

# A METHOD FOR ESTABLISHING CERTAIN TRIGONOMETRIC INEQUALITIES

**MOWAFFAQ HAJJA**

Department of Mathematics  
Yarmouk University  
Irbid, Jordan.  
EMail: [mowhajja@yahoo.com](mailto:mowhajja@yahoo.com)

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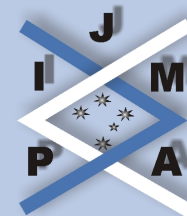
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*Key words:* Geometric inequalities, Equifacial tetrahedra, Solid angle.

*Abstract:* In this note, we describe a method for establishing trigonometric inequalities that involve symmetric functions in the cosines of the angles of a triangle. The method is based on finding a complete set of relations that define the cosines of such angles.



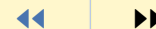
**Trigonometric Inequalities**

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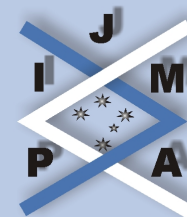
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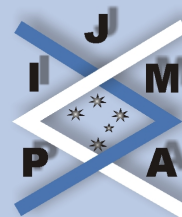
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## 1. Introduction

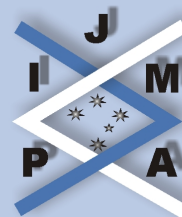
This note is motivated by the desire to find a straightforward proof of the fact that among all equifacial tetrahedra, the regular one has the maximal solid angle sum [9]. This led to a similar desire to find a systematic method for optimizing certain trigonometric functions and for establishing certain trigonometric inequalities.

Let us recall that a tetrahedron is called *equifacial* (or *isosceles*) if its faces are congruent. It is clear that the three angles enclosed by the arms of each corner angle of such a tetrahedron are the same as the three angles of a triangular face. Less obvious is the fact that the faces of an equifacial tetrahedron are necessarily acute-angled [8], [9].

Let us also recall that if  $A$ ,  $B$  and  $C$  are the three angles enclosed by the arms of a solid angle  $V$ , then the *content*  $E$  of  $V$  is defined as the area of the spherical triangle whose vertices are traced by the arms of  $V$  on the unit sphere centered at the vertex of  $V$  [5]. This content  $E$  is given (in [5], for example) by

$$(1.1) \quad \tan \frac{E}{2} = \frac{\sqrt{1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C}}{1 + \cos A + \cos B + \cos C}.$$

The statement made at the beginning of this article is equivalent to saying that the maximum of the quantity (1.1) over all acute triangles  $ABC$  is attained at  $A = B = C = \pi/3$ . To treat (1.1) as a function of three variables  $\cos A$ ,  $\cos B$ , and  $\cos C$ , one naturally raises the question regarding a complete set of relations that define the cosines of the angles of an acute triangle, and similarly for a general triangle. These questions are answered in Theorems 2.2 and 2.3. The statement regarding the maximum of the quantity (1.1) over acute triangles  $ABC$  is established in Theorem 3.1. The methods developed are then used to establish several examples of trigonometric inequalities.

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## 2. Triples that can Serve as the Cosines of the Angles of a Triangle

Our first theorem answers the natural question regarding what real triples qualify as the cosines of the angles of a triangle. For the proof, we need the following simple lemma taken together with its elegant proof from [7]. The lemma is actually true for any number of variables.

**Lemma 2.1.** *Let  $u$ ,  $v$ , and  $w$  be real numbers and let*

$$(2.1) \quad s = u + v + w, \quad p = uv + vw + wu, \quad q = uvw.$$

*Then  $u$ ,  $v$ , and  $w$  are non-negative if and only if  $s$ ,  $p$  and  $q$  are non-negative.*

*Proof.* If  $s$ ,  $p$ , and  $q$  are non-negative, then the polynomial  $f(T) = T^3 - sT^2 + pT - q$  will be negative for every negative value of  $T$ . Thus its roots  $u$ ,  $v$ , and  $w$  (which are assumed to be real) must be non-negative. ■

With reference to (2.1), it is worth recording that the assumption that  $s$ ,  $p$ , and  $q$  are non-negative does not imply that  $u$ ,  $v$ , and  $w$  are real. For example, if  $\zeta$  is a primitive third root of unity, and if  $(u, v, w) = (1, \zeta, \zeta^2)$ , then  $s = p = 0$  and  $q = 1$ . For more on this, see [11].

