - 1 + 2\frac{a_2}{a_1}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - 1}.$$

Now we estimate $\beta_{\frac{3+\delta}{2}}$. Due to Jensen's inequality we have $\left(\beta_{\frac{3+\delta}{2}}\right)^{\frac{4}{3+\delta}} \leq \sigma^2 = 1$. Let $\delta = 1$. Then $\beta_{\frac{3+\delta}{2}} = \sigma^2 = 1$ and so $\frac{a_2}{a_1} = (\gamma^3)^2$. Let $\delta \in \left[\frac{1}{3}, 1\right)$. Then $\frac{5-\delta}{2} \leq 2+\delta$, so $\beta_{\frac{5-\delta}{2}}$ exists. Due to the Cauchy-Schwarz-inequality we have

$$1 = (\sigma^2)^2 \leq \beta_{\frac{3+\delta}{2}}\beta_{\frac{5-\delta}{2}};$$

hence

$$\beta_{\frac{3+\delta}{2}} \geq \frac{1}{\beta_{\frac{5-\delta}{2}}} \geq \frac{1}{\left(\gamma^{2+\delta}\right)^{\frac{5-\delta}{2(2+\delta)}}}.$$

altogether

$$\frac{a_2}{a_1} \leq (\gamma^{2+\delta})^{4\vartheta}.$$

Let $k \in \mathbb{N},\ k \geq 2$. For $\delta \geq \frac{1}{2^k+1}$ it follows that $2 + \frac{1-\delta}{2^k} \leq 2+\delta$; hence $\beta_{2+\frac{1-\delta}{2^k}}$ exists. Due to the Cauchy-Schwarz-inequality we have

$$1 = (\sigma^2)^2 \leq \beta_{2+\frac{1-\delta}{2^k}}\beta_{2-\frac{1-\delta}{2^k}}$$

and

$$\left(\beta_{2-\frac{1-\delta}{2^{j+1}}}\right)^{2^j} = \left(\left(\beta_{2-\frac{1-\delta}{2^{j+1}}}\right)^2\right)^{2^{j-1}} \leq \left(\beta_{2-\frac{1-\delta}{2^j}}\beta_2\right)^{2^{j-1}}$$

for $j \in \{1, \dots, k-1\}$. This yields

$$\beta_{\frac{3+\delta}{2}} \geq \left(\beta_{2-\frac{1-\delta}{2^k}}\right)^{2^{k-1}} \geq \frac{1}{\left(\beta_{2+\frac{1-\delta}{2^k}}\right)^{2^{k-1}}} \geq \frac{1}{\left(\gamma^{2+\delta}\right)^{\frac{2^{k+1}+1-\delta}{2(2+\delta)}}}.$$

Altogether we get $\frac{a_2}{a_1} \leq (\gamma^{2+\delta})^{(2^k+2)\vartheta}$ for $\delta \in \left[\frac{1}{2^k+1}, \frac{1}{2^{k-1}+1}\right)$.

Combining this with (vi) and (vii) we obtain the assertion. □

Remark 2.2. For each $\delta \in (0,1]$ equality holds in Theorem 2.1 iff $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$ (where ε_x denotes the Dirac measure in x).

Proof. (i) Let X be a real random variable with $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$. Then $E(X) = 0$, $\mathrm{Var}(X) = 1,\ \gamma^{2+\delta} = 1$ and so $g_\delta(\gamma^{2+\delta}) = 1$ for all $\delta \in (0,1]$. Since $P(X < 0) = P(X > 0) = \frac{1}{2}$ we get equality.

(ii) In Theorem 2.1 we have

$$\left(\frac{\beta^2_{\frac{3+\delta}{2}}}{2\gamma^{2+\delta}}\right)^{\frac{3+\delta}{2}} \leq \alpha^{\frac{3+\delta}{2}} \leq \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}}\beta_{\frac{3+\delta}{2}}.$$

