Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 69, 2004

SOME INEQUALITIES FOR A CLASS OF GENERALIZED MEANS

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Received 15 March, 2004; accepted 04 April, 2004
Communicated by D. Stefanescu

ABSTRACT. In this paper, we define a symmetric function, show its properties, and establish several analytic inequalities, some of which are "Ky Fan" type inequalities. The harmonic-geometric mean inequality is refined.

Key words and phrases: Symmetric function ; Ky Fan inequality ; Harmonic-geometric mean inequality.

2000 *Mathematics Subject Classification.* 05E05, 26D20.

1. INTRODUCTION

Let $x = (x_1, x_2, \dots, x_n)$ be an n-tuple of positive numbers. The un-weighted arithmetic, geometric and harmonic means of x, denoted by $A_n(x)$, $G_n(x)$, $H_n(x)$, respectively, are defined as follows

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i,\ G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}},\ H_n(x) = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}.$$

Assume that $0 \le x_i < 1$, $1 \le i \le n$ and define $1 - x = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$. Throughout the sequel the symbols $A_n(1-x)$, $G_n(1-x)$, $H_n(1-x)$ will stand for the un-weighted arithmetic, geometric, harmonic means of $1 - x$.

A remarkable new counterpart of the inequality $G_n \le A_n$ has been published in [1].

Theorem 1.1. *If* $0 < x_i \le \frac{1}{2}$, *for all* $i = 1, 2, \dots, n$, *then*

$$\frac{G_n(x)}{G_n(1-x)} \le \frac{A_n(x)}{A_n(1-x)} \tag{1.1}$$

with equality if and only if all the x_i *are equal.*

ISSN (electronic): 1443-5756

A Project Supported by Scientific Research Fund of Hunan Provincial Education Department (China)(granted 03C427).
054-04

This result, commonly referred to as the Ky Fan inequality, has stimulated the interest of many researchers. New proofs, improvements and generalizations of the inequality (1.1) have been found. For more details, interested readers can see [2], [3] and [4].

W.-L. Wang and P.-F. Wang [5] have established a counterpart of the classical inequality $H_n \le G_n \le A_n$. Their result reads as follows.

Theorem 1.2. *If* $0 < x_i \le \frac{1}{2}$, *for all* $i = 1, 2, \dots, n$, *then*

$$\frac{H_n(x)}{H_n(1-x)} \le \frac{G_n(x)}{G_n(1-x)} \le \frac{A_n(x)}{A_n(1-x)}. \tag{1.2}$$

All kinds of means about numbers and their inequalities have stimulated the interest of many researchers. Here we define a new mean, that is:

Definition 1.1. Let $x \in \mathbb{R}_+^n = \{x | x = (x_1, x_2, \dots, x_n) | x_i > 0, i = 1, 2, \dots, n\}$, we define the symmetric function as follows

$$H_n^r(x) = H_n^r(x_1, x_2, \dots, x_n) = \left[\prod_{1 \le i_1 < \cdots i_r \le n} \left(\frac{r}{\sum_{i=1}^r x_{i_j}^{-1}} \right) \right]^{\frac{1}{\binom{n}{r}}}.$$

Clearly $H_n^n(x) = H_n(x)$, $H_n^1(x) = G_n(x)$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

The Schur-convex function was introduced by I. Schur in 1923 [7]. Its definition is as follows:

Definition 1.2. $f : I^n \to \mathbb{R} (n > 1)$ is called Schur-convex if $x \prec y$, then

$$f(x) \le f(y) \tag{1.3}$$

for all $x, y \in I^n = I \times I \times \cdots \times I$ (n copies). It is called strictly Schur-convex if the inequality is strict; f is called Schur-concave (resp. strictly Schur-concave) if the inequality (1.3) is reversed. For more details, interested readers can see [6], [7] and [8].

The paper is organized as follows. A refinement of harmonic-geometric mean inequality is obtained in Section 3. In Section 4, we investigate the Schur-convexity of the symmetric function. Several "Ky Fan" type inequalities are obtained in Section 5.