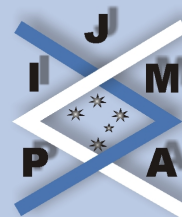
**Theorem 2.2.** *Let  $u$ ,  $v$  and  $w$  be real numbers. Then there exists a triangle  $ABC$  such that  $u = \cos A$ ,  $v = \cos B$ , and  $w = \cos C$  if and only if*

$$(2.2) \quad u + v + w \geq 1$$

$$(2.3) \quad uvw \geq -1$$

$$(2.4) \quad u^2 + v^2 + w^2 + 2uvw = 1.$$

*The triangle is acute, right or obtuse according to whether  $uvw$  is greater than, equal to or less than 0.*

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*Proof.* Let  $A$ ,  $B$ , and  $C$  be the angles of a triangle and let  $u$ ,  $v$ , and  $w$  be their cosines. Then (2.3) is trivial, (2.2) follows from

$$(2.5) \quad \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2},$$

or from Carnot's formula

$$(2.6) \quad \frac{r}{R} = \cos A + \cos B + \cos C - 1,$$

where  $r$  and  $R$  are the inradius and circumradius of  $ABC$ , and (2.4) follows by squaring  $\sin A \sin B = \cos C + \cos A \cos B$  and using  $\sin^2 \theta = 1 - \cos^2 \theta$ . For (2.5), see [4, 678, page 166], for (2.6), see [10], and for more on (2.4), see [6].

Conversely, let  $u$ ,  $v$ , and  $w$  satisfy (2.2), (2.3), and (2.4), and let  $s$ ,  $p$  and  $q$  be as in (2.1). Then (2.2), (2.3), and (2.4) can be rewritten as

$$(2.7) \quad s \geq 1, \quad q \geq -1, \quad s^2 - 2p + 2q = 1.$$

We show first that

$$\alpha = 1 - u^2, \quad \beta = 1 - v^2, \quad \gamma = 1 - w^2$$

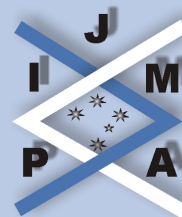
are non-negative. By Lemma 2.1, this is equivalent to showing that  $\alpha + \beta + \gamma$ ,  $\alpha\beta + \beta\gamma + \gamma\alpha$ , and  $\alpha\beta\gamma$  are non-negative. But it is routine to check that

$$\alpha + \beta + \gamma = 2(q + 1) \geq 0,$$

$$4(\alpha\beta + \beta\gamma + \gamma\alpha) = ((s - 1)^2 + 2(q + 1))^2 + 4(s - 1)^3 \geq 0,$$

$$4\alpha\beta\gamma = (s - 1)^2(s^2 + 2s + 1 + 4q) \geq (s - 1)^2(1 + 2 + 1 - 4) \geq 0.$$

Thus  $-1 \leq u, v, w \leq 1$ . Therefore there exist unique  $A$ ,  $B$  and  $C$  in  $[0, \pi]$  such that  $u = \cos A$ ,  $v = \cos B$ , and  $w = \cos C$ . It remains to show that  $A + B + C = \pi$ .

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The sum of  $u + vw$ ,  $v + wu$ , and  $w + uv$  is  $s + p$  and

$$s + p = s + \frac{s^2 + 2q - 1}{2} \geq 1 + \frac{1 - 2 - 1}{2} = 0.$$

Thus at least one of them, say  $w + uv$ , is non-negative. Also, (2.4) implies that

$$(w + uv)^2 = u^2v^2 + 1 - u^2 - v^2 = (1 - u^2)(1 - v^2) = \sin^2 A \sin^2 B.$$

Since  $\sin A$ ,  $\sin B$ , and  $w + uv$  are all non-negative, it follows that  $w + uv = \sin A \sin B$ , and therefore

$$\cos C = w = -uv + \sin A \sin B = -\cos A \cos B + \sin A \sin B = -\cos(A + B).$$

It also follows from (2.2) that

$$2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} = \cos A + \cos B \geq 1 - \cos C \geq 0$$

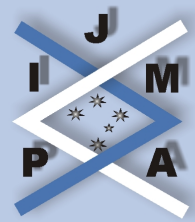
and hence  $0 \leq A + B \leq \pi$ . Thus  $C$  and  $A + B$  are in  $[0, \pi]$ . From  $\cos C = -\cos(A + B)$ , it follows that  $A + B + C = \pi$ , as desired. ■

Now let  $s$ ,  $p$  and  $q$  be given real numbers and let  $u$ ,  $v$ , and  $w$  be the zeros of the cubic  $T^3 - sT^2 + pT - q$ . Thus  $u$ ,  $v$  and  $w$  are completely defined by (2.1). It is well-known [12, Theorem 4.32, page 239] that  $u$ ,  $v$  and  $w$  are real if and only if the discriminant is non-negative, i.e.

$$(2.8) \quad \Delta = -27q^2 + 18spq + p^2s^2 - 4s^3q - 4p^3 \geq 0.$$

If we assume that (2.7) holds, then we can eliminate  $p$  from  $\Delta$  to obtain

$$\Delta = -(s^2 + 2s + 1 + 4q)(s^4 - 2s^3 + 4s^2q - s^2 - 20sq + 4s - 2 + 20q + 4q^2).$$