In the first "$\leq$" there is equality iff $|X|^{1/2}$ and $|X|^{1+\delta/2}$ are linearly dependent P-almost surely, i.e. $P(|X| \in \{0, c\}) = 1$ for some $c > 0$. As $E(X) = 0$, we obtain $P^X = p(\varepsilon_c + \varepsilon_{-c}) + (1 - 2p)\varepsilon_0$. So the inequality is sharp iff $g_\delta(\gamma^{2+\delta}) = 1$. With $1 = E(X^2) = 2c^2p$ we obtain

$$\gamma^{2+\delta} = 2pc^{2+\delta} = \frac{1}{(2p)^{\delta/2}}.$$

The functions $h_\delta : (0, \frac{1}{2}] \to \mathbb{R},\ \delta \in (0,1]$, defined by $h_\delta(p) = g_\delta\left(\frac{1}{(2p)^{\delta/2}}\right)$, are strictly decreasing. Due to $p \in \left(0, \frac{1}{2}\right]$ and $h_\delta\left(\frac{1}{2}\right) = 1$ we get $p = \frac{1}{2},\ r = 0$ and $c = 1$, therefore $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$. □

Since the Central Limit Theorem is concerned with sums of random variables (instead of single variables) we need a corresponding generalization of Theorem 2.1.

Corollary 2.3. *Under assumption (M),*

$$E\left(\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right|^{2+\delta}\right) \leq \frac{\gamma^{2+\delta}-1}{n^{\delta/2}} + n^{2-\delta/2}$$

holds, and therefore,

$$P\left(\sum_{i=1}^{n} X_i < 0\right) \leq g_\delta\left(\frac{\gamma^{2+\delta}-1}{n^{\delta/2}} + n^{2-\delta/2}\right) P\left(\sum_{i=1}^{n} X_i > 0\right).$$

Proof.

$$\begin{aligned}
\left|\sum_{i=1}^{n} X_i\right|^{2+\delta} &= \left(\left|\sum_{i=1}^{n} X_i\right|^3\right)^{\frac{2+\delta}{3}} \\
&\leq \left(\sum_{i=1}^{n} |X_i|^3 + 3\sum_{\substack{k,i=1\\k\neq i}}^{n} |X_k^2 X_i| + \sum_{\substack{i,k,l=1\\i\neq k\neq l\neq i}}^{n} |X_i X_k X_l|\right)^{\frac{2+\delta}{3}} \\
&\leq \sum_{i=1}^{n} |X_i|^{2+\delta} + 3\sum_{\substack{k,i=1\\k\neq i}}^{n} |X_k^2 X_i|^{\frac{2+\delta}{3}} + \sum_{\substack{i,k,l=1\\i\neq k\neq l\neq i}}^{n} |X_i X_k X_l|^{\frac{2+\delta}{3}}.
\end{aligned}$$

Since the X_i are independent and, due to $E|X_i|^2 = 1$ and Jensen's inequality, $E|X_i|^\alpha \leq 1$ for $\alpha \leq 2$, we obtain

$$E\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right|^{2+\delta} \leq \frac{1}{n^{1+\delta/2}}(n\gamma^{2+\delta} + (n^3 - n)) = \frac{\gamma^{2+\delta}-1}{n^{\delta/2}} + n^{2-\delta/2}.$$

From $E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right) = 0,\ E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right)^2 = 1$ and Theorem 2.1 the assertion follows. □

We need a bound which does not depend on the number of summands. For this aim we use for $n < \kappa := (4c_\delta\gamma^{2+\delta})^{2/\delta}$ the asymmetry inequality of Corollary 2.3 and for $n \geq \kappa$ the Berry-Esseen bound (the choice of κ as boundary between "small" and "large" n is somewhat arbitrary).

Corollary 2.4. *Under Assumption (M)*

$$P\left(\sum_{i=1}^{n} X_i < 0\right) \leq (f_\delta(\gamma^{2+\delta}) - 1) P\left(\sum_{i=1}^{n} X_i > 0\right),$$

where

$$f_\delta(y) := \max\left\{3, g_\delta(y),\ g_\delta\left(\frac{y-1}{4c_\delta}\right) + (4c_\delta y)^{\frac{4}{\delta}-1}\right\} + 1.$$

Proof. (i) Let $n < \kappa$. Then Corollary 2.3 yields

$$P\left(\sum_{i=1}^{n} X_i < 0\right) \leq g_\delta\left(\frac{\gamma^{2+\delta}-1}{n^{\delta/2}} + n^{2-\delta/2}\right) P\left(\sum_{i=1}^{n} X_i > 0\right).$$

