

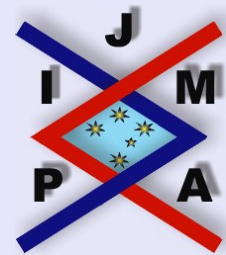
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ON HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract

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Abstract

In the present paper, by introducing some parameters, new forms of Hardy-Hilbert's inequalities are given.

2000 Mathematics Subject Classification: 26D15.

Key words: Hardy-Hilbert's integral inequality.

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1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(t), g(t) \geq 0$, $0 < \int_0^\infty f^p(t)dt < \infty$, and $0 < \int_0^\infty g^q(t)dt < \infty$, then, the Hardy-Hilbert integral inequality is given by:

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(t)dt \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(t)dt \right)^{\frac{1}{q}},$$

where the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is the best possible (see [1]).

Yang [2] and [3] gave the following generalization of (1.1)

$$(1.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y-2\alpha)^\lambda} dx dy \leq K_\lambda^{\frac{1}{p}}(p) K_\lambda^{\frac{1}{q}}(q) \left(\int_0^\infty (t-\alpha)^{1-\lambda} f^p(t)dt \right)^{\frac{1}{p}} \times \left(\int_0^\infty (t-\alpha)^{1-\lambda} g^q(t)dt \right)^{\frac{1}{q}},$$

where

$$K_\lambda(r) = \int_0^\infty \frac{u^{\frac{1}{r}-1}}{(1+u)^\lambda} du = B\left(\frac{1}{r}, \lambda - \frac{1}{r}\right) \quad 0 < \lambda \leq 1, \lambda > \frac{1}{r} > 0,$$

B is the beta function defined by

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad p, q > 0$$



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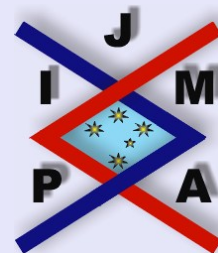
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and, for $0 < T < \infty$,

$$(1.3) \quad \int_0^T \int_0^T \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \\ \leq \beta \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(\int_\alpha^T \left[1 - \frac{1}{2} \left(\frac{t}{T} \right)^{\frac{\lambda}{2}} \right] t^{1-\lambda} f^2(t) dt \right)^{\frac{1}{2}} \\ \times \left(\int_\alpha^T \left[1 - \frac{1}{2} \left(\frac{t}{T} \right)^{\frac{\lambda}{2}} \right] t^{1-\lambda} g^2(t) dt \right)^{\frac{1}{2}}.$$

In the present paper, by introducing some parameters, new forms of Hardy-Hilbert's inequalities are given.



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2. New Results

We state and prove the following:

Lemma 2.1. Let $\lambda > 0$, $p, q, r > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $f(t), g(t), h(t) \geq 0$, $\lambda_i(t) \geq 0$, $i = p, q, r$, and assume that

$$0 < \int_a^b \lambda_p^p(t) f^p(t) dt < \infty,$$

$$0 < \int_c^d \lambda_q^q(t) g^q(t) dt < \infty$$

and

$$0 < \int_c^d \lambda_r^r(t) h^r(t) dt < \infty.$$

Then the two following inequalities are equivalent

$$(2.1) \quad \int_a^b \int_c^d \int_e^k \frac{f(x)g(y)h(z)}{h^\lambda(x, y, z)} dx dy dz$$

$$\leq K \left(\int_a^b \lambda_p^p(t) f^p(t) dt \right)^{\frac{1}{p}} \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{1}{r}},$$

where $K = K(\lambda, p, q, r)$ is a constant, and

$$(2.2) \quad \int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right)^{\frac{qr}{q+r}} dx$$

$$\leq K^{\frac{qr}{q+r}} \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{r}{q+r}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{q}{q+r}}.$$



