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A METHOD FOR ESTABLISHING CERTAIN TRIGONOMETRIC INEQUALITIES

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ABSTRACT. In this note, we describe a method for establishing trigonometric inequalities that involve symmetric functions in the cosines of the angles of a triangle. The method is based on finding a complete set of relations that define the cosines of such angles.

Key words and phrases: Geometric inequalities, Equifacial tetrahedra, Solid angle.

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1. INTRODUCTION

This note is motivated by the desire to find a straightforward proof of the fact that among all equifacial tetrahedra, the regular one has the maximal solid angle sum [9]. This led to a similar desire to find a systematic method for optimizing certain trigonometric functions and for establishing certain trigonometric inequalities.

Let us recall that a tetrahedron is called *equifacial* (or *isosceles*) if its faces are congruent. It is clear that the three angles enclosed by the arms of each corner angle of such a tetrahedron are the same as the three angles of a triangular face. Less obvious is the fact that the faces of an equifacial tetrahedron are necessarily acute-angled [8], [9].

Let us also recall that if A, B and C are the three angles enclosed by the arms of a solid angle V, then the *content* E of V is defined as the area of the spherical triangle whose vertices are traced by the arms of V on the unit sphere centered at the vertex of V [5]. This content E is given (in [5], for example) by

$$\tan\frac{E}{2}=\frac{\sqrt{1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C}}{1+\cos A+\cos B+\cos C}. \tag{1.1}$$

The statement made at the beginning of this article is equivalent to saying that the maximum of the quantity (1.1) over all acute triangles ABC is attained at $A=B=C=\pi/3$. To treat (1.1) as a function of three variables $\cos A$, $\cos B$, and $\cos C$, one naturally raises the question regarding a complete set of relations that define the cosines of the angles of an acute

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triangle, and similarly for a general triangle. These questions are answered in Theorems 2.2 and 2.3. The statement regarding the maximum of the quantity (1.1) over acute triangles ABC is established in Theorem 3.1. The methods developed are then used to establish several examples of trigonometric inequalities.

2. TRIPLES THAT CAN SERVE AS THE COSINES OF THE ANGLES OF A TRIANGLE

Our first theorem answers the natural question regarding what real triples qualify as the cosines of the angles of a triangle. For the proof, we need the following simple lemma taken together with its elegant proof from [7]. The lemma is actually true for any number of variables.

Lemma 2.1. *Let u, v, and w be real numbers and let*

$$s = u + v + w, \quad p = uv + vw + wu, \quad q = uvw. \tag{2.1}$$

Then u, v, and w are non-negative if and only if s, p and q are non-negative.

Proof. If s, p, and q are non-negative, then the polynomial $f(T) = T^3 - sT^2 + pT - q$ will be negative for every negative value of T. Thus its roots u, v, and w (which are assumed to be real) must be non-negative. □

With reference to (2.1), it is worth recording that the assumption that s, p, and q are non-negative does not imply that u, v, and w are real. For example, if ζ is a primitive third root of unity, and if $(u, v, w) = (1, \zeta, \zeta^2)$, then $s = p = 0$ and $q = 1$. For more on this, see [11].

Theorem 2.2. *Let u, v and w be real numbers. Then there exists a triangle ABC such that $u = \cos A$, $v = \cos B$, and $w = \cos C$ if and only if*

$$u + v + w \geq 1 \tag{2.2}$$

$$uvw \geq -1 \tag{2.3}$$

$$u^2 + v^2 + w^2 + 2uvw = 1. \tag{2.4}$$

The triangle is acute, right or obtuse according to whether uvw is greater than, equal to or less than 0.

Proof. Let A, B, and C be the angles of a triangle and let u, v, and w be their cosines. Then (2.3) is trivial, (2.2) follows from

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}, \tag{2.5}$$

or from Carnot's formula

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1, \tag{2.6}$$

where r and R are the inradius and circumradius of ABC, and (2.4) follows by squaring $\sin A \sin B = \cos C + \cos A \cos B$ and using $\sin^2 \theta = 1 - \cos^2 \theta$. For (2.5), see [4, 678, page 166], for (2.6), see [10], and for more on (2.4), see [6].

