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EXPLICIT UPPER BOUNDS FOR THE AVERAGE ORDER OF $d_n(m)$ AND APPLICATION TO CLASS NUMBER

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ABSTRACT. In this paper, we prove some explicit upper bounds for the average order of the generalized divisor function, and, according to an idea of Lenstra, we use them to obtain bounds for the class number of number fields.

Key words and phrases: Multiplicative number theory, Average order, Class number.

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1. INTRODUCTION

Let $\mathbb{K}$ be a number field of degree n, signature (r_1, r_2), discriminant $d(\mathbb{K})$, Minkowski bound $b(\mathbb{K}) := b = \left(\frac{n!}{n^n}\right)\left(\frac{4}{\pi}\right)^{r_2} |d(\mathbb{K})|^{\frac{1}{2}}$ and class number $h(\mathbb{K})$. We denote by $\mathcal{O}_{\mathbb{K}}$ the ring of algebraic integers of $\mathbb{K}$. We are interested here in finding explicit upper bounds for $h(\mathbb{K})$ of the type

$$h(\mathbb{K}) \le \varepsilon(n) |d(\mathbb{K})|^{\frac{1}{2}} (\log |d(\mathbb{K})|)^{n-1},$$

where $\varepsilon(n)$ is a positive constant depending on n, and $\log$ is the natural logarithm.

There are several methods to get such bounds for $h(\mathbb{K})$: Roland Quême in [8] used the geometry of numbers to prove that if $b > 17$,

$$\mathcal{R}(\mathbb{K})\, h(\mathbb{K}) \le w(\mathbb{K}) \left(\frac{2}{\pi}\right)^{r_2} |d(\mathbb{K})|^{\frac{1}{2}} (2\log b)^n,$$

where $\mathcal{R}(\mathbb{K})$ is the regulator of $\mathbb{K}$, and $w(\mathbb{K})$ is the number of roots of unity in $\mathbb{K}$.

In [5], Stéphane Louboutin proved, by using analytic methods, that

$$\mathcal{R}(\mathbb{K})\, h(\mathbb{K}) \le \frac{w(\mathbb{K})}{2} \left(\frac{2}{\pi}\right)^{r_2} |d(\mathbb{K})|^{\frac{1}{2}} \left(\frac{e\log |d(\mathbb{K})|}{4(n-1)}\right)^{n-1},$$

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and, if $\mathbb{K}$ is a totally real abelian extension of $\mathbb{Q}$,

$$\mathcal{R}(\mathbb{K})\, h(\mathbb{K}) \le d(\mathbb{K})^{\frac{1}{2}} \left\{ \frac{\log d(\mathbb{K})}{4(n-1)} + 0.025 \right\}^{n-1}.$$

The methods used to get these bounds are very deep, but it is necessary to compute the regulator (which is usually not easy), or use the Zimmert's lower bound for $\mathcal{R}(\mathbb{K})$ (see [11]):

$$\mathcal{R}(\mathbb{K}) \ge 0.02\, w(\mathbb{K})\, e^{0.46 r_1 + 0.1 r_2}.$$

We want to prove some inequalities involving $h(\mathbb{K})$ in an elementary way: we have

$$h(\mathbb{K}) \le \left| \{ \mathfrak{a} : \text{integral ideal of } \mathcal{O}_{\mathbb{K}},\ \mathcal{N}(\mathfrak{a}) \le b \} \right|,$$

where $\mathcal{N}(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$, and, using an idea of H.W. Lenstra (see [4]), we can see, by considering how prime numbers can split in $\mathbb{K}$, that, for each positive integer m, the number of integral ideals $\mathfrak{a}$ of absolute norm m is bounded by the number of solutions of the equation

$$a_1 a_2 \cdots a_n = m \quad (a_i \in \mathbb{N}^*).$$

Lenstra deduced that

$$h(\mathbb{K}) \le \left| \{ (a_1, \ldots, a_n) \in (\mathbb{N}^*)^n,\ a_1 a_2 \cdots a_n \le b \} \right|. \tag{1.1}$$

Now the idea is to work with the generalized divisor function d_n, since (1.1) is equivalent to:

Lemma 1.1. *Let $\mathbb{K}$ be a number field of degree $n \ge 2$, and b be the Minkowski bound of $\mathbb{K}$. Then:*

$$h(\mathbb{K}) \le \sum_{m \le b} d_n(m).$$

In an oral communication, J.L. Nicolas and G. Tenenbaum proved that, for any integer $n \ge 1$ and any real number $x \ge 1$,

$$\sum_{m \le x} d_n(m) \le \frac{x}{(n-1)!} (\log x + n - 1)^{n-1}. \tag{1.2}$$

(one can prove this inequality by induction).

Hence, by Lemma 1.1 and (1.2), we get Lenstra's result, namely:

$$h(\mathbb{K}) \le \frac{b}{(n-1)!} (\log b + n - 1)^{n-1}.$$

2. NOTATION

We mention here some notation that will be used throughout the paper:

General. m, n, r, s will always denote positive integers, x a real number ≥ 1, and $[x]$ denote the integral part of x, the unique integer satisfying $x - 1 < [x] \le x$.

- $\psi(x) := x - [x] - \frac{1}{2}$, and $e(x) := e^{2i\pi x}$. ψ is 1-periodic and $|\psi(x)| \le \frac{1}{2}$.
- $\gamma \approx 0.5772156649015328606065120900...$ is the Euler constant.
- For any finite set $\mathcal{E}$, $|\mathcal{E}|$ denotes the number of elements in $\mathcal{E}$.