For the function $h : [1,\infty) \to [1,\infty)$ defined by $h(y) := \frac{\gamma^{2+\delta}-1}{y^{\delta/2}} + y^{2-\delta/2}$ we obtain

$$h''(y) = \frac{\delta}{2}\left(1+\frac{\delta}{2}\right)(\gamma^{2+\delta}-1)y^{-2-\delta/2} + \left(2-\frac{\delta}{2}\right)\left(1-\frac{\delta}{2}\right)y^{-\delta/2} > 0,$$

i.e., h is convex. Hence the maximum of

$$h : \{1,\dots,\lfloor\kappa\rfloor\} \to [1,\infty)$$

is attained either for $n = 1$ or for $n = \lfloor\kappa\rfloor$. Since g_δ is strictly increasing this yields

$$g_\delta\left(\frac{\gamma^{2+\delta}-1}{n^{\delta/2}} + n^{2-\delta/2}\right) \le \max\left\{g_\delta(\gamma^{2+\delta}), g_\delta\left(\frac{\gamma^{2+\delta}-1}{4c_\delta\gamma^{2+\delta}} + (4c_\delta\gamma^{2+\delta})^{4/\delta-1}\right)\right\}$$

for all $n < \kappa$.
(ii) Let $n \ge \kappa$, i.e. $n^{\delta/2} \ge 4c_\delta\gamma^{2+\delta}$. Then the (special case of the) theorem of Berry-Esseen yields

$$\left|P\left(\sum_{i=1}^n X_i < 0\right) - \frac{1}{2}\right| \le c_\delta\frac{\gamma^{2+\delta}}{n^{\delta/2}} \le \frac{1}{4}$$

and, therefore,

$$P\left(\sum_{i=1}^n X_i < 0\right)\Bigg/ P\left(\sum_{i=1}^n X_i > 0\right) \le \frac{3/4}{1/4} = 3.$$

Combining (i) and (ii) we obtain the assertion. □

3. SOME FUTHER INEQUALITIES

The next inequality represents a quantification of Lemma 7 by Landers and Rogge [8].

Lemma 3.1. *Under Assumption (M)*

$$P\left(\min_{p\le n\le q}\sum_{i=1}^n X_i \le r\right) - P\left(\max_{p\le n\le q}\sum_{i=1}^n X_i \le r\right)$$
$$\le f_\delta(\gamma^{2+\delta})\left(P\left(\sum_{i=1}^p X_i \le r, \sum_{i=1}^q X_i \ge r\right) + P\left(\sum_{i=1}^p X_i \ge r, \sum_{i=1}^q X_i \le r\right)\right)$$

for all $p, q \in \mathbb{N}$ *s.t.* $p < q$ *and for all* $r \in \mathbb{R}$.

Proof. Using the same notation (A,α,β,A_k) as Landers and Rogge [8, p. 280], we have to prove that

$$P(A) \le f_\delta(\gamma^{2+\delta})(\alpha+\beta).$$

(i) First we show that $P\left(A \cap \{\sum_{i=1}^p X_i \le r\}\right) \le f_\delta(\gamma^{2+\delta})\cdot\alpha$; for this it is sufficient to prove that

$$P\left(A \cap \left\{\sum_{i=1}^p X_i \le r\right\} \cap \left\{\sum_{i=1}^q X_i \le r\right\}\right) \le (f_\delta(\gamma^{2+\delta})-1)\cdot\alpha.$$

But due to the independence of the A_k and $\sum_{i=k+1}^{q} X_i$

$$
\begin{aligned}
P&\left(A \cap \left\{\sum_{i=1}^{p} X_i \le r\right\} \cap \left\{\sum_{i=1}^{q} X_i \le r\right\}\right) \\
&\le \sum_{k=p+1}^{q-1} P(A_k) P\left(\sum_{i=k+1}^{q} X_i \le 0\right) \\
&\le (f_\delta(\gamma^{2+\delta}) - 1) \sum_{k=p+1}^{q-1} P(A_k) P\left(\sum_{i=k+1}^{q} X_i \ge 0\right) \\
&\quad \text{according to Corollary 2.4} \\
&\le (f_\delta(\gamma^{2+\delta}) - 1) \cdot \alpha.
\end{aligned}
$$

(ii) Similarly, it follows that

$$P\left(A \cap \left\{\sum_{i=1}^{k} X_i > r\right\}\right) \le f_\delta(\gamma^{2+\delta}) \cdot \beta.$$

(i) and (ii) yield the assertion. □

A thorough examination of the proof of Lemma 8 of Landers and Rogge [8] allows a generalization and quantification of their result:

Lemma 3.2. *Under Assumption (M),*

(i) $P\left(\sum_{i=1}^{n} X_i \le t, \sum_{i=1}^{n+k} X_i \ge t\right) \le \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{k}{n}}$.