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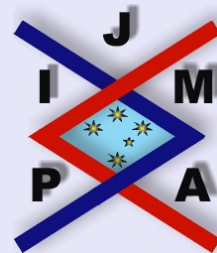
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Proof. Suppose (2.2) is satisfied, then

$$\begin{aligned}
 & \int_a^b \int_c^d \int_e^k \frac{f(x)g(y)h(z)}{h^\lambda(x, y, z)} dx dy dz \\
 &= \int_a^b \lambda_p(x) f(x) \left(\lambda_p^{-1}(x) \int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right) dx \\
 &\leq \left(\int_a^b \lambda_p^p(x) f^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^b \lambda_p^{\frac{qr}{q+r}} \left[\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right]^{\frac{qr}{q+r}} dx \right)^{\frac{q+r}{qr}} \\
 &\leq K \left(\int_a^b \lambda_p^p(t) f^p(t) dt \right)^{\frac{1}{p}} \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{1}{r}}.
 \end{aligned}$$

Now, suppose that (2.1) is satisfied, then

$$\begin{aligned}
 & \int_a^b \lambda_p^{\frac{qr}{q+r}} \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right)^{\frac{qr}{q+r}} dx \\
 &= \int_a^b \int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} \cdot \lambda_p^{\frac{qr}{q+r}} \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right)^{\frac{qr}{q+r}-1} dx dy dz \\
 &\leq K \left(\int_c^d \lambda_q^q(y) g^q(y) dy \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(z) h^r(z) dz \right)^{\frac{1}{r}} \\
 &\quad \times \left(\int_a^b \lambda_p^p(x) \lambda_p^{-p \frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x, y, z)} dy dz \right)^{p \left(\frac{qr}{q+r}-1 \right)} dx \right)^{\frac{1}{p}}
 \end{aligned}$$



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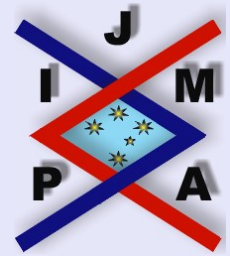


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$$= K \left(\int_c^d \lambda_q^q(y) g^q(y) dy \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(z) h^r(z) dz \right)^{\frac{1}{r}} \\ \times \left(\int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dy dz \right)^{\frac{qr}{q+r}} dx \right)^{\frac{1}{p}},$$

therefore

$$\left(\int_a^b \lambda_p^{\frac{qr}{q+r}}(x) \left(\int_c^d \int_e^k \frac{g(y)h(z)}{h^\lambda(x,y,z)} dy dz \right)^{\frac{qr}{q+r}} dx \right)^{\frac{1}{q} + \frac{1}{r}} \\ \leq k \left(\int_c^d \lambda_q^q(t) g^q(t) dt \right)^{\frac{1}{q}} \left(\int_e^k \lambda_r^r(t) h^r(t) dt \right)^{\frac{1}{r}},$$

and the desired equivalence is proved. \square

Lemma 2.2. (a) Let $0 \leq y \leq 1$, $\alpha > 0$, $f \geq 0$. If we define the function

$$g(y) = y^{-\alpha} \int_0^y f(x) dx,$$

then $g(y) \geq g(1)$.

(b) Let $y \geq 1$, $\alpha > 0$, $f \geq 0$. Defining the function,

$$h(y) = y^{-\alpha} \int_0^y f(x) dx,$$

we have $h(y) \geq h(1)$.

Proof. (a) Let $x = \frac{1}{t}$, then

$$g(y) = y^{-\alpha} \int_{y^{-1}}^{\infty} \frac{f\left(\frac{1}{t}\right) dt}{t^2}.$$

We observe also that

$$g'(y) = y^{-\alpha} [-y^2 f(y)] + \left(\int_{y^{-1}}^{\infty} \frac{f\left(\frac{1}{t}\right) dt}{t^2} \right) (-\alpha) y^{-\alpha-1} \leq 0,$$

therefore g is non-increasing, which implies $g(y) \geq g(1)$.

(b) We obviously have

$$h'(y) = y^\alpha f(y) + \left(\int_0^y f(x) dx \right) \alpha y^{\alpha-1} \geq 0,$$

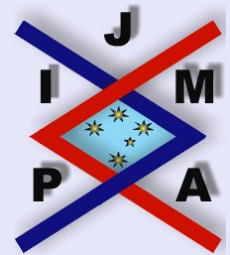
therefore h is non-decreasing, and hence $h(y) \geq h(1)$.

□

The following result may be stated as well.