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Also (2.7) implies that  $s^2 + 2s + 1 + 4q = (s + 1)^2 + 4q \geq 2^2 + 4(-1) \geq 0$ , with equality if and only if  $(s, q) = (1, -1)$ . Moreover, when  $(s, q) = (1, -1)$ , the second factor of  $\Delta$  equals zero. Therefore, with (2.7) assumed, the condition that  $\Delta \geq 0$  is equivalent to the condition

$$(2.9) \quad \Delta^* = -s^4 + 2s^3 - 4s^2q + s^2 + 20sq - 4s + 2 - 20q - 4q^2 \geq 0,$$

with  $\Delta = 0$  if and only if  $\Delta^* = 0$ . From this and from Theorem 2.2, it follows that  $u, v$  and  $w$  are the cosines of the angles of a triangle if and only if (2.7) and (2.8) (or equivalently (2.7) and (2.9)) hold. Also, the discriminant of  $\Delta^*$ , as a polynomial in  $q$ , is  $16(3 - 2s)^3$ . Therefore for (2.9) to be satisfied (for any  $s$  at all), we must have  $s \leq 3/2$ . Solving (2.9) for  $q$ , we re-write (2.9) in the equivalent form

$$(2.10) \quad \begin{cases} f_1(s) \leq q \leq f_2(s), \text{ where} \\ f_1(s) = \frac{-s^2 + 5s - 5 - (3 - 2s)^{3/2}}{2}, \quad f_2(s) = \frac{-s^2 + 5s - 5 + (3 - 2s)^{3/2}}{2}. \end{cases}$$

Figure 1 is a sketch of the region  $\Omega_0$  defined by  $f_1(s) \leq q \leq f_2(s)$ ,  $1 \leq s \leq 1.5$ , using the facts that  $f_1(s)$  and  $f_2(s)$  are increasing (and that  $f_1$  is concave down and  $f_2$  is concave up) on  $s \in [1, 1.5]$ . Note that the points  $(s, q)$  of  $\Omega_0$  satisfy  $q \geq f_1(1) = -1$ , rendering the condition  $q \geq -1$  (in (2.7)) redundant. We summarize this in the following theorem.

**Theorem 2.3.** *Let  $s, p$ , and  $q$  be real numbers. Then the zeros of the cubic  $T^3 - sT^2 + pT - q$  (are real and) can serve as the cosines of the angles of a triangle if and only if  $(s, p, q)$  lies in the region  $\Omega$  defined by*

$$(2.11) \quad s^2 - 2p + 2q - 1 = 0,$$

$$(2.12) \quad 1 \leq s \leq 1.5$$

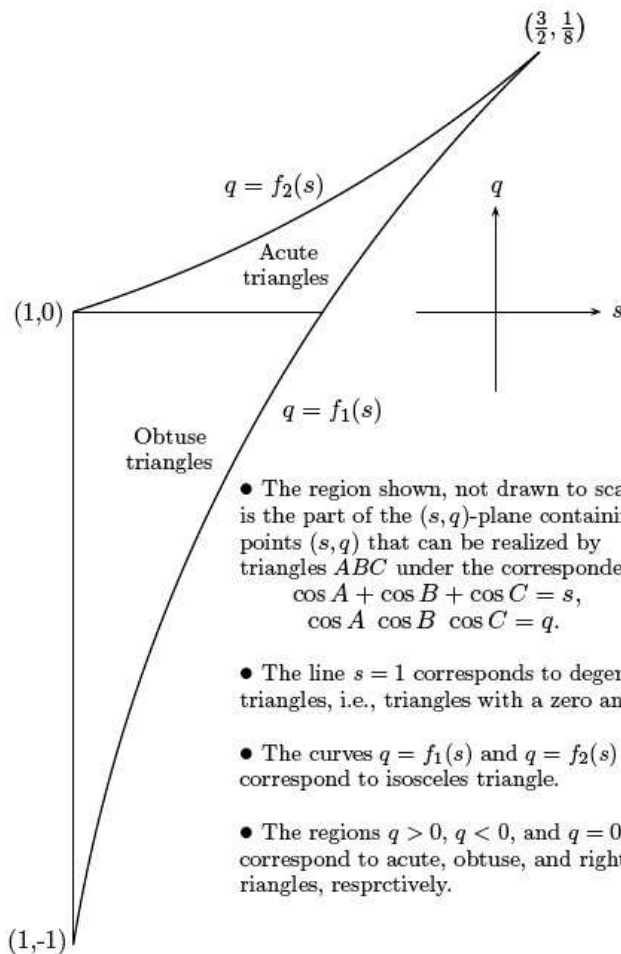
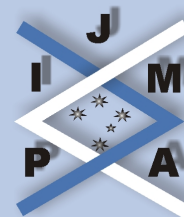


Figure 1.