On number fields. $\mathbb{K}$ is a number field of degree $n \ge 2$, signature (r_1, r_2), discriminant $d(\mathbb{K})$, Minkowski bound $b = \left(\frac{n!}{n^n}\right)\left(\frac{4}{\pi}\right)^{r_2} |d(\mathbb{K})|^{\frac{1}{2}}$, class number $h(\mathbb{K})$.

On arithmetical functions. By $\mathbf{1}$, we mean the arithmetical function defined by $\mathbf{1}(m) = 1$ for any positive integer m.

The generalized divisor function d_n is defined by

$$d_1(m) = 1,\ d_n(m) := \sum_{a_1 a_2 \cdots a_n = m} 1 \ \ (n \geq 2),$$

and, if $n = 2$, we simply denote it by $d(m)$.

If f and g are two arithmetical functions, the Dirichlet convolution product of f and g is defined by

$$(f * g)(m) := \sum_{\delta | m} f(\delta)\, g\left(\frac{m}{\delta}\right).$$

3. BASIC PROPERTIES OF THE GENERALIZED DIVISOR FUNCTION

The properties of the generalized divisor function can be found in [5], [9] and [10]. For our purpose, we only need to know that d_n is multiplicative (i.e. $d_n(rs) = d_n(r)\, d_n(s)$ whenever $\gcd(r,s) = 1$) and, for any prime number p and any non-negative integer l, we have :

$$d_n\left(p^l\right) = \binom{n+l-1}{l},$$

where $\binom{a}{b}$ denotes a binomial coefficient ([9], equality (4)).

It's important to note that we have

$$d_n = \underbrace{\mathbf{1} * \mathbf{1} * \ldots * \mathbf{1}}_{n \text{ times}} \quad (n \geq 1). \tag{3.1}$$

One knows that the average order of $d_n(m)$ is $\sim (\log m)^{n-1} / (n-1)!$: to see this, one can use the following result ([9], equality (18)):

$$\sum_{m \leq x} d_n(m) = x (\log x)^{n-1} \left\{ \frac{1}{(n-1)!} + O\left(\frac{1}{\log x}\right) \right\} \quad (x > 1,\ n \geq 2).$$

Our aim is to compute several constants $\kappa(n)$ depending (or not) on n such that

$$\sum_{m \leq x} d_n(m) \leq \kappa(n)\, x (\log x)^{n-1}.$$

We will need the following lemma:

Lemma 3.1. *Let $x \geq 1$. Then:*

$$\sum_{m \leq x} \frac{1}{m} = \log x + \gamma - \frac{\psi(x)}{x} + \frac{\varepsilon}{x^2} \quad \textit{with} \quad |\varepsilon| \leq \frac{1}{4}.$$

This result is well-known, and a proof can be found in [2].

4. RESULTS

Theorem 4.1. *Let $n \geq 1$ be an integer and $x \geq 1$ a real number. Then:*

$$\sum_{m \leq x} d_n(m) \leq x \left(\log x + \gamma + \frac{1}{x}\right)^{n-1}.$$

Theorem 4.2. *Let $n \geq 1$ be an integer and $x \geq 6$ a real number. Then:*

$$\sum_{m \leq x} d_n(m) \leq 2x (\log x)^{n-1}.$$

5. Application to Class Number

Theorem 5.1. *Let* $\mathbb{K}$ *be a number field of degree* n, *Minkowski bound* b *and class number* $h(\mathbb{K})$. *Then:*

$$h(\mathbb{K}) \leq b\left(\log b + \gamma + b^{-1}\right)^{n-1}.$$

Theorem 5.2. *Let* $\mathbb{K}$ *be a number field of degree* $n \geq 2$, *Minkowski bound* b *and class number* $h(\mathbb{K})$. *Then, if* $b \geq 6$,

$$h(\mathbb{K}) \leq 2b(\log b)^{n-1}.$$

Theorem 5.3. *Let* $\mathbb{K}$ *be a number field of degree* n, *discriminant* $d(\mathbb{K})$ *and class number* $h(\mathbb{K})$. *Then :*

$$h(\mathbb{K}) \leq \frac{2^{n-1}}{(n-1)!}\left|d(\mathbb{K})\right|^{\frac{1}{2}}\left(\log\left|d(\mathbb{K})\right|\right)^{n-1}.$$

More generally, if $a > 0$ *is satisfying* $a \geq 2(n-1)/(\log|d(\mathbb{K})|)$, *then*

$$h(\mathbb{K}) \leq \left(\frac{a+1}{2}\right)^{n-1}\frac{\left|d(\mathbb{K})\right|^{\frac{1}{2}}}{(n-1)!}\left(\log\left|d(\mathbb{K})\right|\right)^{n-1}.$$

6. Proofs of the Theorems

In the following proofs, we set

$$S_n(x) := \sum_{m \leq x} d_n(m).$$

Proof of Theorem 4.1.

$$\begin{aligned} S_n(x) &= \sum_{m \leq x} \sum_{a_1 \ldots a_n = m} 1 \\ &= \sum_{a_1 \leq x} \sum_{a_2 \leq x} \cdots \sum_{a_n \leq x/(a_1 \ldots a_{n-1})} 1 \\ &\leq \sum_{a_1 \leq x} \cdots \sum_{a_{n-1} \leq x} \frac{x}{a_1 \ldots a_{n-1}} \\ &= x\left(\sum_{a \leq x} \frac{1}{a}\right)^{n-1}, \end{aligned}$$