Conversely, let u, v, and w satisfy (2.2), (2.3), and (2.4), and let s, p and q be as in (2.1). Then (2.2), (2.3), and (2.4) can be rewritten as

$$s \geq 1,\ q \geq -1,\ s^2 - 2p + 2q = 1. \tag{2.7}$$

We show first that

$$\alpha = 1 - u^2,\ \beta = 1 - v^2,\ \gamma = 1 - w^2$$

are non-negative. By Lemma 2.1, this is equivalent to showing that $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$, and $\alpha\beta\gamma$ are non-negative. But it is routine to check that

$$\alpha+\beta+\gamma = 2(q+1) \ge 0,$$
$$4(\alpha\beta+\beta\gamma+\gamma\alpha) = ((s-1)^2+2(q+1))^2+4(s-1)^3 \ge 0,$$
$$4\alpha\beta\gamma = (s-1)^2(s^2+2s+1+4q) \ge (s-1)^2(1+2+1-4) \ge 0.$$

Thus $-1 \le u, v, w \le 1$. Therefore there exist unique A, B and C in $[0,\pi]$ such that $u = \cos A$, $v = \cos B$, and $w = \cos C$. It remains to show that $A+B+C=\pi$.

The sum of $u+vw$, $v+wu$, and $w+uv$ is $s+p$ and

$$s+p = s+\frac{s^2+2q-1}{2} \ge 1+\frac{1-2-1}{2} = 0.$$

Thus at least one of them, say $w+uv$, is non-negative. Also, (2.4) implies that

$$(w+uv)^2 = u^2v^2+1-u^2-v^2 = (1-u^2)(1-v^2) = \sin^2 A\sin^2 B.$$

Since $\sin A$, $\sin B$, and $w+uv$ are all non-negative, it follows that $w+uv = \sin A\sin B$, and therefore

$$\cos C = w = -uv+\sin A\sin B = -\cos A\cos B+\sin A\sin B = -\cos(A+B).$$

It also follows from (2.2) that

$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = \cos A+\cos B \ge 1-\cos C \ge 0$$

and hence $0 \le A+B \le \pi$. Thus C and $A+B$ are in $[0,\pi]$. From $\cos C = -\cos(A+B)$, it follows that $A+B+C=\pi$, as desired. □

Now let s, p and q be given real numbers and let u, v, and w be the zeros of the cubic T^3-sT^2+pT-q. Thus u, v and w are completely defined by (2.1). It is well-known [12, Theorem 4.32, page 239] that u, v and w are real if and only if the discriminant is non-negative, i.e.

$$\Delta = -27q^2+18spq+p^2s^2-4s^3q-4p^3 \ge 0. \tag{2.8}$$

If we assume that (2.7) holds, then we can eliminate p from Δ to obtain

$$\Delta = -(s^2+2s+1+4q)(s^4-2s^3+4s^2q-s^2-20sq+4s-2+20q+4q^2).$$

Also (2.7) implies that $s^2+2s+1+4q = (s+1)^2+4q \ge 2^2+4(-1) \ge 0$, with equality if and only if $(s,q)=(1,-1)$. Moreover, when $(s,q)=(1,-1)$, the second factor of Δ equals zero. Therefore, with (2.7) assumed, the condition that $\Delta \ge 0$ is equivalent to the condition

$$\Delta^* = -s^4+2s^3-4s^2q+s^2+20sq-4s+2-20q-4q^2 \ge 0, \tag{2.9}$$

with $\Delta = 0$ if and only if $\Delta^* = 0$. From this and from Theorem 2.2, it follows that u, v and w are the cosines of the angles of a triangle if and only if (2.7) and (2.8) (or equivalently (2.7) and (2.9)) hold. Also, the discriminant of Δ^*, as a polynomial in q, is $16(3-2s)^3$. Therefore for (2.9) to be satisfied (for any s at all), we must have $s \le 3/2$. Solving (2.9) for q, we re-write (2.9) in the equivalent form

$$\begin{cases} f_1(s) \le q \le f_2(s), \text{ where} \\ f_1(s) = \dfrac{-s^2+5s-5-(3-2s)^{3/2}}{2}, \quad f_2(s) = \dfrac{-s^2+5s-5+(3-2s)^{3/2}}{2}. \end{cases} \tag{2.10}$$