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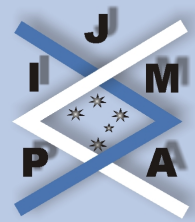
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and any of the equivalent conditions (2.8), (2.9) and (2.10) hold. The boundary of  $\Omega$  consists of the line segment defined by

$$s = 1, \quad q = p \in [-1, 0]$$

and corresponding to degenerate triangles (i.e. triangles with a zero angle), and the curve parametrized by

$$(2.13) \quad s = 2t + 1 - 2t^2, \quad q = t^2(1 - 2t^2), \quad p = t^2 + 2t(1 - 2t^2), \quad 0 \leq t \leq 1$$

and corresponding to isosceles triangles having angles  $(\theta, \theta, \pi - 2\theta)$ , where  $\theta = \cos^{-1} t$ . It is clear that  $\pi - 2\theta$  is acute for  $0 < t < 1/\sqrt{2}$  and obtuse for  $1/\sqrt{2} < t < 1$ . Acute and obtuse triangles correspond to  $q > 0$  and  $q < 0$  (respectively), and right triangles are parametrized by

$$q = 0, \quad p = \frac{s^2 - 1}{2}, \quad s \in [1, \sqrt{2}].$$

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### 3. Maximizing the Sum of the Contents of the Corner Angles of an Equifacial Tetrahedron

We now turn to the optimization problem mentioned at the beginning.

**Theorem 3.1.** *Among all acute triangles  $ABC$ , the quantity (1.1) attains its maximum at  $A = B = C = \pi/3$ . Therefore among all equifacial tetrahedra, the regular one has a vertex solid angle of maximum measure.*

*Proof.* Note that (1.1) is not defined for obtuse triangles. Squaring (1.1) and using (2.7), we see that our problem is to maximize

$$f(s, q) = \frac{4q}{(s+1)^2}$$

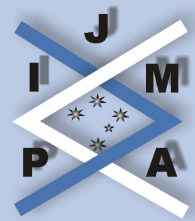
over  $\Omega$ . Clearly, for a fixed  $s$ ,  $f$  attains its maximum when  $q$  is largest. Thus we confine our search to the part of (2.13) defined by  $0 \leq t \leq 1/\sqrt{2}$ . Therefore our objective function is transformed to the one-variable function

$$g(t) = \frac{t^2(1-2t^2)}{(t^2-t-1)^2}, \quad 0 \leq t \leq 1/\sqrt{2}.$$

From

$$g'(t) = \frac{2t(2t-1)(t+1)^2}{(t^2-t-1)^3},$$

we see that  $g$  attains its maximum at  $t = 1/2$ , i.e at the equilateral triangle. ■



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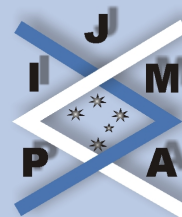


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## 4. A Method for Optimizing Certain Trigonometric Expressions

Theorem 3.1 above describes a systematic method for optimizing certain symmetric functions in  $\cos A$ ,  $\cos B$ , and  $\cos C$ , where  $A$ ,  $B$ , and  $C$  are the angles of a general triangle. If such a function can be written in the form  $H(s, p, q)$ , where  $s$ ,  $p$ , and  $q$  are as defined in (2.1), then one can find its optimum values as follows:

1. One finds the interior critical points of  $H$  by solving the system

$$\begin{aligned}\frac{\partial H}{\partial s} + \frac{\partial H}{\partial p} s &= \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p} q = 0, \\ s^2 - 2p + 2q &= 1, \\ 1 < s < 1.5,\end{aligned}$$

$$\Delta = -27q^2 + 18spq + p^2s^2 - 4s^3q - 4p^3 > 0.$$

Equivalently, one uses  $s^2 - 2p + 2q = 1$  to write  $H$  as a function of  $s$  and  $q$ , and then solve the system

$$\begin{aligned}\frac{\partial H}{\partial s} &= \frac{\partial H}{\partial q} = 0, \\ 1 < s < 1.5,\end{aligned}$$

$$\Delta^* = -s^4 + 2s^3 - 4s^2q + s^2 + 20sq - 4s + 2 - 20q - 4q^2 > 0.$$

Usually, no such interior critical points exist.

2. One then optimizes  $H$  on degenerate triangles, i.e., on

$$s = 1, \quad p = q, \quad q \in [-1, 0].$$

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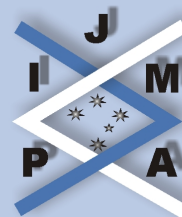


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3. One finally optimizes  $H$  on isosceles triangles, i.e., on

$$s = 2t + 1 - 2t^2, \quad p = t^2 + 2t(1 - 2t^2), \quad q = t^2(1 - 2t^2), \quad t \in [0, 1].$$

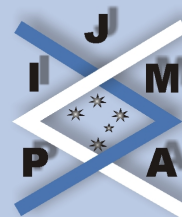
If the optimization is to be done on acute triangles only, then

1. Step 1 is modified by adding the condition  $q > 0$ ,
2. Step 2 is discarded,
3. in Step 3,  $t$  is restricted to the interval  $[0, 1/\sqrt{2}]$ ,
4. a fourth step is added, namely, optimizing  $H$  on right triangles, i.e., on

$$(4.1) \quad p = \frac{s^2 - 1}{2}, \quad q = 0, \quad s \in [1, \sqrt{2}].$$

For obtuse triangles,

1. Step 1 is modified by adding the condition  $q < 0$ ,
2. Step 2 remains,
3. in Step 3,  $t$  is restricted to the interval  $[1/\sqrt{2}, 1]$ ,
4. the fourth step described in (4.1) is added.