and we use Lemma 3.1 to conclude the proof. □

Proof of Theorem 4.2. (1) We first note that, since

$$S_n(t) = \begin{cases} 1, & \text{if } 1 \le t < 2, \\ n+1, & \text{if } 2 \le t < 3, \\ 2n+1, & \text{if } 3 \le t < 4, \\ \dfrac{(n^2+5n+2)}{2}, & \text{if } 4 \le t < 5, \\ \dfrac{(n^2+7n+2)}{2}, & \text{if } 5 \le t < 6, \\ \dfrac{(3n^2+7n+2)}{2}, & \text{if } 6 \le t < 7, \\ \dfrac{(3n^2+9n+2)}{2}, & \text{if } 7 \le t < 8, \end{cases}$$

then

$$\int_1^{e^2} t^{-2} S_n(t)\, dt = \left(\frac{7}{24} - \frac{3e^{-2}}{2}\right) n^2 + \left(\frac{1093}{840} - \frac{9e^{-2}}{2}\right) n + 1 - e^{-2},$$

and then, if $n \ge 2$,

$$\int_1^{e^2} t^{-2} S_n(t)\, dt < \frac{2n^2}{3}. \tag{6.1}$$

(2) Let $x \ge 6$, $n \ge 1$. The theorem is true if $n = 1$, since $S_1(x) = [x] \le x$, so we prove the result for $n \ge 2$.

We first check that the theorem is true when $6 \le x < e^2$. Indeed, in this case, we have

$$2x(\log x)^{n-1} \ge 12(\log 6)^{n-1} > 4n^2 \ge \frac{3n^2+9n+2}{2} = S_n\left(e^2\right) \ge S_n(x).$$

so we can suppose that $x \ge e^2$ and $n \ge 2$.

We prove the inequality by induction : if $n = 2$,

$$S_2(x) = \sum_{r \le x} \sum_{s \le x/r} 1 \le x \log x + x \le 2x \log x.$$

Assume it is true for some $n \ge 2$. By (3.1), we have:

$$\begin{aligned} S_{n+1}(x) &= \sum_{m \le x} (d_n * \mathbf{1})(m) \\ &= \sum_{m \le x} \sum_{\delta \mid m} d_n(\delta) \\ &= \sum_{\delta \le x} d_n(\delta) \left[\frac{x}{\delta}\right] \end{aligned}$$

$$\begin{aligned}
&\leq x \sum_{\delta \leq x} \frac{d_n(\delta)}{\delta} \\
&= x \int_{1_-}^{x} t^{-1} d\left(S_n(t)\right) \\
&= S_n(x) + x \int_1^x t^{-2} S_n(t)\, dt \\
&= S_n(x) + x \int_1^{e^2} t^{-2} S_n(t)\, dt + x \int_{e^2}^{x} t^{-2} S_n(t)\, dt.
\end{aligned}$$

Using (6.1) and induction hypothesis, we get

$$\begin{aligned}
S_{n+1}(x) &\leq 2x(\log x)^{n-1} + \frac{2n^2 x}{3} + 2x \int_{e^2}^{x} t^{-1} (\log t)^{n-1}\, dt \\
&= \frac{2x}{n} (\log x)^n + x\left\{2(\log x)^{n-1} + \frac{2n^2}{3} - \frac{2^{n+1}}{n}\right\} \\
&= 2x(\log x)^n - x f_n(x),
\end{aligned}$$

where

$$f_n(x) := \left(2 - \frac{2}{n}\right)(\log x)^n - \left\{2(\log x)^{n-1} + \frac{2n^2}{3} - \frac{2^{n+1}}{n}\right\}.$$

Now we have

$$f_n(x) \geq f_n\left(e^2\right) = 2^n - \frac{2n^2}{3} \geq 0,$$

hence

$$S_{n+1}(x) \leq 2x(\log x)^n.$$

This concludes the proof of Theorem 4.2. □

Proof of Theorems 5.1 & 5.2. Direct applications of Theorems 4.1 and 4.2. □

Proof of Theorem 5.3. Let $a > 0$, and suppose $x \geq e^{(n-1)/a}$. Then $n - 1 \leq a \log x$, and, using (1.2),

$$S_n(x) \leq \frac{(a+1)^{n-1}}{(n-1)!} x (\log x)^{n-1}. \tag{6.2}$$

Now, Since $b < |d(\mathbb{K})|^{\frac{1}{2}}$, we have, by Lemma 1.1,

$$h(\mathbb{K}) \leq \sum_{m \leq |d(\mathbb{K})|^{1/2}} d_n(m).$$

We then use the inequality ([6], Lemma 10)

$$|d(\mathbb{K})| \geq e^{2(n-1)/3}$$

and (6.2) with $a = 3$ to get the first part of Theorem 5.3.

The 2nd part comes directly from (6.2). This concludes the proof of Theorem 5.3. □

7. Using the Convolution Relation in a Different Way

We now want to prove another bound, using the Dirichlet hyperbola principle:

Theorem 7.1. *Let $\mathbb{K}$ be a number field of degree n, Minkowski bound b and class number $h(\mathbb{K})$. Then, if $b \geq 36$,*

(i) $n = 2p$ $(p \geq 1)$,

$$h(\mathbb{K}) \leq \frac{b}{2^{p-2}(p-1)!} (\log b)^p (\log b + p - 1)^{p-1},$$

(ii) $n = 2p+1$ $(p \geq 1)$,

$$h(\mathbb{K}) \leq \frac{b}{2^p (p-1)!} (\log b)^p \left\{ \log b (\log b + p - 1)^{p-1} + \binom{2}{p} (\log b + p)^p \right\}.$$

We first need the following result:

Lemma 7.2. *Let $x \geq 6$ be a real number and $k \geq 1$ an integer. Then:*

$$\sum_{m \leq x} \frac{d_k(m)}{m} \leq 2 (\log x)^k.$$

Proof. The result is true if $k = 1$, so we suppose $k \geq 2$. Suppose first that $x \geq e^2$. By partial summation, we can write, using Theorem 4.2,

$$\begin{aligned} \sum_{m \leq x} \frac{d_k(m)}{m} &= x^{-1} S_k(x) + \int_1^x t^{-2} S_k(t)\, dt \\ &\leq 2(\log x)^{k-1} + \int_1^{e^2} t^{-2} S_k(t)\, dt + 2 \int_{e^2}^x t^{-1} (\log t)^{k-1}\, dt \\ &< \frac{2}{k} (\log x)^k + 2 (\log x)^{k-1} + \frac{2k^2}{3} - \frac{2^{k+1}}{k}, \end{aligned}$$

and one can check that

$$2(\log x)^{k-1} + \frac{2k^2}{3} - \frac{2^{k+1}}{k} \leq \left(\frac{3}{2} - \frac{1}{k} \right) (\log x)^k$$

if $x \geq e^2$ and $k \geq 2$, hence

$$\sum_{m \leq x} \frac{d_k(m)}{m} \leq \left(\frac{3}{2} + \frac{1}{k} \right) (\log x)^k \leq 2 (\log x)^k.$$

Now, if $6 \leq x < e^2$ and $k \geq 2$, we get

$$2(\log x)^k \geq 2(\log 6)^k > \frac{6k^2}{5} > \frac{1}{840}\left(245k^2 + 1093k + 840\right) = \sum_{m \leq e^2} \frac{d_k(m)}{m} \geq \sum_{m \leq x} \frac{d_k(m)}{m},$$

which concludes the proof of Lemma 7.2. □

Proof of Theorem 7.1. Let $x \geq 36$ be a real number. If $n = 2p$ is even, using (3.1) again, we can write:

$$\sum_{m \leq x} d_n(m) = \sum_{m \leq x} \left(d_{n/2} * d_{n/2} \right)(m) = \sum_{m \leq x} (d_p * d_p)(m),$$

and, by the Dirichlet hyperbola principle, we get, for any real number T satisfying $1 \leq T \leq x$,

$$\sum_{m \leq x} d_n(m) \leq \sum_{m \leq T} d_p(m) \sum_{r \leq x/m} d_p(r) + \sum_{m \leq x/T} d_p(m) \sum_{r \leq x/m} d_p(r),$$

and then, using (1.2),

$$\sum_{m \leq x} d_n(m) \leq \frac{x}{(p-1)!} \left\{ \sum_{m \leq T} \frac{d_p(m)}{m} \left(\log \frac{x}{m} + p - 1 \right)^{p-1} \right.$$
$$\left. + \sum_{m \leq x/T} \frac{d_p(m)}{m} \left(\log \frac{x}{m} + p - 1 \right)^{p-1} \right\},$$

and, with Lemma 7.2, if $\min\left(T, \frac{x}{T}\right) \geq 6$, we get

$$\sum_{m \leq x} d_n(m) \leq \frac{2x(\log x + p - 1)^{p-1}}{(p-1)!} \left\{ (\log T)^p + \left(\log \left(\frac{x}{T} \right) \right)^p \right\},$$

and we choose $T = x^{\frac{1}{2}}$ (so $\min\left(T, \frac{x}{T}\right) = x^{\frac{1}{2}} \geq 6$) to conclude the proof.

If $n = 2p+1$ is odd, then we write:

$$\sum_{m \leq x} d_n(m) = \sum_{m \leq x} \left(d_{(n-1)/2} * d_{(n+1)/2} \right)(m) = \sum_{m \leq x} (d_p * d_{p+1})(m).$$

□

8. CASE OF QUADRATIC FIELDS

We suppose in this section that $\mathbb{K} = \mathbb{Q}\left(\sqrt{d}\right)$, where $d \in \mathbb{Z} \setminus \{0, 1\}$ is supposed to be squarefree. We denote here Δ the discriminant and $h(d)$ the class number. We recall that:

$$\Delta = \begin{cases} d, & \text{if } d \equiv 1 \pmod 4, \\ 4d, & \text{if } d \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

The problem of the class number is in this case utterly resolved: for example, if $d < -4$, we have (see [1], Corollary 5.3.13)

$$h(d) = \left\{ 2 - \left(\frac{d}{2} \right) \right\}^{-1} \sum_{1 \leqslant k < |d|/2} \left(\frac{d}{k} \right),$$

where $\left(\frac{d}{k}\right)$ represents the Kronecker-Jacobi symbol. Nevertheless, we think it would be interesting to have upper bounds for $h(d)$.

We also note that, by [3], we can replace, in Lemma 1.1, the Minkowski bound b by the bound β defined by:

$$\beta := \begin{cases} \sqrt{\Delta/8}, & \text{if } \Delta \geq 8, \\ \sqrt{-\Delta/3}, & \text{if } \Delta < 0. \end{cases}$$

We can see that the problem of the class number of a quadratic field is then connected with that of having good estimations of the error-term

$$R(x) := \sum_{m \leq x} d(m) - x(\log x + 2\gamma - 1)$$

(Dirichlet divisor problem).

One can prove in an elementary way that $R(x) = O\left(x^{\frac{1}{2}}\right)$ (see below). Voronoï proved that $R(x) = O\left(x^{\frac{1}{3}} \log x\right)$. If we use the technique of exponent pairs (see [2]), we can have

$R(x) = O\left(x^{\frac{27}{82}}\right)$. By using very sophisticated technics, Huxley succeeded in proving that $R(x) = O\left(x^{\frac{23}{73}}(\log x)^{\frac{461}{146}}\right)$.