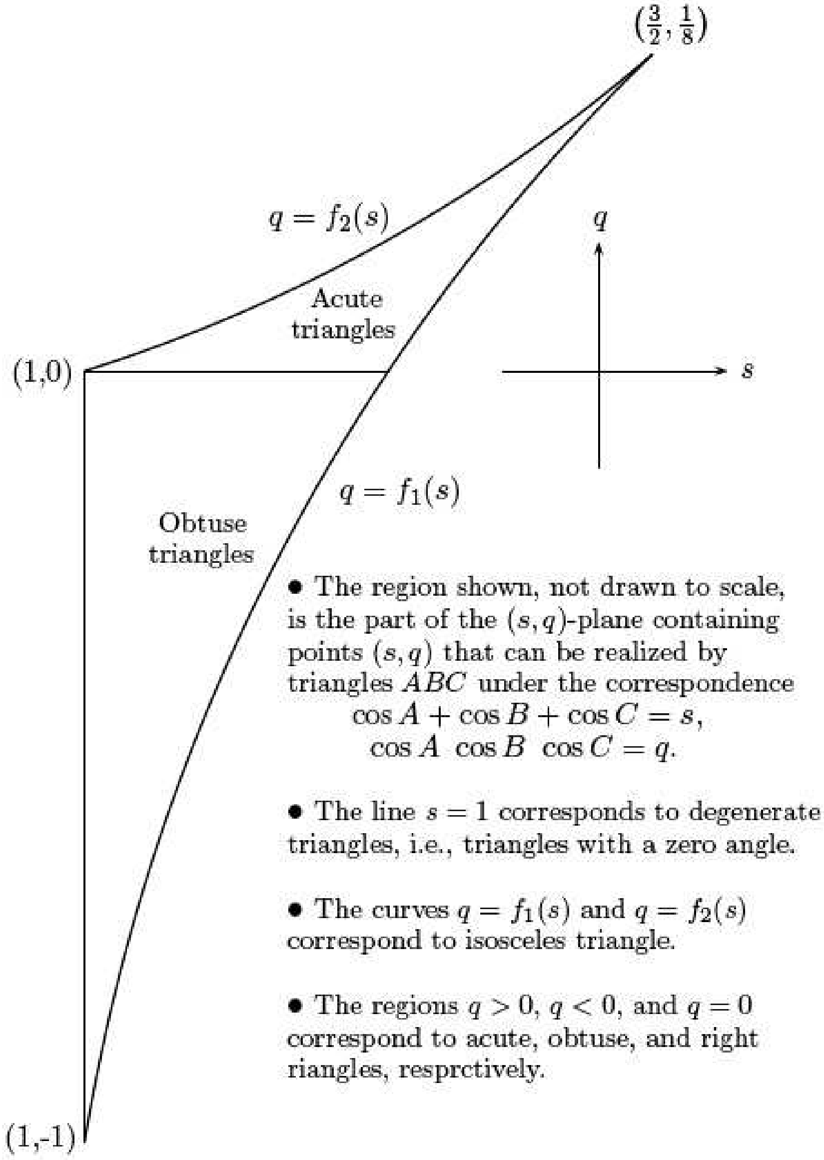

Figure 1.

Figure 1 is a sketch of the region Ω_0 defined by $f_1(s) \leq q \leq f_2(s)$, $1 \leq s \leq 1.5$, using the facts that $f_1(s)$ and $f_2(s)$ are increasing (and that f_1 is concave down and f_2 is concave up) on $s \in [1, 1.5]$. Note that the points (s, q) of Ω_0 satisfy $q \geq f_1(1) = -1$, rendering the condition $q \geq -1$ (in (2.7)) redundant. We summarize this in the following theorem.