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## 5. Examples

The following examples illustrate the method.

*Example 5.1.* The inequality

$$(5.1) \quad \sum \sin B \sin C \leq \left( \sum \cos A \right)^2$$

is proved in [3], where the editor *wonders if there is a nicer way* of proof. In answer to the editor's request, Bager gave another proof in [1, page 20]. We now use our routine method.

Using  $\sin A \sin B - \cos A \cos B = \cos C$ , one rewrites this inequality as

$$s + p \leq s^2.$$

It is clear that  $H = s^2 - s - p$  has no interior critical points, since  $\partial H / \partial p + \partial H / \partial q = -1$ . For degenerate triangles,  $s = 1$  and  $H = -p = -q$  and takes all the values in  $[0, 1]$ . For isosceles triangles,

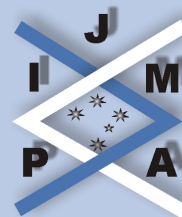
$$(5.2) \quad H = (2t + 1 - 2t^2)^2 - (2t + 1 - 2t^2) - (t^2 + 2t(1 - 2t^2)) = t^2(2t - 1)^2 \geq 0.$$

Thus  $H \geq 0$  for all triangles and our inequality is established.

One may like to establish a reverse inequality of the form  $s + p \geq s^2 - c$  and to separate the cases of acute and obtuse triangles. For this, note that on right triangles,  $q = 0$ , and  $H = s^2 - s - (s^2 - 1)/2$  increases with  $s$ , taking all values in  $[0, 1/8]$ . Also, with reference to (5.2), note that

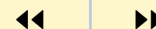
$$\frac{d}{dt}(t^2(2t - 1)^2) = 2(4t - 1)t(2t - 1).$$

Thus  $t^2(2t - 1)^2$  increases on  $[0, 1/4]$ , decreases on  $[1/4, 1/2]$ , and increases on  $[1/2, 1]$ .



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The first three tables below record the critical points of  $H$  and its values at those points, and the last one records the maximum and minimum of  $H$  on the sets of all acute and all obtuse triangles separately. Note that the numbers  $1/8$  and  $1.5 - \sqrt{2}$  are quite close, but one can verify that  $1/8$  is the larger one. Therefore the maximum of  $s^2 - s - p$  is  $1/8$  on acute triangles and  $1$  on obtuse triangles, and we have proved the following stronger version of (5.1):

$$\sum \sin B \sin C \leq \left( \sum \cos A \right)^2 \leq \frac{1}{8} + \sum \sin B \sin C \text{ for acute triangles}$$

$$\sum \sin B \sin C \leq \left( \sum \cos A \right)^2 \leq 1 + \sum \sin B \sin C \text{ for obtuse triangles}$$

	Isosceles				
	Acute			Obtuse	
$t$	0	1/4	1/2	$\sqrt{2}/2$	1
$H$	0	1/64	0	$1.5 - \sqrt{2}$	1

	Degenerate	
	$s = 1$	
$q$	-1	0
$H$	1	0

	Right	
	$q = 0$	
$s$	1	1.5
$H$	0	1/8

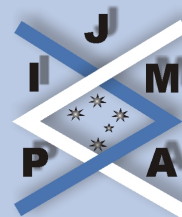
	Acute	Obtuse	All
$\max H$	1/8	1	1
$\min H$	0	0	0

We may also consider the function

$$G = \frac{s + p}{s^2}.$$

Again,  $G$  has no interior critical points since  $\partial G / \partial p = 1/s^2$ . On degenerate triangles,  $s = 1$  and  $G = 1 + q$  and takes all values in  $[0, 1]$ . On right triangles,  $q = 0$  and we have

$$G = \frac{s^2 + 2s - 1}{2s^2}, \quad \frac{dG}{ds} = \frac{1 - s}{s^3} \leq 0.$$



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Therefore  $G$  is decreasing for  $s \in [1, 1.5]$  and takes all values in  $[17/18, 1]$ . It remains to work on isosceles triangles. There,

$$G = \frac{(1-t)(1+t)(4t+1)}{(2t^2-2t-1)^2} \quad \text{and} \quad \frac{dG}{dt} = \frac{2t(2t-1)(2t^2+4t-1)}{(2t^2-2t-1)^3}.$$

Let  $r = (-2 + \sqrt{6})/2$  be the positive zero of  $2t^2 + 4t - 1$ . Then  $0 < r < 1/2$  and  $G$  decreases on  $[0, r]$ , increases on  $[r, 1/2]$ , decreases on  $[1/2, 1]$ . Its values at significant points and its extremum values are summarized in the tables below.