(ii) $P\left(\sum_{i=1}^{n} X_i \ge t, \sum_{i=1}^{n+k} X_i \le t\right) \le \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{k}{n}}$.

Proof. (ii) follows from (i) by replacing X_i by $-X_i$. Analogously to the proof of [8], Lemma 8, we obtain

$$
\begin{aligned}
P&\left(\sum_{i=1}^{n} X_i \le t, \sum_{i=1}^{n+k} X_i \ge t\right) \\
&\le \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \int \left| \Phi\left(\frac{t}{\sqrt{n}}\right) - \Phi\left(\frac{t}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^{k} x_i\right) \right| Q(dx_1, \dots, dx_k) \\
&\le \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{k}{n}} \int \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{k} x_i \right| Q(dx_1, \dots, dx_k) \\
&\le \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{k}{n}},
\end{aligned}
$$

since

$$E\left|\frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i\right| \le \sqrt{E\left|\frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i\right|^2} = 1.$$

□

4. A BERRY-ESSEEN THEOREM FOR RANDOM SUMS

As an important application of these inequalities we state our main result:

Theorem 4.1 (Berry-Esseen type result for random sums)**.** *Let* $\{X_n, n \geq 1\}$ *be i.i.d. random variables with* $EX_n = 0$, $\operatorname{Var} X_n = 1$ *and* $\gamma^{2+\delta} := E|X_n|^{2+\delta} < \infty$ *for some* $\delta \in (0,1]$*; let* $\{N_n,\ n \in \mathbb{N}\}$ *be integer-valued random variables and* $\{\zeta_n,\ n \in \mathbb{N}\}$ *real numbers with* $\lim_{n\to\infty} \zeta_n = 0$ *such that there exist* $d, \tau > 0$ *with*

$$P\left(\left|\frac{N_n}{n\tau} - 1\right| > \zeta_n\right) \leq d\sqrt{\zeta_n}.$$

Then[1]

(i) $$\sup_{t\in\mathbb{R}} \left| P\left(\frac{1}{\sqrt{n\tau}} \sum_{i=1}^{N_n} X_i \leq t\right) - \Phi(t)\right| \leq c_\delta \gamma^{2+\delta}(1 + 2^{2+\delta/2} f_\delta(\gamma^{2+\delta}))\frac{1}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} + 2f_\delta(\gamma^{2+\delta})\sqrt{\frac{2+1/\tau}{\pi}\max\left\{\frac{1}{n}, \zeta_n\right\}} + 2d\sqrt{\zeta_n}.$$

(ii) $$\sup_{t\in\mathbb{R}} \left| P\left(\frac{1}{\sqrt{N_n}} \sum_{i=1}^{N_n} X_i \leq t\right) - \Phi(t)\right| \leq c_\delta \gamma^{2+\delta}(1 + 2^{2+\delta/2} f_\delta(\gamma^{2+\delta}))\frac{1}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} + 2f_\delta(\gamma^{2+\delta})\sqrt{\frac{2+1/\tau}{\pi}\max\left\{\frac{1}{n}, \zeta_n\right\}} + (3d+1)\sqrt{\zeta_n}$$

for all $n \in \mathbb{N}$ *s.t.* $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$.

Since $\frac{n\tau}{2} - n\tau\zeta_n \underset{n\to\infty}{\longrightarrow} \infty$ there exists an n_0 s.t. $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$ for all $n \geq n_0$.