Theorem 2.3. *Let s, p, and q be real numbers. Then the zeros of the cubic $T^3 - sT^2 + pT - q$ (are real and) can serve as the cosines of the angles of a triangle if and only if (s, p, q) lies in the region Ω defined by*

$$s^2 - 2p + 2q - 1 = 0, \tag{2.11}$$
$$1 \leq s \leq 1.5 \tag{2.12}$$

and any of the equivalent conditions (2.8), (2.9) and (2.10) hold. The boundary of Ω consists of the line segment defined by

$$s = 1, \quad q = p \in [-1, 0]$$

and corresponding to degenerate triangles (i.e. triangles with a zero angle), and the curve parametrized by

(2.13) $$s = 2t + 1 - 2t^2,\ q = t^2(1 - 2t^2), \quad p = t^2 + 2t(1 - 2t^2), \quad 0 \le t \le 1$$

and corresponding to isosceles triangles having angles $(\theta, \theta, \pi - 2\theta)$*, where* $\theta = \cos^{-1} t$*. It is clear that* $\pi - 2\theta$ *is acute for* $0 < t < 1/\sqrt{2}$ *and obtuse for* $1/\sqrt{2} < t < 1$*. Acute and obtuse triangles correspond to* $q > 0$ *and* $q < 0$ *(respectively), and right triangles are parametrized by*

$$q = 0, \quad p = \frac{s^2 - 1}{2},\ s \in [1, \sqrt{2}].$$

3. MAXIMIZING THE SUM OF THE CONTENTS OF THE CORNER ANGLES OF AN EQUIFACIAL TETRAHEDRON

We now turn to the optimization problem mentioned at the beginning.

Theorem 3.1. *Among all acute triangles* ABC*, the quantity (1.1) attains its maximum at* $A = B = C = \pi/3$*. Therefore among all equifacial tetrahedra, the regular one has a vertex solid angle of maximum measure.*

Proof. Note that (1.1) is not defined for obtuse triangles. Squaring (1.1) and using (2.7), we see that our problem is to maximize

$$f(s, q) = \frac{4q}{(s + 1)^2}$$

over Ω. Clearly, for a fixed s, f attains its maximum when q is largest. Thus we confine our search to the part of (2.13) defined by $0 \le t \le 1/\sqrt{2}$. Therefore our objective function is transformed to the one-variable function

$$g(t) = \frac{t^2(1 - 2t^2)}{(t^2 - t - 1)^2}, \quad 0 \le t \le 1/\sqrt{2}.$$

From

$$g'(t) = \frac{2t(2t - 1)(t + 1)^2}{(t^2 - t - 1)^3},$$

we see that g attains its maximum at $t = 1/2$, i.e at the equilateral triangle. □

4. A METHOD FOR OPTIMIZING CERTAIN TRIGONOMETRIC EXPRESSIONS

Theorem 3.1 above describes a systematic method for optimizing certain symmetric functions in $\cos A$, $\cos B$, and $\cos C$, where A, B, and C are the angles of a general triangle. If such a function can be written in the form $H(s, p, q)$, where s, p, and q are as defined in (2.1), then one can find its optimum values as follows:

(1) One finds the interior critical points of H by solving the system

$$\frac{\partial H}{\partial s} + \frac{\partial H}{\partial p} s = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p} = 0,$$
$$s^2 - 2p + 2q = 1,$$
$$1 < s < 1.5,$$
$$\Delta = -27q^2 + 18spq + p^2 s^2 - 4s^3 q - 4p^3 > 0.$$

Equivalently, one uses $s^2 - 2p + 2q = 1$ to write H as a function of s and q, and then solve the system

$$\frac{\partial H}{\partial s} = \frac{\partial H}{\partial q} = 0,$$
$$1 < s < 1.5,$$
$$\Delta^* = -s^4 + 2s^3 - 4s^2 q + s^2 + 20sq - 4s + 2 - 20q - 4q^2 > 0.$$

Usually, no such interior critical points exist.