2. Lemmas

In this section, we give the following lemmas for the proofs of our main results.

Lemma 2.1. ([5]) *If* $0 < x_i \le \frac{1}{2}$, *for all* $i = 1, 2, \dots, n$, *then*

$$\frac{\sum_{i=1}^n \frac{1}{1-x_i}}{\sum_{i=1}^n \frac{1}{x_i}} \le \left[\frac{\prod_{i=}^n \frac{1}{1-x_i}}{\prod_{i=}^n \frac{1}{x_i}} \right]^{\frac{1}{n}} \text{ or } \frac{H_n(x)}{H_n(1-x)} \le \frac{G_n(x)}{G_n(1-x)}. \tag{2.1}$$

Lemma 2.2. *If* $0 < x_i \le \frac{1}{2}$, *for all* $i = 1, 2, \dots, n+1$, *and* $S_{n+1} = \sum_{i=1}^{n+1} \frac{1}{x_i}$, $\overline{S}_{n+1} = \sum_{i=1}^{n+1} \frac{1}{1-x_i}$, *then*

$$\left[\frac{\sum_{i=1}^{n+1} \left(\overline{S}_{n+1} - \frac{1}{1-x_i} \right)}{\sum_{i=1}^{n+1} \left(S_{n+1} - \frac{1}{x_i} \right)} \right]^n \le \left[\frac{\prod_{i=1}^{n+1} \left(\overline{S}_{n+1} - \frac{1}{1-x_i} \right)}{\prod_{i=1}^{n+1} \left(S_{n+1} - \frac{1}{x_i} \right)} \right]^{\frac{1}{n+1}}. \tag{2.2}$$

Proof. Inequality (2.2) is equivalent to the following

$$n \ln \frac{\overline{S}_{n+1}}{S_{n+1}} \le \frac{1}{n+1} \ln \left[\frac{\prod_{i=1}^{n+1} \left(\overline{S}_{n+1} - \frac{1}{1-x_i} \right)}{\prod_{i=1}^{n+1} (S_{n+1} - \frac{1}{x_i})} \right]$$

Since $0 < x_i \le \frac{1}{2}$, and $1 - x_i \ge x_i$, it follows that

$$\begin{aligned} \frac{\overline{S}_{n+1} - \frac{1}{1-x_j}}{S_{n+1} - \frac{1}{x_j}} &= \frac{\frac{1}{1-x_1} + \cdots + \frac{1}{1-x_{j-1}} + \frac{1}{1-x_{j+1}} + \cdots + \frac{1}{1-x_{n+1}}}{\frac{1}{x_1} + \cdots + \frac{1}{x_{j-1}} + \frac{1}{x_{j+1}} + \cdots + \frac{1}{x_{n+1}}} \\ &\ge \frac{\frac{1}{1-x_1} \cdots \frac{1}{1-x_{j-1}} \frac{1}{1-x_{j+1}} \cdots \frac{1}{1-x_{n+1}}}{\frac{1}{x_1} \cdots \frac{1}{x_{j-1}} \frac{1}{x_{j+1}} \cdots \frac{1}{x_{n+1}}}. \end{aligned}$$

By the above inequality and Lemma 2.1, we have

$$\begin{aligned} \frac{1}{n+1} \ln \prod_{i=1}^{n+1} \frac{\overline{S}_{n+1} - \frac{1}{1-x_i}}{S_{n+1} - \frac{1}{x_i}} &\ge \frac{1}{n+1} \ln \prod_{i=1}^{n+1} \left[\left(\frac{1}{1-x_i} \right) \Big/ \left(\frac{1}{x_i} \right) \right]^n \\ &= n \ln \prod_{i=1}^{n+1} \left[\left(\frac{1}{1-x_i} \right) \Big/ \left(\frac{1}{x_i} \right) \right]^{n+1} \\ &\ge n \ln \frac{\frac{1}{1-x_1} + \cdots + \frac{1}{1-x_{n+1}}}{\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}}}, \end{aligned}$$

or

$$n \ln \frac{\overline{S}_{n+1}}{S_{n+1}} \le \frac{1}{n+1} \ln \left[\frac{\prod_{i=1}^{n+1} \left(\overline{S}_{n+1} - \frac{1}{1-x_i} \right)}{\prod_{i=1}^{n+1} \left(S_{n+1} - \frac{1}{x_i} \right)} \right].$$