Proof. (i) Let $n \in \mathbb{N}$ fulfill $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$ and define as Landers and Rogge [8, p. 271]

$$b_n(t) := t\sqrt{n\tau} \text{ and } I_n := \{k \in \mathbb{N} : \lfloor n\tau - n\tau\zeta_n\rfloor \leq k \leq \lfloor n\tau + n\tau\zeta_n\rfloor\}.$$

Due to the assumption on N_n we have

$$P(N_n \notin I_n) \leq d\sqrt{\zeta_n}.$$

For

$$A_n(t) := \left\{\max_{k\in I_n} \sum_{i=1}^{k} X_i \leq b_n(t)\right\}, \qquad B_n(t) := \left\{\min_{k\in I_n} \sum_{i=1}^{k} X_i \leq b_n(t)\right\}$$

follows (see Landers and Rogge [8, p. 272]) for each $t \in \mathbb{R}$

$$P(A_n(t) \cap \{N_n \in I_n\}) \leq P\left(\sum_{i=1}^{N_n} X_i \leq b_n(t), N_n \in I_n\right) \leq P(B_n(t))$$

[1] $\lfloor x\rfloor := \max\{n \in \mathbb{N} : n \leq x\}$.

and

$$P(A_n(t)) \leq P\left(\sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq b_n(t)\right) \leq P(B_n(t)).$$

Using the Berry-Esseen theorem and the result (3.3) of Petrov [10, p. 114], we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left| P\left(\sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq b_n(t)\right) - \Phi(t) \right| \\
&\leq \sup_{t\in\mathbb{R}} \left| P\left(\frac{1}{\sqrt{\lfloor n\tau \rfloor}} \sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) - \Phi\left(t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) \right| + \sup_{t\in\mathbb{R}} \left| \Phi\left(t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) - \Phi(t) \right| \\
&\leq \frac{c_\delta \gamma^{2+\delta}}{\lfloor n\tau \rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e \lfloor n\tau \rfloor}}.
\end{aligned}$$

For $p(n) := \lfloor n\tau - n\tau\zeta_n \rfloor,\ q(n) := \lfloor n\tau + n\tau\zeta_n \rfloor$ we obtain from Lemma 3.1

$$\begin{aligned}
&P(B_n(t)) - P(A_n(t)) \\
&\quad \leq f_\delta(\gamma^{2+\delta}) \cdot \left(P\left(\sum_{i=1}^{p(n)} X_i \leq b_n(t) \leq \sum_{i=1}^{q(n)} X_i\right) + P\left(\sum_{i=1}^{p(n)} X_i \geq b_n(t) \geq \sum_{i=1}^{q(n)} X_i\right)\right).
\end{aligned}$$

According to Lemma 3.2 it follows that

$$\begin{aligned}
\sup_{t\in\mathbb{R}} P\left(\sum_{i=1}^{p(n)} X_i \leq b_n(t) \leq \sum_{i=1}^{q(n)} X_i\right) &\leq \frac{2c_\delta \gamma^{2+\delta}}{(p(n))^{\delta/2}} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{q(n)-p(n)}{p(n)}} \\
&\leq \frac{2^{1+\delta/2} c_\delta \gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2+1/\tau}{\pi} \max\left\{\frac{1}{n}, \zeta_n\right\}},
\end{aligned}$$

since $p(n) \geq n\tau - n\tau\zeta_n - 1 \geq n\tau/2$ and, therefore,

$$\sqrt{\frac{q(n)-p(n)}{p(n)}} \leq \sqrt{\frac{2n\tau\zeta_n + 1}{n\tau/2}} = \sqrt{4\zeta_n + \frac{2}{n\tau}};$$

analogously

$$\sup_{t\in T} P\left(\sum_{i=1}^{p(n)} X_i \geq b_n(t) \geq \sum_{i=1}^{q(n)} X_i\right) \leq \frac{2^{1+\delta/2} c_\delta \gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2+1/\tau}{\pi} \max\left\{\frac{1}{n}, \zeta_n\right\}}.$$