(2) One then optimizes H on degenerate triangles, i.e., on

$$s = 1, \quad p = q, \quad q \in [-1, 0].$$

(3) One finally optimizes H on isosceles triangles, i.e., on

$$s = 2t + 1 - 2t^2, \quad p = t^2 + 2t(1 - 2t^2), \quad q = t^2(1 - 2t^2)), \quad t \in [0, 1].$$

If the optimization is to be done on acute triangles only, then

(1) Step 1 is modified by adding the condition $q > 0$,
(2) Step 2 is discarded,
(3) in Step 3, t is restricted to the interval $[0, 1/\sqrt{2}]$,
(4) a fourth step is added, namely, optimizing H on right triangles, i.e., on

(4.1) $$p = \frac{s^2 - 1}{2}, \quad q = 0, \ s \in [1, \sqrt{2}].$$

For obtuse triangles,

(1) Step 1 is modified by adding the condition $q < 0$,
(2) Step 2 remains,
(3) in Step 3, t is restricted to the interval $[1/\sqrt{2}, 1]$,
(4) the fourth step described in (4.1) is added.

5. EXAMPLES

The following examples illustrate the method.

Example 5.1. The inequality

(5.1) $$\sum \sin B \sin C \le \left(\sum \cos A\right)^2$$

is proved in [3], where the editor *wonders if there is a nicer way* of proof. In answer to the editor's request, Bager gave another proof in [1, page 20]. We now use our routine method.

Using $\sin A \sin B - \cos A \cos B = \cos C$, one rewrites this inequality as

$$s + p \le s^2.$$

It is clear that $H = s^2 - s - p$ has no interior critical points, since $\partial H/\partial p + \partial H/\partial q = -1$. For degenerate triangles, $s = 1$ and $H = -p = -q$ and takes all the values in $[0, 1]$. For isosceles triangles,

(5.2) $$H = (2t + 1 - 2t^2)^2 - (2t + 1 - 2t^2) - (t^2 + 2t(1 - 2t^2)) = t^2(2t - 1)^2 \ge 0.$$

Thus $H \ge 0$ for all triangles and our inequality is established.

One may like to establish a reverse inequality of the form $s + p \ge s^2 - c$ and to separate the cases of acute and obtuse triangles. For this, note that on right triangles, $q = 0$, and $H =$

$s^2 - s - (s^2 - 1)/2$ increases with s, taking all values in $[0, 1/8]$. Also, with reference to (5.2), note that

$$\frac{d}{dt}(t^2(2t-1)^2) = 2(4t-1)t(2t-1).$$

Thus $t^2(2t-1)^2$ increases on $[0, 1/4]$, decreases on $[1/4, 1/2]$, and increases on $[1/2, 1]$.

The first three tables below record the critical points of H and its values at those points, and the last one records the maximum and minimum of H on the sets of all acute and all obtuse triangles separately. Note that the numbers 1/8 and $1.5 - \sqrt{2}$ are quite close, but one can verify that 1/8 is the larger one. Therefore the maximum of $s^2 - s - p$ is $1/8$ on acute triangles and 1 on obtuse triangles, and we have proved the following stronger version of (5.1):

$$\sum \sin B \sin C \le \left(\sum \cos A\right)^2 \le \frac{1}{8} + \sum \sin B \sin C \text{ for acute triangles}$$
$$\sum \sin B \sin C \le \left(\sum \cos A\right)^2 \le 1 + \sum \sin B \sin C \text{ for obtuse triangles}$$

	Isosceles				
	Acute		Obtuse		
t	0	1/4	1/2	$\sqrt{2}$/2	1
H	0	1/64	0	1.5-$\sqrt{2}$	1

	Degenerate	
	$s = 1$	
q	-1	0
H	1	0

	Right	
	$q = 0$	
s	1	1.5
H	0	1/8

	Acute	Obtuse	All
$\max H$	1/8	1	1
$\min H$	0	0	0

We may also consider the function

$$G = \frac{s+p}{s^2}.$$

Again, G has no interior critical points since $\partial G/\partial p = 1/s^2$. On degenerate triangles, $s = 1$ and $G = 1 + q$ and takes all values in $[0, 1]$. On right triangles, $q = 0$ and we have

$$G = \frac{s^2 + 2s - 1}{2s^2}, \quad \frac{dG}{ds} = \frac{1-s}{s^3} \le 0.$$

Therefore G is decreasing for $s \in [1, 1.5]$ and takes all values in $[17/18, 1]$. It remains to work on isosceles triangles. There,

$$G = \frac{(1-t)(1+t)(4t+1)}{(2t^2 - 2t - 1)^2} \quad \text{and} \quad \frac{dG}{dt} = \frac{2t(2t-1)(2t^2+4t-1)}{(2t^2-2t-1)^3}.$$

Let $r = (-2 + \sqrt{6})/2$ be the positive zero of $2t^2 + 4t - 1$. Then $0 < r < 1/2$ and G decreases on $[0, r]$, increases on $[r, 1/2]$, decreases on $[1/2, 1]$. Its values at significant points and its extremum values are summarized in the tables below.