The following result is well-known, but, to make our exposition self-contained, we include the proof:

Lemma 8.1. *Let $x \geq 1$. Then :*

$$\sum_{m \leq x} d(m) \leq x(\log x + 2\gamma - 1) + 2\left|\sum_{m \leq x^{1/2}} \psi\left(\frac{x}{m}\right)\right| + \frac{3}{4}.$$

Proof. By the Dirichlet hyperbola principle, we have:

$$\begin{aligned}
\sum_{m \leq x} d(m) &= \sum_{rs \leq x} 1 \\
&= \sum_{r \leq x^{1/2}} \sum_{s \leq x/r} 1 + \sum_{s \leq x^{1/2}} \sum_{r \leq x/s} 1 - \sum_{r \leq x^{1/2}} \sum_{s \leq x^{1/2}} 1 \\
&= 2 \sum_{r \leq x^{1/2}} \sum_{s \leq x/r} 1 - \left[\sqrt{x}\right]^2 \\
&= 2 \sum_{r \leq x^{1/2}} [x/r] - \left(\sqrt{x} - \psi\left(\sqrt{x}\right) - \frac{1}{2}\right)^2 \\
&= 2 \sum_{r \leq x^{1/2}} \left(x/r - \psi(x/r) - \frac{1}{2}\right) - x - \psi^2\left(\sqrt{x}\right) \\
&\quad -\frac{1}{4} + 2\psi\left(\sqrt{x}\right)\sqrt{x} + \sqrt{x} - \psi\left(\sqrt{x}\right),
\end{aligned}$$

and, by using Lemma 3.1, we get

$$\begin{aligned}
\sum_{m \leq x} d(m) &= 2x\left(\frac{1}{2}\log x + \gamma - x^{-\frac{1}{2}}\psi\left(\sqrt{x}\right) + \varepsilon x^{-1}\right) - 2\sum_{r \leq x^{1/2}} \psi\left(\frac{x}{r}\right) - \sqrt{x} + \psi\left(\sqrt{x}\right) \\
&\quad + \frac{1}{2} - x - \psi^2\left(\sqrt{x}\right) - \frac{1}{4} + 2\psi\left(\sqrt{x}\right)\sqrt{x} + \sqrt{x} - \psi\left(\sqrt{x}\right) \\
&= x(\log x + 2\gamma - 1) + 2\varepsilon + \frac{1}{4} - \psi^2\left(\sqrt{x}\right) - 2\sum_{r \leq x^{1/2}} \psi\left(\frac{x}{r}\right),
\end{aligned}$$

and we conclude by noting that $|\varepsilon| \leq \frac{1}{4}$ and $\left|\frac{1}{4} - \psi^2\left(\sqrt{x}\right)\right| \leq \frac{1}{4}$ if $x \geq 1$. □

Corollary 8.2. *Let $x \geq 1$. Then:*

$$\sum_{m \leq x} d(m) \leq x(\log x + 2\gamma - 1) + \sqrt{x} + \frac{3}{4}.$$

Proof. Use $|\psi(t)| \leq \frac{1}{2}$ in Lemma 8.1. □

We get, using Lemma 1.1:

Corollary 8.3. *Let* $\mathbb{K} = \mathbb{Q}\left(\sqrt{d}\right)$ *be a quadratic field of discriminant* Δ. *Then :*

$$h(d) \le \begin{cases} \sqrt{\frac{\Delta}{8}}\left\{\frac{1}{2}\log\Delta + 2\gamma - 1 - \frac{3}{2}\log 2\right\} + \left(\frac{\Delta}{8}\right)^{\frac{1}{4}} + \frac{3}{4}, & \text{if } \Delta \ge 8, \\ \sqrt{-\frac{\Delta}{3}}\left\{\frac{1}{2}\log(-\Delta) + 2\gamma - 1 - \frac{1}{2}\log 3\right\} + \left(-\frac{\Delta}{3}\right)^{\frac{1}{4}} + \frac{3}{4}, & \text{if } \Delta < 0. \end{cases}$$

Example 8.1. If $d = 13693$, then, using PARI system (see [1]), we get $h(d) = 15$. The bound of Corollary 8.3 gives

$$h(d) < 166.$$

Example 8.2. If $d = -300119$, then we have $h(d) = 781$, and Corollary 8.3 gives

$$h(d) < 1889.$$

For bigger discriminants, it could be interesting to have a lower exponent on the error-term. We want to prove this explicit version of Voronoï's theorem:

Lemma 8.4. *Let* $x \ge 3$. *Then:*

$$\left|\sum_{m \le x^{1/2}} \psi\left(\frac{x}{m}\right)\right| < 6\,x^{\frac{1}{3}}\log x.$$

We first need an effective version of Van Der Corput inequality:

Lemma 8.5. *Let* $f \in C^2((N;2N] \mapsto \mathbb{R})$. *If there exist real numbers* $c \ge 1$ *and* $\lambda_2 > 0$ *satisfying*

$$\lambda_2 \le f''(x) \le c\lambda_2 \quad (N < x \le 2N),$$

then:

$$\left|\sum_{N<m\le 2N} e(\pm f(m))\right| \le 4\pi^{-\frac{1}{2}}\left\{cN\lambda_2^{\frac{1}{2}} + 2\lambda_2^{-\frac{1}{2}}\right\}.$$

Proof. We first prove the following result:

Let $f \in C^2([N;2N] \mapsto \mathbb{R})$ *satisfying*

(i) $f'(x) \notin \mathbb{Z}$ *if* $N < x < 2N$,

(ii) *there exists* $\lambda_2 \in \left(0; \frac{1}{\pi}\right]$ *verifying* $f''(x) \ge \lambda_2 \quad (N \le x \le 2N)$.