Altogether we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left|P\left(\sum_{i=1}^{N_n} X_i \le b_n(t)\right) - \Phi(t)\right| \\
&\quad\le \sup_{t\in\mathbb{R}}\left|P\left(\sum_{i=1}^{N_n} X_i \le b_n(t), N_n \in I_n\right) - \Phi(t)\right| + P(N_n \notin I_n) \\
&\quad\le P(B_n(t)) - P(A_n(t)) + \frac{c_\delta \gamma^{2+\delta}}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} + 2d\sqrt{\zeta_n} \\
&\quad\le 2f_\delta\left(\gamma^{2+\delta}\right)\left(\frac{2^{1+\delta}c_\delta\gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2+1/\tau}{\pi}\max\left\{\frac{1}{n},\zeta_n\right\}}\right) \\
&\qquad\quad + \frac{c_\delta \gamma^{2+\delta}}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} + 2d\sqrt{\zeta_n} \\
&\quad\le c_\delta\gamma^{2+\delta}(1 + 2^{2+\delta/2}f_\delta(\gamma^{2+\delta}))\frac{1}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} \\
&\qquad\quad + 2f_\delta(\gamma^{2+\delta})\sqrt{\frac{2+1/\tau}{\pi}\max\left\{\frac{1}{n},\zeta_n\right\}} + 2d\sqrt{\zeta_n}.
\end{aligned}$$

(ii) Applying Lemma 1 of Michel and Pfanzagl [9] for

$$r = \zeta_n, \quad f = \frac{1}{\sqrt{n\tau}}\sum_{i=1}^{N_n} X_i, \quad g = \sqrt{\frac{N_n}{n\tau}}$$

and using the fact that $\left|\sqrt{\frac{N_n}{n\tau}} - 1\right| > \sqrt{\zeta_n}$ implies $\left|\frac{N_n}{n\tau} - 1\right| > \zeta_n$, hence

$$P\left(\left|\sqrt{\frac{N_n}{n\tau}} - 1\right| > \sqrt{\zeta_n}\right) \le P\left(\left|\frac{N_n}{n\tau} - 1\right| > \zeta_n\right) \le d\sqrt{\zeta_n},$$

we obtain from part (i)

$$\begin{aligned}
&\sup_{t\in T}\left|P\left(\frac{1}{\sqrt{N_n}}\sum_{i=1}^{N_n} X_i \le t\right) - \Phi(t)\right| \\
&\quad\le c_\delta\gamma^{2+\delta}(1 + 2^{2+\delta/2}f_\delta(\gamma^{2+\delta}))\frac{1}{\lfloor n\tau\rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e\lfloor n\tau\rfloor}} \\
&\qquad\quad + 2f_\delta(\gamma^{2+\delta})\sqrt{\frac{2+1/\tau}{\pi}\max\left\{\frac{1}{n},\zeta_n\right\}} + (3d+1)\sqrt{\zeta_n}.
\end{aligned}$$

□

REFERENCES

[1] F.J. ANSCOMBE, Large sample theory of sequential estimation, *Proc. Cambridge Philos. Soc.*, **48** (1952), 600–607.

[2] P. VAN BEEK, An application of Fourier methods to the problem of sharpening the Berry-Esseen inequality, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **23** (1972), 187–196.

[3] J.R. BLUM, D.L. HANSON AND J.I. ROSENBLATT, On the central limit theorem for the sum of a random number of independent random variables,. *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **1** (1963), 389–393.

[4] H. CALLAERT AND P. JANSSEN, A note on the convergence rate of random sums, *Rev. Roum. Math. Pures et Appl.*, **28** (1983), 147–151.

[5] Y.S. CHOW AND H. TEICHER, *Probability Theory: Independence, Interchangeability, Martingales*. Springer, New York, 1978.

[6] H. KLÄVER, *Ein Berry-Esseen-Satz für Zufallssummen*. Diploma Thesis, Univ. Münster, 2002

[7] A. KRAJKA AND Z. RYCHLIK, The order of approximation in the central limit theorem for random summation, *Acta Math. Hungar.*, **51** (1988), 109–115.

[8] D. LANDERS AND L. ROGGE, The exact approximation order in the central limit theorem for random summation, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **36** (1976), 269–283.

[9] R. MICHEL AND J. PFANZAGL, The accuracy of the normal approximations for minimum contrast estimates, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **18** (1971), 73–84.

[10] V.V. PETROV, *Sums of Independent Random Variables.* Springer, Heidelberg, 1975.

[11] A. RENYI, On the central limit theorem for the sum of a random number of independent random variables. *Acta Math. Acad. Sci. Hung.*, **11** (1960), 97–102.

[12] W. TYSIAK, *Gleichmäßige Berry-Esseen-Abschätzungen*. Ph. Thesis, Univ. Wuppertal, 1983.