	Isosceles				
	Acute		Obtuse		
t	0	$(-2+\sqrt{6})/2$	1/2	$\sqrt{2}$/2	1
G	1	$(7+2\sqrt{6})/12$	1	$(1+2\sqrt{2})/4$	0

	Degenerate	
	$s = 1$	
q	-1	0
G	0	1

	Right	
	$q = 0$	
s	1	1.5
G	1	17/18

	Acute	Obtuse	All
$\max G$	1	1	1
$\min G$	17/18	0	0

Here we have used the delicate inequalities

$$\frac{17}{18} < \frac{1+2\sqrt{2}}{4} < \frac{7+2\sqrt{6}}{12} < 1.$$

As a result, we have proved the following addition to (5.1):

$$\frac{17}{18}\left(\sum \cos A\right)^2 \leq \sum \sin B \sin C \leq \left(\sum \cos A\right)^2 \text{ for acute triangles.}$$

Example 5.2. In [1], the inequality (8) (page 12) reads $p \geq 6q$. To prove this, we take

$$H = \frac{p}{q} = \frac{s^2 - 1 + 2q}{2q}.$$

It is clear that H has no interior critical points since $\partial H/\partial s$ is never 0. On the set of degenerate triangles, $s = 1$ and H is identically 1. On the set of right triangles, we note that as $q \to 0^+$, $H \to +\infty$, and as $q \to 0^-$, $H \to -\infty$. On the set of isosceles triangles,

$$H = \frac{t^2 + 2t(1-2t^2)}{t^2(1-2t^2)} = \frac{1}{1-2t^2} + \frac{2}{t}$$

$$\frac{dH}{dt} = \frac{4t}{(1-2t^2)^2} - \frac{2}{t^2} = \frac{-2(2t-1)(2t^3-2t-1)}{t^2(1-2t^2)^2}$$

Since $2t^3 - 2t - 1 = 2t(t^2-1) - 1$ is negative on $[0,1]$, it follows that H decreases from ∞ to 6 on $[0,1/2]$, increases from 6 to ∞ on $[1/2, 1/\sqrt{2}]$, and increases from $-\infty$ to 1 on $[1/\sqrt{2}, 1]$. Therefore the minimum of H is 6 on acute triangles and 1 on obtuse triangles. Thus we have the better conclusion that

$$p \geq 6q \text{ for acute triangles}$$
$$p \geq q \text{ for obtuse triangles}$$

It is possible that the large amount of effort spent by Bager in proving the weak statement that $p \geq 6q$ for obtuse triangles is in fact due to the weakness of the statement, not being the best possible.

One may also take $G = p - 6q$. Again, it is clear that we have no interior critical points. On degenerate triangles, $G = -5q$, $q \in [-1, 0]$, and thus G takes all the values between 0 and 5. On right triangles, $G = p = (s^2-1)/2$ and G takes all values between 0 and 5/8. On isosceles triangles,

$$G = t^2 + 2t(1-2t^2) - 6t^2(1-2t^2) \text{ and } \frac{dG}{dt} = 2(2t-1)(12t^2+3t-1).$$

If r denotes the positive zero of $12t^2 + 3t - 1$, then $r \leq 0.2$, $G(r) \leq 0.2$ and G increases from 0 to $G(r)$ on $[0, r]$, decreases from $G(r)$ to 0 on $[r, 1/2]$ increases from 0 to 1/2 on $[1/2, 1/\sqrt{2}]$, and increases from 1/2 to 5 on $[1/\sqrt{2}, 1]$. Therefore $G \geq 0$ for all triangles, and $G \leq 5/8$ for acute triangles and $G \leq 5$ for obtuse triangles; and we have the stronger inequality

$$6q + \frac{5}{8} \geq p \ \geq \ 6q \text{ for acute triangles}$$
$$6q + 5 \geq p \ \geq \ 6q \text{ for obtuse triangles}$$