Theorem 2.3. Let $f(t), g(t), h(t) \geq 0, p, q, r > 1, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, 2 < \lambda < 3,$
 $\gamma > \mu \max \{p, q, r\}$, and

$$\max \left\{ -\frac{1}{p}, -\frac{1}{q}, -\frac{1}{r} \right\} < \mu < \min \left\{ \frac{\lambda - 1}{p}, \frac{\lambda - 1}{q}, \frac{\lambda - 1}{r} \right\}.$$



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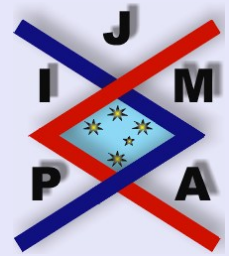


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$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} f^p(t) dt < \infty,$$

$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} g^q(t) dt < \infty$$

and

$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} h^r(t) dt < \infty,$$

then

$$(2.3) \quad \int_{\alpha}^T \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz$$

$$\leq \left(\int_{\alpha}^T \phi(t, \mu, \lambda, p)(t - \alpha)^{2-\lambda} f^p(t) dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\alpha}^T \phi(t, \mu, \lambda, q)(t - \alpha)^{2-\lambda} g^q(t) dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_{\alpha}^T \phi(t, \mu, \lambda, r)(t - \alpha)^{2-\lambda} h^r(t) dt \right)^{\frac{1}{r}},$$

where

$$\phi(t, \mu, \lambda, j) = B(\lambda - \mu j - 1, \mu j + 1) \left\{ B(\lambda - 2, 1 - \mu j) \right.$$

$$\left. - \left(\frac{t - \alpha}{T - \alpha} \right)^{\gamma} \int_0^1 \frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu j-1}} du \right\} - \left(\frac{t - \alpha}{T - \alpha} \right)^{2\gamma}$$

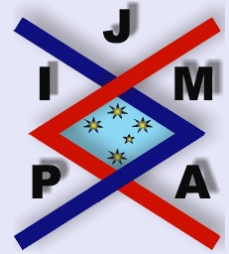
$$\times \int_0^1 \frac{u^{\lambda-\mu j-2}}{(1+u)^\lambda} du \int_0^1 \frac{u^{\gamma-\mu j-2}}{(1+u)^{\lambda-\mu j-1}} du, \quad j = p, q, r.$$

Proof. The proof is as follows. We have

$$\begin{aligned} & \int_\alpha^T \int_\alpha^T \int_\alpha^T \frac{f(x)g(x)h(z)}{(x+y+z)^\lambda} dx dy dz \\ &= \int_\alpha^T \int_\alpha^T \int_\alpha^T \frac{f(x) \left(\frac{z-\alpha}{y-\alpha}\right)^\mu}{(x+y+z)^{\lambda/p}} \frac{g(y) \left(\frac{x-\alpha}{z-\alpha}\right)^\mu}{(x+y+z)^{\lambda/q}} \frac{h(z) \left(\frac{y-\alpha}{x-\alpha}\right)^\mu}{(x+y+z)^{\lambda/r}} dx dy dz \\ &\leq \left(\int_\alpha^T \int_\alpha^T \int_\alpha^T \frac{f^p(x) \left(\frac{z-\alpha}{y-\alpha}\right)^{\mu p}}{(x+y+z)^\lambda} dx dy dz \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_\alpha^T \int_\alpha^T \int_\alpha^T \frac{g^q(y) \left(\frac{x-\alpha}{z-\alpha}\right)^{\mu q}}{(x+y+z)^\lambda} dx dy dz \right)^{\frac{1}{q}} \\ &\quad \times \left(\int_\alpha^T \int_\alpha^T \int_\alpha^T \frac{h^r(z) \left(\frac{y-\alpha}{x-\alpha}\right)^{\mu r}}{(x+y+z)^\lambda} dx dy dz \right)^{\frac{1}{r}} \\ &= F^{\frac{1}{p}} G^{\frac{1}{q}} H^{\frac{1}{r}}, \text{ say.} \end{aligned}$$

Then we have

$$F = \int_\alpha^T (x-\alpha)^{2-\lambda} f^p(x) dx \int_\alpha^T \frac{\left(\frac{y-\alpha}{x-\alpha}\right)^{-\mu p}}{\left(1 + \frac{y-\alpha}{x-\alpha}\right)^{\lambda-\mu p-1}} \frac{dy}{x-\alpha}$$