□

Lemma 2.3. [6, p. 259]. *Let $f(x) = f(x_1, x_2, \dots, x_n)$ be symmetric and have continuous partial derivatives on I^n, where I is an open interval. Then $f : I^n \to \mathbb{R}$ is Schur-convex if and only if*

$$(x_i - x_j) \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right) \ge 0 \tag{2.3}$$

on I^n. It is strictly Schur-convex if (2.3) is a strict inequality for $x_i \ne x_j$, $1 \le i, j \le n$.

Since $f(x)$ is symmetric, Schur's condition can be reduced as [7, p. 57]

$$(x_1 - x_2) \left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \ge 0, \tag{2.4}$$

and f is strictly Schur-convex if (2.4) is a strict inequality for $x_1 \ne x_2$. The Schur condition that guarantees a symmetric function being Schur-concave is the same as (2.3) or (2.4) except the direction of the inequality.

In Schur's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n. Lemma 2.3 remains true if we replace I^n by a set $A \subseteq \mathbb{R}^n$ with the following properties ([7, p. 57]):

(i) A is convex and has a nonempty interior;
(ii) A is symmetric in the sense that $x \in A$ implies $Px \in A$ for any $n \times n$ permutation matrix P.

3. REFINEMENT OF THE HARMONIC-GEOMETRIC MEAN INEQUALITY

The goal of this section is to obtain the basic inequality of $H_n^r(x)$, and give a refinement of the Harmonic-Geometric mean inequality.

Theorem 3.1. *Let* $x \in \mathbb{R}_+^n = \{x | x = (x_1, x_2, \dots, x_n) | x_i > 0, i = 1, 2, \dots, n\}$, *then*

$$H_n^{r+1}(x) \le H_n^r(x), \qquad r = 1, 2, \dots, n-1. \tag{3.1}$$

Proof. By the arithmetic-geometric mean inequality and the monotonicity of the function $y = \ln x$, we have

$$\begin{aligned}
\binom{n}{r+1} \ln H_n^{r+1}(x) &= \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \ln \frac{r+1}{\sum_{j=1}^{r+1} x_{i_j}^{-1}} \\
&= \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \ln \left[\frac{(r+1)r}{(r+1)\sum_{k=1}^{r+1} x_{i_k}^{-1} - \sum_{j=1}^{r+1} x_{i_j}^{-1}} \right] \\
&= \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \ln \left\{ \frac{r}{\left[\sum_{j=1}^{r+1} \left(\sum_{k=1}^{r+1} x_{i_k}^{-1} - x_{i_j}^{-1} \right)\right] \Big/ (r+1)} \right\} \\
&\le \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \ln \left[\frac{r}{\left(\prod_{j=1}^{r+1} \left(\sum_{k=1}^{r+1} x_{i_k}^{-1} - x_{i_j}^{-1} \right) \right)^{\frac{1}{r+1}}} \right] \\
&= \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \ln \left[\prod_{j=1}^{r+1} \frac{r}{\sum_{k=1}^{r+1} x_{i_k}^{-1} - x_{i_j}^{-1}} \right]^{\frac{1}{r+1}} \\
&= \frac{1}{r+1} \sum_{1 \le i_1 < \cdots < i_{r+1} \le n} \left[\sum_{j=1}^{r+1} \ln \frac{r}{\sum_{k=1}^{r+1} x_{i_k}^{-1} - x_{i_j}^{-1}} \right] \\
&= \frac{1}{r+1} \sum_{j=1}^{n} \sum_{1 \le i_1 < \cdots < i_r \le n}^{i_1, \dots, i_r \ne j} \ln \frac{r}{\sum_{k=1}^{r} x_{i_k}^{-1}}.
\end{aligned}$$