Then:

$$\left|\sum_{N\le m\le 2N} e(\pm f(m))\right| \le 4\pi^{-\frac{1}{2}}\,\lambda_2^{-\frac{1}{2}}. \tag{8.1}$$

Since

$$\left|\sum_{N\le m\le 2N} e(-f(m))\right| = \left|\sum_{N\le m\le 2N} e(f(m))\right|,$$

we shall prove (8.1) just for f, and since $f''(x) > 0$ for $x \in [N;2N]$, f' is a strictly increasing function.

Let x be a real number satisfying $0 < x < \frac{1}{2}$. By (i), we can define real numbers u, v, N_1, N_2 such that $u := f'(N)$, $v := f'(2N)$, and $f'(N_1) = [u] + x$, $f'(N_2) = [u] + 1 - x$. We have :

$$\sum_{N\le m\le 2N} e(f(m)) = \sum_{N\le m<N_1} e(f(m)) + \sum_{N_1\le m\le N_2} e(f(m)) + \sum_{N_2<m\le 2N} e(f(m)),$$

with

$$\left|\sum_{N\le m<N_1} e\left(f\left(m\right)\right)\right| \le \max\left\{N_1-N,1\right\} = \max\left\{\frac{f'\left(N_1\right)-f'\left(N\right)}{f''\left(\xi\right)},1\right\}$$

for some real number $\xi\in\left(N;N_1\right)$, then, by $\left(ii\right)$,

$$\left|\sum_{N\le m<N_1} e\left(f\left(m\right)\right)\right| \le \max\left\{\frac{[u]+x-u}{\lambda_2},1\right\} \le \max\left\{\frac{x}{\lambda_2},1\right\},$$

and we have the same for

$$\left|\sum_{N_2<m\le 2N} e\left(f\left(m\right)\right)\right| \le \max\left\{\frac{v+x-[u]-1}{\lambda_2},1\right\} \le \max\left\{\frac{x}{\lambda_2},1\right\},$$

and we use Kusmin-Landau inequality (see [7]) to get

$$\left|\sum_{N_1\le m\le N_2} e\left(f\left(m\right)\right)\right| \le \cot\left(\frac{\pi x}{2}\right) \le \frac{2}{\pi x}.$$

We then have:

$$\left|\sum_{N\le m\le 2N} e\left(f\left(m\right)\right)\right| \le 2\max\left\{\frac{x}{\lambda_2},1\right\} + \frac{2}{\pi x}.$$

We then choose $x=\left(\frac{\lambda_2}{\pi}\right)^{\frac{1}{2}}$, so $\frac{x}{\lambda_2}=\left(\pi\lambda_2\right)^{-\frac{1}{2}}\ge 1$ if $\lambda_2\le\pi^{-1}$, and we get

$$\left|\sum_{N\le m\le 2N} e\left(f\left(m\right)\right)\right| \le 4\pi^{-\frac{1}{2}}\lambda_2^{-\frac{1}{2}}.$$

We are now ready to prove Lemma 8.5:

If $\lambda_2>\frac{1}{\pi}$, then $4\pi^{-\frac{1}{2}}cN\lambda_2^{\frac{1}{2}}>4\pi^{-1}N>N$, so we suppose $\lambda_2\le\frac{1}{\pi}$.
We take u,v as above, and we define

$$[u;v]\cap\mathbb{Z} := \left\{l+1,...,l+K\right\}$$

for some integer l and positive integer K, and define

$$J_k := \left\{m\in\mathbb{Z},\ l+k-1<m\le l+k\right\}\cap[u;v] \quad \left(1\le k\le K+1\right).$$

We have, by (8.1),

$$\left|\sum_{N<m\le 2N} e\left(f\left(m\right)\right)\right| \le \sum_{k=1}^{K+1}\left|\sum_{m\in J_k} e\left(f\left(m\right)\right)\right| \le 4\pi^{-\frac{1}{2}}\left(K+1\right)\lambda_2^{-\frac{1}{2}},$$

and, by the mean value theorem,

$$K-1\le v-u=f'\left(2N\right)-f'\left(N\right)\le cN\lambda_2,$$

thus

$$\left|\sum_{N<m\le 2N} e\left(f\left(m\right)\right)\right| \le 4\pi^{-\frac{1}{2}}\left(cN\lambda_2+2\right)\lambda_2^{-\frac{1}{2}}.$$

This concludes the proof of Lemma 8.5. □

Proof of Lemma 8.4. We write

$$\left|\sum_{m\leq x^{\frac{1}{2}}}\psi\left(\frac{x}{m}\right)\right|=\left|\sum_{m\leq 2x^{1/3}}\psi\left(\frac{x}{m}\right)+\sum_{2x^{1/3}<m\leq x^{\frac{1}{2}}}\psi\left(\frac{x}{m}\right)\right|\leq x^{\frac{1}{3}}+\left|\Sigma\right|.$$

We then split the interval $\left(2x^{\frac{1}{3}};x^{\frac{1}{2}}\right]$ into sub-intervals of the form $(N;2N]$ with $2x^{\frac{1}{3}}<N\leq x^{\frac{1}{2}}$: the number J of such intervals satisfies

$$2^{J-1}N\leq x^{\frac{1}{2}}<2^{J}N,$$

and since $N>2x^{\frac{1}{3}}$, we have

$$J=\left[\frac{\log\left(x^{\frac{1}{2}}/N\right)}{\log 2}+1\right]<\frac{\log x}{6\log 2}.$$

We then have :

$$\left|\Sigma\right|\leq\max_{2x^{1/3}<N\leq x^{1/2}}\left|\sum_{N<m\leq 2N}\psi\left(\frac{x}{m}\right)\right|\frac{\log x}{6\log 2}.$$