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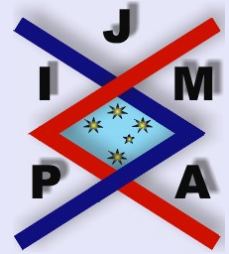
$$\times \int_{\alpha}^T \frac{\left(\frac{z-\alpha}{x+y-2\alpha}\right)^{-\mu p}}{\left(1 + \frac{z-\alpha}{x+y-2\alpha}\right)^{\lambda}} \frac{dz}{x+y-2\alpha}.$$

Now by Lemma 2.2, we can state that

$$\begin{aligned} & \int_0^T \frac{\left(\frac{z-\alpha}{x+y-2\alpha}\right)^{\mu p}}{\left(1 + \frac{z-\alpha}{x+y-2\alpha}\right)^{\lambda}} \frac{dz}{x+y-2\alpha} \\ &= \int_0^{\frac{T-\alpha}{x+y-2\alpha}} \frac{u^{\mu p}}{(1+u)^{\lambda}} du \\ &\leq \int_0^{\frac{T-\alpha}{y-\alpha}} \frac{u^{\mu p}}{(1+u)^{\lambda}} du \\ &= \int_{\frac{y-\alpha}{T-\alpha}}^{\infty} \frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}} du \\ &= \left(\int_0^{\infty} - \left(\frac{y-\alpha}{T-\alpha}\right)^{\gamma} \left(\frac{y-\alpha}{T-\alpha}\right)^{-\gamma} \int_0^{\frac{y-\alpha}{T-\alpha}} \right) \frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}} du \\ &\leq \left[B(\lambda - \mu p - 1, \mu p + 1) - \left(\frac{y-\alpha}{T-\alpha}\right)^{\gamma} \int_0^1 \frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}} du \right]. \end{aligned}$$

Therefore

$$F \leq \int_{\alpha}^T (x-\alpha)^{2-\lambda} f^p(x) dx \int_{\alpha}^T \frac{\left(\frac{y-\alpha}{x-\alpha}\right)^{-\mu p}}{\left(1 + \frac{y-\alpha}{x-\alpha}\right)^{\lambda-\mu p-1}} \frac{dy}{x-\alpha}$$



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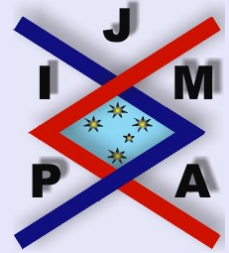


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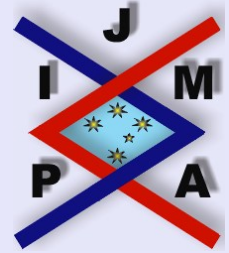
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$$\begin{aligned}
 & \times \left[B(\lambda - \mu p - 1, \mu p + 1) - \left(\frac{y - \alpha}{T - \alpha} \right)^\gamma \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \right] \\
 = & \int_\alpha^T (x - \alpha)^{2 - \lambda} f^p(x) dx \int_0^{\frac{T - \alpha}{x - \alpha}} \frac{u^{-\mu p}}{(1 + u)^{\lambda - \mu p - 1}} du \\
 & \times \left[B(\lambda - \mu p - 1, \mu p + 1) - \left(\frac{x - \alpha}{T - \alpha} \right)^\gamma u^\gamma \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \right] \\
 = & \int_\alpha^T (x - \alpha)^{2 - \lambda} f^p(x) dx \times \left[B(\lambda - \mu p - 1, \mu p + 1) - \int_0^{\frac{T - \alpha}{x - \alpha}} \frac{u^{-\mu p}}{(1 + u)^{\lambda - \mu p - 1}} \right. \\
 & \left. - \left(\frac{x - \alpha}{T - \alpha} \right)^\gamma \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \int_0^{\frac{T - \alpha}{x - \alpha}} \frac{u^{\gamma - \mu p}}{(1 + u)^{\lambda - \mu p - 1}} du \right] \\
 = & \int_\alpha^T (x - \alpha)^{2 - \lambda} f^p(x) dx \times \left[B(\lambda - \mu p - 1, \mu p + 1) - \int_{\frac{x - \alpha}{T - \alpha}}^\infty \frac{u^{\lambda - 3}}{(1 + u)^{\lambda - \mu p - 1}} du \right. \\
 & \left. - \left(\frac{x - \alpha}{T - \alpha} \right)^\gamma \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \int_0^{\frac{T - \alpha}{x - \alpha}} \frac{u^{\gamma - \mu p}}{(1 + u)^{\lambda - \mu p - 1}} du \right] \\
 = & \int_\alpha^T (x - \alpha)^{2 - \lambda} f^p(x) dx \\
 & \times \left[B(\lambda - \mu p - 1, \mu p + 1) - \left(\int_0^\infty - \int_0^{\frac{x - \alpha}{T - \alpha}} \right) \frac{u^{\lambda - 3}}{(1 + u)^{\lambda - \mu p - 1}} du \right. \\
 & \left. - \left(\frac{x - \alpha}{T - \alpha} \right)^\gamma \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \int_0^{\frac{T - \alpha}{x - \alpha}} \frac{u^{\gamma - \mu p}}{(1 + u)^{\lambda - \mu p - 1}} du \right]
 \end{aligned}$$