Let

$$S_j = \sum_{1 \le i_1 < \cdots < i_r \le n}^{i_1, \dots, i_r \ne j} \ln \frac{r}{\sum_{k=1}^{r} x_{i_k}^{-1}}, \qquad j = 1, 2, \dots, n.$$

We can easily get

$$\sum_{j=1}^{n} S_j = (n-r) \sum_{1 \le i_1 < \cdots < i_r \le n} \ln \frac{r}{\sum_{k=1}^{r} x_{i_k}^{-1}} = (n-r) \binom{n}{r} \ln H_n^r(x).$$

Thus

$$\binom{n}{r+1} \ln H_n^{r+1}(x) \le \frac{n-r}{r+1} \binom{n}{r} \ln H_n^r(x) = \binom{n}{r+1} \ln H_n^r(x),$$

or

$$H_n^{r+1}(x) \le H_n^r(x), \qquad r = 1, 2, \dots, n-1.$$

□

Corollary 3.2. *Let* $x \in \mathbb{R}_+^n = \{x | x = (x_1, x_2, \dots, x_n) | x_i > 0, i = 1, 2, \dots, n\}$, *then*

$$H_n(x) \le H_n^{n-1}(x) \le \cdots \le H_n^2(x) \le H_n^1(x) = G_n(x). \tag{3.2}$$

Remark 3.3. The corollary refines the harmonic-geometric mean inequality.

4. SCHUR-CONVEXITY OF THE FUNCTION $H_n^r(x)$

In this section, we investigate the Schur-convexity of the function $H_n^r(x)$, and establish several analytic inequalities by use of the theory of majorization.

Theorem 4.1. *Let* $\mathbb{R}_+^n = \{x|x = (x_1, x_2, \dots, x_n)|x_i > 0, i = 1, 2, \dots, n\}$*, then the function* $H_n^r(x)$ *is Schur-concave in* $\mathbb{R}_+^n$*.*

Proof. It is clear that $H_n^r(x)$ is symmetric and has continuous partial derivatives on $\mathbb{R}_+^n$. By Lemma 2.3, we only need to prove

$$(x_1 - x_2)\left(\frac{\partial H_n^r(x)}{\partial x_1} - \frac{\partial H_n^r(x)}{\partial x_2}\right) \le 0.$$

As matter of fact, we can easily derive

$$\ln H_n^r(x) = \frac{1}{\binom{n}{r}} \sum_{2 \le i_1 < \cdots < i_r \le n} \ln \frac{r}{\sum_{j=1}^{r} x_{i_j}^{-1}} + \sum_{2 \le i_1 < \cdots < i_{r-1} \le n} \ln \frac{r}{x_1^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}}.$$

Differentiating $\ln H_n^r(x)$ with respect to x_1, we have

$$\begin{aligned}
\frac{\partial H_n^r(x)}{\partial x_1} &= \frac{H_n^r(x)}{\binom{n}{r}} \left(\sum_{2 \le i_1 < \cdots < i_{r-1} \le n} \frac{1}{x_1^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}} \right) \cdot \frac{1}{x_1^2} \\
&= \frac{H_n^r(x)}{\binom{n}{r}} \cdot \frac{1}{x_1^2} \left[\left(\sum_{3 \le i_1 < \cdots < i_{r-1} \le n} \frac{1}{x_1^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}} \right) \right. \\
&\qquad \left. + \sum_{3 \le i_1 < \cdots < i_{r-2} \le n} \frac{1}{\left(x_1^{-1} + x_2^{-1} + \sum_{j=1}^{r-2} x_{i_j}^{-1}\right)} \right].
\end{aligned}$$