		Isosceles			
		Acute		Obtuse	
$t$	0	$(-2 + \sqrt{6})/2$	1/2	$\sqrt{2}/2$	1
$G$	1	$(7 + 2\sqrt{6})/12$	1	$(1 + 2\sqrt{2})/4$	0

		Degenerate	
		$s = 1$	
$q$	-1	0	
$G$	0	1	

		Right	
		$q = 0$	
$s$	1	1.5	
$G$	1	17/18	

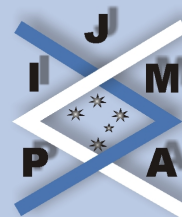
	Acute	Obtuse	All
$\max G$	1	1	1
$\min G$	17/18	0	0

Here we have used the delicate inequalities

$$\frac{17}{18} < \frac{1 + 2\sqrt{2}}{4} < \frac{7 + 2\sqrt{6}}{12} < 1.$$

As a result, we have proved the following addition to (5.1):

$$\frac{17}{18} \left( \sum \cos A \right)^2 \leq \sum \sin B \sin C \leq \left( \sum \cos A \right)^2 \quad \text{for acute triangles.}$$

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*Example 5.2.* In [1], the inequality (8) (page 12) reads  $p \geq 6q$ . To prove this, we take

$$H = \frac{p}{q} = \frac{s^2 - 1 + 2q}{2q}.$$

It is clear that  $H$  has no interior critical points since  $\partial H/\partial s$  is never 0. On the set of degenerate triangles,  $s = 1$  and  $H$  is identically 1. On the set of right triangles, we note that as  $q \rightarrow 0^+$ ,  $H \rightarrow +\infty$ , and as  $q \rightarrow 0^-$ ,  $H \rightarrow -\infty$ . On the set of isosceles triangles,

$$H = \frac{t^2 + 2t(1 - 2t^2)}{t^2(1 - 2t^2)} = \frac{1}{1 - 2t^2} + \frac{2}{t}$$
$$\frac{dH}{dt} = \frac{4t}{(1 - 2t^2)^2} - \frac{2}{t^2} = \frac{-2(2t - 1)(2t^3 - 2t - 1)}{t^2(1 - 2t^2)^2}$$

Since  $2t^3 - 2t - 1 = 2t(t^2 - 1) - 1$  is negative on  $[0, 1]$ , it follows that  $H$  decreases from  $\infty$  to 6 on  $[0, 1/2]$ , increases from 6 to  $\infty$  on  $[1/2, 1/\sqrt{2}]$ , and increases from  $-\infty$  to 1 on  $[1/\sqrt{2}, 1]$ . Therefore the minimum of  $H$  is 6 on acute triangles and 1 on obtuse triangles. Thus we have the better conclusion that

$$p \geq 6q \text{ for acute triangles}$$

$$p \geq q \text{ for obtuse triangles}$$

It is possible that the large amount of effort spent by Bager in proving the weak statement that  $p \geq 6q$  for obtuse triangles is in fact due to the weakness of the statement, not being the best possible.

One may also take  $G = p - 6q$ . Again, it is clear that we have no interior critical points. On degenerate triangles,  $G = -5q$ ,  $q \in [-1, 0]$ , and thus  $G$  takes all the values between 0 and 5. On right triangles,  $G = p = (s^2 - 1)/2$  and  $G$  takes all





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values between 0 and 5/8. On isosceles triangles,

$$G = t^2 + 2t(1 - 2t^2) - 6t^2(1 - 2t^2) \quad \text{and} \quad \frac{dG}{dt} = 2(2t - 1)(12t^2 + 3t - 1).$$

If  $r$  denotes the positive zero of  $12t^2 + 3t - 1$ , then  $r \leq 0.2$ ,  $G(r) \leq 0.2$  and  $G$  increases from 0 to  $G(r)$  on  $[0, r]$ , decreases from  $G(r)$  to 0 on  $[r, 1/2]$  increases from 0 to 1/2 on  $[1/2, 1/\sqrt{2}]$ , and increases from 1/2 to 5 on  $[1/\sqrt{2}, 1]$ . Therefore  $G \geq 0$  for all triangles, and  $G \leq 5/8$  for acute triangles and  $G \leq 5$  for obtuse triangles; and we have the stronger inequality

$$6q + \frac{5}{8} \geq p \geq 6q \quad \text{for acute triangles}$$

$$6q + 5 \geq p \geq 6q \quad \text{for obtuse triangles}$$

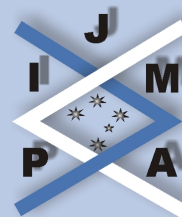
	Acute triangles		Obtuse triangles		
	Right	Isosceles	Degenerate	Right	Isosceles
max $G$	5/8	1/2	5	5/8	5
min $G$	0	0	0	0	1/2