Moreover, using Erdös-Turán inequality (see Appendix A), we get, for any positive integer H,

$$\left|\sum_{N<m\leq 2N}\psi\left(\frac{x}{m}\right)\right|\leq\frac{N}{2H}+\frac{1}{\pi}\left\{\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{N<m\leq 2N}e\left(\frac{hx}{m}\right)\right|+H\sum_{h>H}\frac{1}{h^2}\left|\sum_{N<m\leq 2N}e\left(\frac{hx}{m}\right)\right|\right\},$$

so, by Lemma 8.5, with $\lambda_2=hx/\left(4N^3\right)$ and $c=8$, we get

$$\begin{aligned}\left|\sum_{N<m\leq 2N}\psi\left(\frac{x}{m}\right)\right|&\leq\frac{N}{2H}+16\pi^{-\frac{3}{2}}\left\{\sum_{h=1}^{H}\left(x^{\frac{1}{2}}\left(Nh\right)^{-\frac{1}{2}}+\left(Nh^{-1}\right)^{\frac{3}{2}}x^{-\frac{1}{2}}\right)\right.\\&\qquad\left.+H\sum_{h>H}\left(x^{\frac{1}{2}}\left(Nh^3\right)^{-\frac{1}{2}}+N^{\frac{3}{2}}x^{-\frac{1}{2}}h^{-5/2}\right)\right\}\\&\leq\frac{N}{2H}+16\pi^{-\frac{3}{2}}\left\{2\left(xHN^{-1}\right)^{\frac{1}{2}}+\zeta\left(\frac{3}{2}\right)N^{\frac{3}{2}}x^{-\frac{1}{2}}\right.\\&\qquad\left.+H\int_{H}^{\infty}\left(\left(\frac{x}{N}\right)^{\frac{1}{2}}t^{-\frac{3}{2}}+\left(\frac{N^3}{x}\right)^{\frac{1}{2}}t^{-5/2}\right)dt\right\}\\&\leq\frac{N}{2H}+16\pi^{-\frac{3}{2}}\left\{4\left(xHN^{-1}\right)^{\frac{1}{2}}+\left(\zeta\left(\frac{3}{2}\right)+2/3\right)N^{\frac{3}{2}}x^{-\frac{1}{2}}\right\},\end{aligned}$$

where $\zeta\left(\frac{3}{2}\right):=\sum_{k=1}^{\infty}k^{-\frac{3}{2}}$. The well-known bound $\zeta\left(\sigma\right)\leq\sigma/\left(\sigma-1\right)\ \left(\sigma>1\right)$ gives $\zeta\left(\frac{3}{2}\right)+\frac{2}{3}\leq\frac{11}{3}<4$, hence

$$\left|\sum_{N<m\leq 2N}\psi\left(\frac{x}{m}\right)\right|\leq\frac{N}{2H}+64\pi^{-\frac{3}{2}}\left\{\left(xHN^{-1}\right)^{\frac{1}{2}}+N^{\frac{3}{2}}x^{-\frac{1}{2}}\right\}.$$

We then choose

$$H=\left[2^{-1}Nx^{-\frac{1}{3}}\right].$$

Considering the inequality $1/[y] \leq 2/y$ $(y \geq 1)$, we get

$$\left|\sum_{N<m\leq 2N} \psi\left(\frac{x}{m}\right)\right| \leq \left(64\pi^{-\frac{3}{2}}2^{-\frac{1}{2}}+2\right)x^{\frac{1}{3}}+64\pi^{-\frac{3}{2}}N^{\frac{3}{2}}x^{-\frac{1}{2}},$$

and

$$|\Sigma| \leq \left\{\left(64\pi^{-\frac{3}{2}}2^{-\frac{1}{2}}+2\right)x^{\frac{1}{3}}+64\pi^{-\frac{3}{2}}x^{\frac{1}{4}}\right\}\frac{\log x}{6\log 2},$$

and since $x \geq 3$, $x^{\frac{1}{4}} \leq 3^{-1/12}x^{\frac{1}{3}}$, then

$$|\Sigma| \leq \left\{64\pi^{-\frac{3}{2}}\left(2^{-\frac{1}{2}}+3^{-\frac{1}{12}}\right)+2\right\}\frac{x^{\frac{1}{3}}\log x}{6\log 2} < 5\,x^{\frac{1}{3}}\log x.$$

□

We obtain with Lemma 1.1:

Corollary 8.6. *Let $\mathbb{K}=\mathbb{Q}\left(\sqrt{d}\right)$ be a quadratic field of discriminant Δ. Then:*

$$h(d) \leq \begin{cases} \sqrt{\Delta/8}\left\{\frac{1}{2}\log\Delta+2\gamma-1-\frac{3}{2}\log 2\right\}+6\left(\Delta/8\right)^{1/6}\log\left(\Delta/8\right)+\frac{3}{4}, & \text{if } \Delta \geq 72, \\ \sqrt{-\Delta/3}\left\{\frac{1}{2}\log\left(-\Delta\right)+2\gamma-1-\frac{1}{2}\log 3\right\} & \\ \qquad +6\left(-\Delta/3\right)^{1/6}\log\left(-\Delta/3\right)+\frac{3}{4}, & \text{if } \Delta < -27. \end{cases}$$

APPENDIX A.