	Acute triangles		Obtuse triangles		
	Right	Isosceles	Degenerate	Right	Isosceles
$\max G$	5/8	1/2	5	5/8	5
$\min G$	0	0	0	0	1/2

Example 5.3. Here, we settle a conjecture in [1, Cj1, page 18)] which was solved in [2]. In our terminology, the conjecture reads

$$pQ \geq \frac{9\sqrt{3}}{4}q, \tag{5.3}$$

where $Q = \sin A \sin B \sin C$. The case $q > 0, p < 0$ cannot occur since $p \geq q$. Also, in the case $q > 0, p < 0$, the inequality is vacuous. So we restrict our attention to the cases when p and q have the same sign and we optimize $H = p^2Q^2/q^2$. From

$$Q^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C) = 1 - s^2 + 2p + p^2 - 2sq - q^2,$$

it follows that

$$\begin{aligned} H &= \frac{p^2(1 - s^2 + 2p + p^2 - 2sq - q^2)}{q^2} \\ &= \frac{p^2(p + 1 + q + s)(p + 1 - q - s)}{q^2} \\ &= \frac{(s^2 - 1 + 2q)^2(s^2 + 2s + 1 + 4q)(s - 1)^2}{16q^2} \\ \frac{\partial H}{\partial q} &= \frac{-(s - 1)^2(s^2 - 1 + 2q)(2qs^2 - 2q - 4q^2 + s^4 + 2s^3 - 2s - 1)}{8q^3} \\ \frac{\partial H}{\partial s} &= \frac{-(s - 1)(s^2 - 1 + 2q)(4qs^2 - q + 2q^2 - qs + s^4 + s^3 - s^2 - s)}{2q^2} \end{aligned}$$

At interior critical points (if any) at which $s^2 - 1 + 2q = 0$, $H = 0$. For other interior critical points, we have

$$\begin{aligned} E_1 &:= 2qs^2 - 2q - 4q^2 + s^4 + 2s^3 - 2s - 1 = 0 \\ E_2 &:= 4qs^2 - q + 2q^2 - qs + s^4 + s^3 - s^2 - s = 0 \\ E_3 &:= E_1 - 2E_2 = -2(5s^2 - s - 2)q - (3s + 1)(s - 1)(s + 1)^2 = 0 \end{aligned}$$

If $5s^2-s-2 = 0$, then $(3s+1)(s-1)(s+1)^2 = 0$, which is impossible. Therefore $5s^2-s-2 \neq 0$ and

$$q = \frac{-(3s + 1)(s - 1)(s + 1)^2}{2(5s^2 - s - 2)}$$

This with E_1 imply that $(s-1)(s-3)(s+1)^2(s^2-s-1) = 0$, which has no feasible solutions.

We move to the boundary. As $q \to 0^{\pm}$, $H \to +\infty$. On $s = 1$, $H = 0$. It remains to work on isosceles triangles. There

$$\begin{aligned} H &= \frac{2(4t^2 - t - 2)^2(1 - t)^3(1 + t)^3}{(1 - 2t^2)^2} \\ \frac{dH}{dt} &= \frac{8(1 - t)^2(1 + t)^2(4t^2 - t - 2)(2t - 1)(12t^4 + 4t^3 - 10t^2 - 4t + 1)}{(1 - 2t^2)^3} \end{aligned}$$

Let $\rho = (1 + \sqrt{33})/8$ be the positive zero of $4t^2 - t - 2$. Then $q < 0, p > 0$ for $t \in (\sqrt{2}/2, \rho)$. By Descartes' rule of signs [13, page 121], the polynomial

$$g(t) = 12t^4 + 4t^3 - 10t^2 - 4t + 1$$

has at most two positive zeros. Since

$$g(0) = 1 > 0 \text{ and } g(1/2) = \frac{-9}{4} < 0$$

then one of the zeros, say r_1 is in $(0, 1/2)$. Also,

$$g(t) = (4t^2 - t - 2)\left(3t^2 + \frac{7}{4}t - \frac{9}{16}\right) - \frac{17}{16}t - \frac{1}{8}.$$