*Example 5.3.* Here, we settle a conjecture in [1, Cj1, page 18)] which was solved in [2]. In our terminology, the conjecture reads

$$(5.3) \quad pQ \geq \frac{9\sqrt{3}}{4}q,$$

where  $Q = \sin A \sin B \sin C$ . The case  $q > 0, p < 0$  cannot occur since  $p \geq q$ . Also, in the case  $q > 0, p < 0$ , the inequality is vacuous. So we restrict our attention to the cases when  $p$  and  $q$  have the same sign and we optimize  $H = p^2Q^2/q^2$ . From

$$Q^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C) = 1 - s^2 + 2p + p^2 - 2sq - q^2,$$

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it follows that

$$\begin{aligned}H &= \frac{p^2(1 - s^2 + 2p + p^2 - 2sq - q^2)}{q^2} \\&= \frac{p^2(p + 1 + q + s)(p + 1 - q - s)}{q^2} \\&= \frac{(s^2 - 1 + 2q)^2(s^2 + 2s + 1 + 4q)(s - 1)^2}{16q^2} \\ \frac{\partial H}{\partial q} &= \frac{-(s - 1)^2(s^2 - 1 + 2q)(2qs^2 - 2q - 4q^2 + s^4 + 2s^3 - 2s - 1)}{8q^3} \\ \frac{\partial H}{\partial s} &= \frac{-(s - 1)(s^2 - 1 + 2q)(4qs^2 - q + 2q^2 - qs + s^4 + s^3 - s^2 - s)}{2q^2}\end{aligned}$$

At interior critical points (if any) at which  $s^2 - 1 + 2q = 0$ ,  $H = 0$ . For other interior critical points, we have

$$E_1 := 2qs^2 - 2q - 4q^2 + s^4 + 2s^3 - 2s - 1 = 0$$

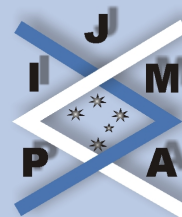
$$E_2 := 4qs^2 - q + 2q^2 - qs + s^4 + s^3 - s^2 - s = 0$$

$$E_3 := E_1 - 2E_2 = -2(5s^2 - s - 2)q - (3s + 1)(s - 1)(s + 1)^2 = 0$$

If  $5s^2 - s - 2 = 0$ , then  $(3s + 1)(s - 1)(s + 1)^2 = 0$ , which is impossible. Therefore  $5s^2 - s - 2 \neq 0$  and

$$q = \frac{-(3s + 1)(s - 1)(s + 1)^2}{2(5s^2 - s - 2)}$$

This with  $E_1$  imply that  $(s - 1)(s - 3)(s + 1)^2(s^2 - s - 1) = 0$ , which has no feasible solutions.

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We move to the boundary. As  $q \rightarrow 0^\pm$ ,  $H \rightarrow +\infty$ . On  $s = 1$ ,  $H = 0$ . It remains to work on isosceles triangles. There

$$H = \frac{2(4t^2 - t - 2)^2(1-t)^3(1+t)^3}{(1-2t^2)^2}$$

$$\frac{dH}{dt} = \frac{8(1-t)^2(1+t)^2(4t^2 - t - 2)(2t-1)(12t^4 + 4t^3 - 10t^2 - 4t + 1)}{(1-2t^2)^3}$$

Let  $\rho = (1 + \sqrt{33})/8$  be the positive zero of  $4t^2 - t - 2$ . Then  $q < 0, p > 0$  for  $t \in (\sqrt{2}/2, \rho)$ . By Descartes' rule of signs [13, page 121], the polynomial

$$g(t) = 12t^4 + 4t^3 - 10t^2 - 4t + 1$$

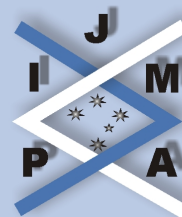
has at most two positive zeros. Since

$$g(0) = 1 > 0 \quad \text{and} \quad g(1/2) = \frac{-9}{4} < 0$$

then one of the zeros, say  $r_1$  is in  $(0, 1/2)$ . Also,

$$g(t) = (4t^2 - t - 2) \left( 3t^2 + \frac{7}{4}t - \frac{9}{16} \right) - \frac{17}{16}t - \frac{1}{8}.$$

Therefore  $g(\rho) < 0$ . Since  $g(1) = 3 > 0$ , it follows that the other positive zero, say  $r_2$ , of  $g$  is in  $(\rho, 1)$ . Therefore  $H$  increases on  $(0, r_1)$ , decreases on  $(r_1, 1/2)$  and then increases on  $(1/2, \sqrt{2}/2)$ . Its maximum on acute triangles is  $\infty$  and its minimum is  $\min\{H(0), H(1/2)\} = \max\{16, 243/16 = 15.1875\} = 243/16$ . This proves (5.3) in the acute case. In the obtuse case with  $p < 0$ , we see that  $H$  increases on  $(\rho, r_2)$  and decreases on  $(r_2, 1)$ . Its minimum is 0 and its maximum is  $H(r_2)$ . This is summarized in the following table.