We want to show here this special form of the Erdös-Turán inequality used in this paper:

Theorem A.1. *Let H, N be positive integers, and $f:(N;2N]\mapsto\mathbb{R}$ be any function. Then:*

$$\left|\sum_{N<m\leq 2N}\psi\left(f(m)\right)\right| \leq \frac{N}{2H}+\frac{1}{\pi}\left\{\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{N<m\leq 2N}e\left(hf(m)\right)\right|+H\sum_{h>H}\frac{1}{h^2}\left|\sum_{N<m\leq 2N}e\left(hf(m)\right)\right|\right\}.$$

Proof. For any positive integers h and H, we set

$$c(h,H) := \frac{H}{2\pi ih}\int_0^{1/H} e(-ht)\,dt.$$

(1) We first note that

$$|c(h,H)| \leq \frac{1}{2\pi}\min\left(\frac{1}{h},\frac{H}{h^2}\right). \tag{A.1}$$

Indeed, if $h \leq H$, then

$$|c(h,H)| \leq \frac{H}{2\pi h}\int_0^{1/H}|e(-ht)|\,dt = \frac{1}{2\pi h},$$

and if $h > H$, then the first derivative test gives

$$|c(h,H)| \leq \frac{H}{2\pi h}\left|\int_0^{1/H}e(-ht)\,dt\right| \leq \frac{H}{2\pi h}\cdot\frac{2}{\pi h} = \frac{H}{(\pi h)^2} < \frac{H}{2\pi h^2}.$$

(2) Let x, t be any real numbers. Since $\psi(x) \leq \psi(x-t)+t$, we get

$$\int_0^{1/H} \psi(x)\,dt \leq \int_0^{1/H} (\psi(x-t)+t)\,dt,$$

and then

$$\psi(x) \leq H \int_0^{1/H} \psi(x-t)\,dt + \frac{1}{2H}. \tag{A.2}$$

The partial sums of the series $\sum_{h\geq 1} \{-\sin(2\pi hx)/(h\pi)\}$ are uniformly bounded, hence

$$\begin{aligned}
\int_0^{1/H} \psi(x-t)\,dt &= -\frac{1}{\pi}\sum_{h=1}^{\infty}\frac{1}{h}\int_0^{1/H} \sin(2\pi h(x-t))\,dt \\
&= -\frac{1}{2\pi i}\sum_{h=1}^{\infty}\frac{1}{h}\int_0^{1/H} \{e(hx)e(-ht) - e(-hx)e(ht)\}\,dt \\
&= -\frac{1}{2\pi i}\sum_{h=1}^{\infty}\frac{e(hx)}{h}\int_0^{1/H} e(-ht)\,dt - \frac{1}{2\pi i}\sum_{h=1}^{\infty}\frac{e(-hx)}{-h}\int_0^{1/H} e(ht)\,dt \\
&= -\sum_{h\in\mathbb{Z},\, h\neq 0}\frac{e(hx)}{2\pi i h}\int_0^{1/H} e(-ht)\,dt = -\frac{1}{H}\sum_{h\in\mathbb{Z},\, h\neq 0} c(h,H)\,e(hx),
\end{aligned}$$

hence, using (A.2),

$$\psi(x) \leq \frac{1}{2H} - \sum_{h\in\mathbb{Z},\, h\neq 0} c(h,H)\,e(hx),$$

and

$$\begin{aligned}
\sum_{N<m\leq 2N} \psi(f(m)) &\leq \frac{N}{2H} - \sum_{h\in\mathbb{Z},\, h\neq 0} c(h,H) \sum_{N<m\leq 2N} e(hf(m)) \\
&\leq \frac{N}{2H} + 2\left|\sum_{h=1}^{\infty} c(h,H) \sum_{N<m\leq 2N} e(hf(m))\right|,
\end{aligned}$$

hence

$$\sum_{N<m\leq 2N} \psi(f(m)) \leq \frac{N}{2H} + 2\sum_{h=1}^{\infty} |c(h,H)| \left|\sum_{N<m\leq 2N} e(hf(m))\right|. \tag{A.3}$$

(3) Since we also have $\psi(x) \geq \psi(x+t)-t$, we get in the same way

$$\begin{aligned}
\sum_{N<m\leq 2N} \psi(f(m)) &\geq -\frac{N}{2H} + \sum_{h\in\mathbb{Z},\, h\neq 0} c(h,H) \sum_{N<m\leq 2N} e(-hf(m)) \\
&\geq -\frac{N}{2H} - 2\left|\sum_{h=1}^{\infty} c(h,H) \sum_{N<m\leq 2N} e(-hf(m))\right| \\
&\geq -\frac{N}{2H} - 2\sum_{h=1}^{\infty} |c(h,H)| \left|\sum_{N<m\leq 2N} e(-hf(m))\right|,
\end{aligned}$$

and since $e\left(-hf\left(m\right)\right)=\overline{e\left(hf\left(m\right)\right)}$, we obtain

$$\sum_{N<m\leq 2N}\psi\left(f\left(m\right)\right)\geq -\frac{N}{2H}-2\sum_{h=1}^{\infty}\left|c\left(h,H\right)\right|\left|\sum_{N<m\leq 2N}e\left(hf\left(m\right)\right)\right|. \tag{A.4}$$

The inequalities (A.3) and (A.4) give

$$\begin{aligned}\left|\sum_{N<m\leq 2N}\psi\left(f\left(m\right)\right)\right| &\leq \frac{N}{2H}+2\sum_{h=1}^{\infty}\left|c\left(h,H\right)\right|\left|\sum_{N<m\leq 2N}e\left(hf\left(m\right)\right)\right| \\ &= \frac{N}{2H}+2\left\{\sum_{h=1}^{H}\left|c\left(h,H\right)\right|\left|\sum_{N<m\leq 2N}e\left(hf\left(m\right)\right)\right|\right. \\ &\qquad \left.+\sum_{h>H}\left|c\left(h,H\right)\right|\left|\sum_{N<m\leq 2N}e\left(hf\left(m\right)\right)\right|\right\},\end{aligned}$$

and we use (A.1).

□

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