Similar to the above, we can also obtain

$$\begin{aligned}
\frac{\partial H_n^r(x)}{\partial x_2} &= \frac{H_n^r(x)}{\binom{n}{r}} \left(\sum_{2 \le i_1 < \cdots < i_{r-1} \le n} \frac{1}{x_2^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}} \right) \cdot \frac{1}{x_2^2} \\
&= \frac{H_n^r(x)}{\binom{n}{r}} \cdot \frac{1}{x_2^2} \left[\left(\sum_{3 \le i_1 < \cdots < i_{r-1} \le n} \frac{1}{x_2^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}} \right) \right. \\
&\qquad \left. + \sum_{3 \le i_1 < \cdots < i_{r-2} \le n} \frac{1}{\left(x_1^{-1} + x_2^{-1} + \sum_{j=1}^{r-2} x_{i_j}^{-1}\right)} \right].
\end{aligned}$$

Thus

$$
\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial H_n^r(x)}{\partial x_1} - \frac{\partial H_n^r(x)}{\partial x_2}\right) \\
&= (x_1 - x_2)\frac{H_n^r(x)}{\binom{n}{r}}\left[\left(\sum_{2\le i_1<\cdots<i_{r-1}\le n} \frac{1}{x_1^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}}\right)\cdot \frac{1}{x_1^2}\right. \\
&\qquad - \left(\sum_{2\le i_1<\cdots<i_{r-1}\le n} \frac{1}{x_2^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}}\right)\cdot \frac{1}{x_2^2} \\
&\qquad \left. + \sum_{3\le i_1<\cdots<i_{r-2}\le n} \frac{1}{\left(x_1^{-1} + x_2^{-1} + \sum_{j=1}^{r-2} x_{i_j}^{-1}\right)}\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)\right] \\
&= -(x_1 - x_2)^2\left[\frac{(x_1 + x_2)}{x_1^2 x_2^2} \sum_{3\le i_1<\cdots<i_{r-2}\le n} \frac{1}{x_1^{-1} + x_2^{-1} + \sum_{j=1}^{r-2} x_{i_j}^{-1}}\right. \\
&\qquad \left. + \sum_{2\le i_1<\cdots<i_{r-1}\le n} \frac{1 + (x_1 + x_2)\sum_{j=1}^{r-1} x_{i_j}^{-1}}{x_1^2 x_2^2 \left(x_1^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}\right)\left(x_2^{-1} + \sum_{j=1}^{r-1} x_{i_j}^{-1}\right)}\right] \\
&\le 0.
\end{aligned}
$$

□

Corollary 4.2. *Let $x_i > 0$, $i = 1, 2, \ldots, n$, $n \ge 2$,and $\sum_{i=1}^{n} x_i = s$, $c > 0$, then*

$$\frac{H_n^r(c + x)}{H_n^r(x)} \ge \left(\frac{nc}{s} + 1\right)^{\frac{1}{\binom{n}{r}}}, \qquad r = 1, 2, \ldots, n, \tag{4.1}$$

where $c + x = (c + x_1, c + x_2, \ldots, c + x_n)$.

Proof. By [9], we have

$$\frac{c + x}{nc + s} = \left(\frac{c + x_1}{nc + s}, \ldots, \frac{c + x_n}{nc + s}\right) \prec \left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}\right) = \frac{x}{s}.$$

Using Theorem 4.1, we obtain

$$H_n^r\left(\frac{c + x}{nc + s}\right) \ge H_n^r\left(\frac{x}{s}\right).$$

Or

$$\frac{H_n^r(c + x)}{H_n^r(x)} \ge \left(\frac{nc}{s} + 1\right)^{\frac{1}{\binom{n}{r}}}.$$

□

Corollary 4.3. *Let $x_i > 0, i = 1, 2, \ldots, n$, $n \ge 2$, and $\sum_{i=1}^{n} x_i = s$, $c \ge s$, then*

$$\frac{H_n^r(c - x)}{H_n^r(x)} \ge \left(\frac{nc}{s} - 1\right)^{\frac{1}{\binom{n}{r}}}, \; r = 1, 2, \ldots, n, \tag{4.2}$$

where $c - x = (c - x_1, c - x_2, \ldots, c - x_n)$.