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$$\begin{aligned}
 &= \int_{\alpha}^T (x - \alpha)^{2-\lambda} f^p(x) dx \\
 &\quad \times \left[B(\lambda - \mu p - 1, \mu p + 1) \left\{ B(\lambda - 2, 1 - \mu p) - \left(\frac{x - \alpha}{T - \alpha} \right)^{\gamma} \left(\frac{x - \alpha}{T - \alpha} \right)^{-\gamma} \right. \right. \\
 &\quad \times \left. \int_0^{\frac{x-\alpha}{T-\alpha}} \frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}} du \right\} - \left(\frac{x - \alpha}{T - \alpha} \right)^{2\gamma} \int_0^1 \frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}} du \\
 &\quad \times \left. \left(\frac{T - \alpha}{x - \alpha} \right)^{\gamma} \int_0^{\frac{T-\alpha}{x-\alpha}} \frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}} \right] \\
 &\leq \int_{\alpha}^T (x - \alpha)^{2-\lambda} f^p(x) dx \\
 &\quad \times \left[B(\lambda - \mu p - 1, \mu p + 1) \left\{ B(\lambda - 2, 1 - \mu p) - \left(\frac{x - \alpha}{T - \alpha} \right)^{\gamma} \right. \right. \\
 &\quad \times \left. \int_0^1 \frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}} du \right\} - \left(\frac{x - \alpha}{T - \alpha} \right)^{2\gamma} \int_0^1 \frac{u^{\lambda-\mu p-2}}{(1+u)^{\lambda}} du \int_0^1 \frac{u^{\gamma-\mu p}}{(1+u)^{\lambda-\mu p-1}} \right] \\
 &= \int_{\alpha}^T \phi(x, \lambda, \mu, p) (x - \alpha)^{2-\lambda} f^p(x) dx,
 \end{aligned}$$

where

$$\begin{aligned}
 \phi(x, \lambda, \mu, p) = &\left[B(\lambda - \mu p - 1, \mu p + 1) \left\{ B(\lambda - 2, 1 - \mu p) \right. \right. \\
 &\quad \left. \left. - \left(\frac{x - \alpha}{T - \alpha} \right)^{\gamma} \int_0^1 \frac{u^{\lambda-3}}{(1+u)^{\lambda-\mu p-1}} du \right\} \right]
 \end{aligned}$$

$$- \left(\frac{x - \alpha}{T - \alpha} \right)^{2\gamma} \int_0^1 \frac{u^{\lambda - \mu p - 2}}{(1 + u)^\lambda} du \int_0^1 \frac{u^{\gamma - \mu p}}{(1 + u)^{\lambda - \mu p - 1}} du \Big].$$

Similarly

$$G = \int_\alpha^T \phi(y, \lambda, \mu, q)(y - \alpha)^{2 - \lambda} g^q(y) dy,$$

and

$$H = \int_\alpha^T \phi(z, \lambda, \mu, r)(z - \alpha)^{2 - \lambda} h^r(z) dz.$$