		Isosceles					
		Acute		Obtuse, $p > 0$	Obtuse, $p > 0$		
$t$	0	$r_1$	1/2	$\sqrt{2}/2$	$\rho$	$r_2$	1
$H$	16	17.4	15.1875	$\infty$	0	0.01	0

The critical points together with the corresponding values of  $H$  are given below:

$t$	0	.18	.5	$\sqrt{2}/2^-$	$\sqrt{2}/2^+$	.85	.9	1
$H(t)$	16	17.4	15.1875	$+\infty$	$+\infty$	0	.01	0

	Acute triangles		Obtuse triangles with $p < 0$	
	Right	Isosceles	Degenerate	Isosceles
$\max H$	$\infty$	$\infty$	0	0.01
$\min H$	$\infty$	15.1875	0	0

*Example 5.4.* Finally, we prove inequality (33) in [1, page 17]. In our terminology, it reads

$$(5.4) \quad p \leq \frac{2}{\sqrt{3}}Q,$$

where  $Q = \sin A \sin B \sin C$ . Clearly, we must restrict our attention to the triangles with  $p > 0$  and minimize  $H = Q^2/p^2$ . Since  $H$  tends to  $+\infty$  as  $p$  tends to 0, we are not concerned with the behaviour of  $H$  near the curve  $p = s^2 - 1 + 2q = 0$ .

From

$$Q^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C) = 1 - s^2 + 2p + p^2 - 2sq - q^2,$$

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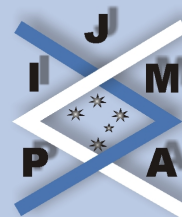
it follows that

$$\begin{aligned}
 H &= \frac{1 - s^2 + 2p + p^2 - 2sq - q^2}{p^2} \\
 &= \frac{(p + 1 + q + s)(p + 1 - q - s)}{p^2} \\
 &= \frac{(s - 1)^2(s^2 + 2s + 1 + 4q)}{(s^2 - 1 + 2q)^2} \\
 \frac{\partial H}{\partial q} &= \frac{-8(s - 1)^2(s + q + 1)}{(s^2 - 1 + 2q)^3} \\
 \frac{\partial H}{\partial s} &= \frac{8q(s - 1)(3s + 2q - 1)}{(s^2 - 1 + 2q)^3}
 \end{aligned}$$

It is clear that no interior critical points exist. At  $q = 0$ ,  $H = 1$ . At  $s = 1$ ,  $p = q < 0$ . On isosceles triangles,

$$H = \frac{4(1 - t)^3(1 + t)^3}{(4t^2 - t - 2)^2} \quad \text{and} \quad \frac{dH}{dt} = \frac{8(1 - t)^2(1 + t)^2(1 - 2t)(2t^2 + 1)}{(4t^2 - t - 2)^3}.$$

Then  $p > 0$  for  $t \in (0, \rho)$ , where  $\rho = (1 + \sqrt{33})/8$  is the positive zero of  $4t^2 - t - 2$ . On this interval, the minimum of  $H$  is  $H(1/2) = 3/4$ . Hence  $H \geq 3/4$  and the result follows by taking square roots.



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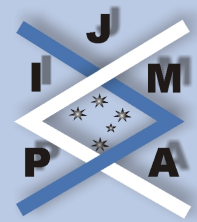
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## 6. Limitations of the Method described in Section 4

The method described in Section 4 deals only with polynomials (and polynomial-like functions) in the variables  $\cos A$ ,  $\cos B$ , and  $\cos C$  that are symmetric in these variables. There is an algorithm which writes such functions in terms of the elementary symmetric polynomials  $s$ ,  $p$ , and  $q$ , and consequently in terms of  $s$  and  $q$  using (2.11). Finding the interior critical points in the  $(s, q)$  domain  $\Omega$  involves solving a system of algebraic equations. Here, there is no algorithm for solving such systems.

For functions in  $\sin A$ ,  $\sin B$ , and  $\sin C$ , one needs to develop a parallel method. This is a worse situation since the algebraic relation among  $\sin A$ ,  $\sin B$ , and  $\sin C$  is more complicated; see [6, Theorem 5]. It is degree 4 and it is not linear in any of the variables. Things are even worse for inequalities that involve both the sines and cosines of the angles of a triangle. Here, one may need the theory of multisymmetric functions.

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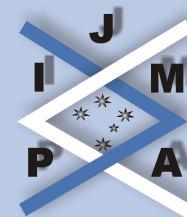
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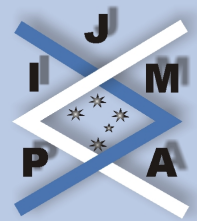
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