Proof. By [9], we have

$$\frac{c-x}{nc-s} = \left(\frac{c-x_1}{nc-s}, \dots, \frac{c-x_n}{nc-s}\right) \prec \left(\frac{x_1}{s}, \dots, \frac{x_n}{s}\right) = \frac{x}{s}.$$

Using Theorem 4.1, we obtain

$$H_n^r\left(\frac{c-x}{nc-s}\right) \geq H_n^r\left(\frac{x}{s}\right),$$

or

$$\frac{H_n^r(c-x)}{H_n^r(x)} \geq \left(\frac{nc}{s}-1\right)^{\frac{1}{\binom{n}{r}}}.$$

□

Remark 4.4. Let $c = s = 1$, we can obtain

$$\frac{H_n^r(1-x)}{H_n^r(x)} \geq (n-1)^{\frac{1}{\binom{n}{r}}}, \qquad r = 1, 2, \dots, n.$$

In particular,

$$\frac{H_n(1-x)}{H_n(x)} \geq (n-1), \qquad \frac{G_n(1-x)}{G_n(x)} \geq \sqrt[n]{n-1}.$$

5. SOME "KY FAN" TYPE INEQUALITIES

In this section, some "Ky Fan" type inequalities are established, the Ky Fan inequality is generalized.

Theorem 5.1. *Assume that* $0 < x_i \leq \frac{1}{2}$, $i = 1, 2, \dots, n$, *then*

$$\frac{H_n^{r+1}(x)}{H_n^{r+1}(1-x)} \leq \left[\frac{H_n^r(x)}{H_n^r(1-x)}\right]^{\frac{1}{r}}, \qquad r = 1, 2, \dots, n-1. \tag{5.1}$$

Proof. Set

$$\varphi_r = \frac{H_n^r(x)}{H_n^r(1-x)} = \prod_{1 \leq i_1 < \dots < i_r \leq n} \left[\frac{\sum_{j=1}^r \frac{1}{1-x_{i_j}}}{\sum_{j=1}^r \frac{1}{x_{i_j}}}\right]^{\frac{1}{\binom{n}{r}}}.$$

By Lemma 2.2 and the monotonicity of the function $y = \ln x$, we have

$$\begin{aligned}
\binom{n}{r+1} \ln \phi_{r+1} &= \sum_{1 \leq i_1 < \dots < i_{r+1} \leq n} \ln \frac{\sum_{j=1}^{r+1} \frac{1}{1-x_{i_j}}}{\sum_{j=1}^{r+1} \frac{1}{x_{i_j}}} \\
&= \sum_{1 \leq i_1 < \dots < i_{r+1} \leq n} \ln \frac{\sum_{j=1}^{r+1} \left(\sum_{k=1}^{r+1} \frac{1}{1-x_{i_k}} - \frac{1}{1-x_{i_j}}\right)}{\sum_{j=1}^{r+1} \left(\sum_{k=1}^{r+1} \frac{1}{x_{i_k}} - \frac{1}{x_{i_j}}\right)} \\
&\leq \sum_{1 \leq i_1 < \dots < i_{r+1} \leq n} \ln \left[\prod_{j=1}^{r+1} \frac{\sum_{k=1}^{r+1} \frac{1}{1-x_{i_k}} - \frac{1}{1-x_{i_j}}}{\sum_{k=1}^{r+1} \frac{1}{x_{i_k}} - \frac{1}{x_{i_j}}}\right]^{\frac{1}{r(r+1)}} \\
&= \frac{1}{r(r+1)} \sum_{j=1}^{n} \sum_{1 \leq i_1 < \dots < i_r \leq n}^{i_1, \dots, i_r \neq j} \ln \frac{\sum_{k=1}^r \frac{1}{1-x_{i_k}}}{\sum_{k=1}^r \frac{1}{x_{i_k}}}.
\end{aligned}$$

Similar to Theorem 3.1, we can derive

$$\binom{n}{r+1}\ln\phi_{r+1} \leq \frac{1}{r(r+1)}(n-r)\binom{n}{r}\ln\phi_r = \frac{1}{r}\binom{n}{r+1}\ln\phi_r.$$