This completes the proof. \square

Corollary 2.4. Let $f(t), g(t), h(z) \geq 0$, $p, q, r > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $2 < \lambda < 3$, and

$$\max \left\{ -\frac{1}{p}, -\frac{1}{q}, -\frac{1}{r} \right\} < \mu < \min \left\{ \frac{\lambda - 1}{p}, \frac{\lambda - 1}{q}, \frac{\lambda - 1}{r} \right\}.$$

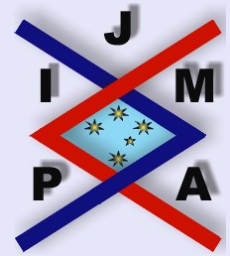
If

$$0 < \int_\alpha^\infty (t - \alpha)^{2 - \lambda} f^p(t) dt < \infty,$$

$$0 < \int_\alpha^\infty (t - \alpha)^{2 - \lambda} g^q(t) dt < \infty$$

and

$$0 < \int_\alpha^\infty (t - \alpha)^{2 - \lambda} h^r(t) dt < \infty,$$



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then we have the inequality

$$(2.4) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz$$

$$\leq K \left(\int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} g^q(t) dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} h^r(t) dt \right)^{\frac{1}{r}},$$

where

$$K = \prod_{j=p,q,r} B^{1/j}(\lambda - \mu_j - 1, \mu_j + 1) B^{1/j}(\lambda - 2, 1 - \mu_j)$$

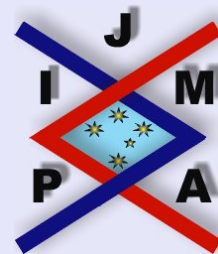
and

$$(2.5) \quad \int_{\alpha}^{\infty} (x-\alpha)^{\frac{qr(\lambda-2)}{p(q+r)}} \left(\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz \right)^{\frac{qr}{q+r}} dx$$

$$\leq K^{\frac{qr}{q+r}} \left(\int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} g^q(t) dt \right)^{\frac{r}{q+r}} \left(\int_{\alpha}^{\infty} (t-\alpha)^{2-\lambda} h^r(t) dt \right)^{\frac{q}{q+r}}.$$

The inequalities (2.4) and (2.5) are equivalent.

Proof. Follows from Theorem 2.3 and Lemma 2.1, on choosing $\gamma = 1$, $T = \infty$, and $\lambda_j(t) = (t-\alpha)^{\frac{2-\lambda}{j}}$. We omit the details. \square



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Corollary 2.5. Let $f(t), g(t), h(t) \geq 0$, $2 < \lambda < 3$, $\gamma > 3\mu$, and $-\frac{1}{3} < \mu < \frac{\lambda-1}{3}$. If

$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} f^3(t) dt < \infty,$$

$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} g^3(t) dt < \infty$$

and

$$0 < \int_{\alpha}^T (t - \alpha)^{2-\lambda} h^3(t) dt < \infty,$$

then

$$(2.6) \quad \int_{\alpha}^T \int_{\alpha}^T \int_{\alpha}^T \frac{f(x)g(x)h(z)}{(x+y+z)^{\lambda}} dx dy dz$$

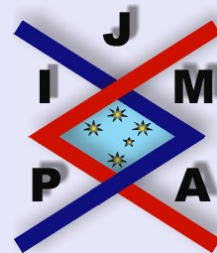
$$\leq \left(\int_{\alpha}^T \phi(t, \lambda, \mu, 3)(t - \alpha)^{2-\lambda} f^3(t) dt \right)^{\frac{1}{3}}$$

$$\times \left(\int_{\alpha}^T \phi(t, \lambda, \mu, 3)(t - \alpha)^{2-\lambda} g^3(t) dt \right)^{\frac{1}{3}}$$

$$\times \left(\int_{\alpha}^T \phi(t, \lambda, \mu, 3)(t - \alpha)^{2-\lambda} h^3(t) dt \right)^{\frac{1}{3}},$$

and

$$(2.7) \quad \int_{\alpha}^T \phi^{-\frac{1}{2}}(t, \lambda, \mu, 3)(x - \alpha)^{\frac{\lambda}{2}-1} \left(\int_{\alpha}^T \int_{\alpha}^T \frac{g(y)h(z)}{(x+y+z)^{\lambda}} dy dz \right)^{\frac{1}{2}} dx$$



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$$\leq \left(\int_{\alpha}^T \phi(t, \lambda, \mu, 3)(t - \alpha)^{2-\lambda} g^3(t) dt \right)^{\frac{1}{3}} \\ \times \left(\int_{\alpha}^T \phi(t, \lambda, \mu, 3)(t - \alpha)^{2-\lambda} h^3(t) dt \right)^{\frac{1}{3}}.$$

The inequalities (2.6) and (2.7) are equivalent.

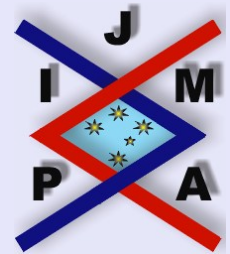
Proof. Follows from Theorem 2.3 and Lemma 2.1, by putting $p = q = r = 3$, and $\lambda_j(t) = (t - \alpha)^{\frac{2-\lambda}{3}} \phi^{\frac{1}{3}}(t, \lambda, \mu, 3)$, $j = p, q, r$. \square

Note. In Corollary 2.5, we may take as a special case $\mu = 1 - \frac{\lambda}{3}$ to obtain

$$\phi(t, \lambda, 1 - \lambda/3, 3) \\ = B(2\lambda - 4, 4 - \lambda)B(\lambda - 2, \lambda - 2) \left\{ 1 - \frac{1}{2} \left(\frac{t - \alpha}{T - \alpha} \right)^{\gamma} \right\} \\ - \left(\frac{t - \alpha}{T - \alpha} \right)^{2\gamma} \int_0^1 \frac{u^{2\lambda-5}}{(1+u)^{\lambda}} du \int_0^1 \frac{u^{\gamma+\lambda-3}}{(1+u)^{2\lambda-4}} du.$$

Corollary 2.6. Let $f(t), g(t), h(t) \geq 0$, $2 < \lambda < 3$, and $-\frac{1}{3} < \mu < \frac{\lambda-1}{3}$. If

$$0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} f^3(t) dt < \infty, \\ 0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^3(t) dt < \infty$$



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and

$$0 < \int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^3(t) dt < \infty,$$

then

$$(2.8) \quad \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^{\lambda}} dx dy dz \\ \leq K \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} f^3(t) dt \right)^{\frac{1}{3}} \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^3(t) dt \right)^{\frac{1}{3}} \\ \times \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^3(t) dt \right)^{\frac{1}{3}},$$

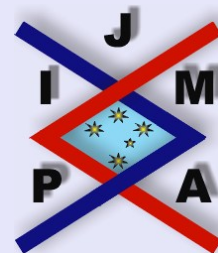
where

$$K = (B(\lambda - 3\mu - 1, 3\mu + 1)B(\lambda - 2, 1 - 3\mu)$$

and

$$(2.9) \quad \int_{\alpha}^{\infty} (x - \alpha)^{-\frac{1}{2}} \left(\int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \frac{g(y)h(z)}{(x+y+z)^{\lambda}} dy dz \right)^{\frac{1}{2}} dx \\ \leq K^{3/2} \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} g^3(t) dt \right)^{\frac{1}{2}} \\ \times \left(\int_{\alpha}^{\infty} (t - \alpha)^{2-\lambda} h^3(t) dt \right)^{\frac{1}{2}}.$$

The inequalities (2.8) and (2.9) are equivalent.



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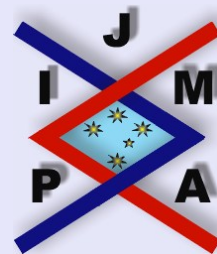
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Proof. Follows from Corollary 2.4 and Lemma 2.1, by putting $p = q = r = 3$,
 $T = \infty$. □



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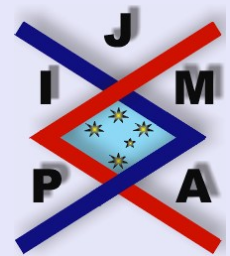
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