Thus

$$(\phi_r)^{\frac{1}{r}} \geq \phi_{r+1},$$

or

$$\frac{H_n^{r+1}(x)}{H_n^{r+1}(1-x)} \leq \left[\frac{H_n^r(x)}{H_n^r(1-x)}\right]^{\frac{1}{r}}, \qquad r = 1, 2, \ldots, n-1.$$

□

Remark 5.2. By Theorem 5.1, we can obtain

$$\frac{H_n^2(x)}{H_n^2(1-x)} \leq \frac{H_n^1(x)}{H_n^1(1-x)} = \frac{G_n(x)}{G_n(1-x)} \leq \frac{A_n(x)}{A_n(1-x)}. \tag{5.2}$$

This is a generalization of the "Ky Fan" inequality.

By Lemma 2.1 and the proof of Theorem 3.1, we have the following

Theorem 5.3. *If* $0 < x_i \leq \frac{1}{2}$, $i = 1, 2, \ldots, n$, *then*

$$\frac{\prod_{i=1}^n (x_i)}{\prod_{i=1}^n (1-x_i)} \leq \frac{H_n(x)}{H_n(1-x)} \leq \frac{G_n(x)}{G_n(1-x)} \leq \frac{A_n(x)}{A_n(1-x)}. \tag{5.3}$$

The inequality (5.3) generalizes the inequality (1.2).

Theorem 5.4. *If* $0 < x_i \leq \frac{1}{2}$, $i = 1, 2, \ldots, n$, *then*

$$\frac{H_n^r(x)}{H_n^r(1-x)} \leq \frac{H_n^1(x)}{H_n^1(1-x)} = \frac{G_n(x)}{G_n(1-x)} \leq \frac{A_n(x)}{A_n(1-x)}, \quad r = 2, \ldots, n. \tag{5.4}$$

Proof. Set

$$\varphi_r = \frac{H_n^r(x)}{H_n^r(1-x)} = \prod_{1 \leq i_1 < \cdots < i_r \leq n} \left[\frac{\sum_{j=1}^r \frac{1}{1-x_{i_j}}}{\sum_{j=1}^r \frac{1}{x_{i_j}}}\right]^{\frac{1}{\binom{n}{r}}}.$$

By Lemma 2.1 and the monotonicity of the function $y = \ln x$, we have

$$\begin{aligned}
\binom{n}{r}\ln\phi_r &= \sum_{1 \leq i_1 < \cdots < i_{r+1} \leq n} \ln \frac{\sum_{j=1}^r \frac{1}{1-x_{i_j}}}{\sum_{j=1}^r \frac{1}{x_{i_j}}} \\
&\leq \sum_{1 \leq i_1 < \cdots < i_{r+1} \leq n} \ln \left[\frac{\prod_{j=1}^r \frac{1}{1-x_{i_j}}}{\prod_{j=1}^r \frac{1}{x_{i_j}}}\right]^{\frac{1}{r}} \\
&= \frac{1}{r} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \sum_{j=1}^r \ln \frac{\frac{1}{1-x_{i_j}}}{\frac{1}{x_{i_j}}}.
\end{aligned}$$

By knowledge of combination, we can easily find

$$\begin{aligned}\binom{n}{r}\ln\phi_r &\le \frac{1}{r}\ln\left[\prod_{i=1}^{n}\frac{\frac{1}{1-x_i}}{\frac{1}{x_i}}\right]^{\binom{n-1}{r-1}}\\ &= \frac{1}{r}\binom{n-1}{r-1}\ln\left[\prod_{i=1}^{n}\frac{\frac{1}{1-x_i}}{\frac{1}{x_i}}\right]\\ &= \frac{1}{r}\binom{n-1}{r-1}\ln\phi_1 = \binom{n}{r}\ln\phi_1.\end{aligned}$$

Thus

$$\phi_r \le \phi_1,\ r = 2,\dots,n,$$

or

(5.5) $$\frac{H_n^r(x)}{H_n^r(1-x)} \le \frac{G_n(x)}{G_n(1-x)},\ r = 2,\dots,n.$$

□

The inequality (5.5) generalizes the “Ky Fan” inequality.

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