Therefore $g(\rho) < 0$. Since $g(1) = 3 > 0$, it follows that the other positive zero, say r_2, of g is in $(\rho, 1)$. Therefore H increases on $(0, r_1)$, decreases on $(r_1, 1/2)$ and then increases on $(1/2, \sqrt{2}/2)$. Its maximum on acute triangles is ∞ and its minimum is $\min\{H(0), H(1/2)\}$ = $\max\{16, 243/16 = 15.1875\}$ = 243/16. This proves (5.3) in the acute case. In the obtuse case with $p < 0$, we see that H increases on (ρ, r_2) and decreases on $(r_2, 1)$. Its minimum is 0 and its maximum is $H(r_2)$. This is summarized in the following table.

	Isosceles						
	Acute		Obtuse, $p > 0$		Obtuse, $p > 0$		
t	0	r_1	1/2	$\sqrt{2}/2$	ρ	r_2	1
H	16	17.4	15.1875	∞	0	0.01	0

The critical points together with the corresponding values of H are given below:

t	0	.18	.5	$\sqrt{2}/2^-$	$\sqrt{2}/2^+$	.85	.9	1
$H(t)$	16	17.4	15.1875	+ ∞	+ ∞	0	.01	0

	Acute triangles		Obtuse triangles with $p < 0$	
	Right	Isosceles	Degenerate	Isosceles
$\max H$	∞	∞	0	0.01
$\min H$	∞	15.1875	0	0

Example 5.4. Finally, we prove inequality (33) in [1, page 17]. In our terminology, it reads

$$p \leq \frac{2}{\sqrt{3}}Q, \tag{5.4}$$

where $Q = \sin A \sin B \sin C$. Clearly, we must restrict our attention to the triangles with $p > 0$ and minimize $H = Q^2/p^2$. Since H tends to $+\infty$ as p tends to 0, we are not concerned with the behaviour of H near the curve $p = s^2 - 1 + 2q = 0$.

From

$$Q^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C) = 1 - s^2 + 2p + p^2 - 2sq - q^2,$$

it follows that

$$\begin{aligned} H &= \frac{1 - s^2 + 2p + p^2 - 2sq - q^2}{p^2} \\ &= \frac{(p + 1 + q + s)(p + 1 - q - s)}{p^2} \\ &= \frac{(s-1)^2(s^2 + 2s + 1 + 4q)}{(s^2 - 1 + 2q)^2} \\ \frac{\partial H}{\partial q} &= \frac{-8(s-1)^2(s + q + 1)}{(s^2 - 1 + 2q)^3} \\ \frac{\partial H}{\partial s} &= \frac{8q(s-1)(3s + 2q - 1)}{(s^2 - 1 + 2q)^3} \end{aligned}$$

It is clear that no interior critical points exist. At $q = 0$, $H = 1$. At $s = 1$, $p = q < 0$. On isosceles triangles,

$$H = \frac{4(1-t)^3(1+t)^3}{(4t^2-t-2)^2} \text{ and } \frac{dH}{dt} = \frac{8(1-t)^2(1+t)^2(1-2t)(2t^2+1)}{(4t^2-t-2)^3}.$$

Then $p > 0$ for $t \in (0, \rho)$, where $\rho = (1+\sqrt{33})/8$ is the positive zero of $4t^2 - t - 2$. On this interval, the minimum of H is $H(1/2) = 3/4$. Hence $H \geq 3/4$ and the result follows by taking square roots.

6. LIMITATIONS OF THE METHOD DESCRIBED IN SECTION 4

The method described in Section 4 deals only with polynomials (and polynomial-like functions) in the variables $\cos A$, $\cos B$, and $\cos C$ that are symmetric in these variables. There is an algorithm which writes such functions in terms of the elementary symmetric polynomials s, p, and q, and consequently in terms of s and q using (2.11). Finding the interior critical points in the (s, q) domain Ω involves solving a system of algebraic equations. Here, there is no algorithm for solving such systems.

For functions in $\sin A$, $\sin B$, and $\sin C$, one needs to develop a parallel method. This is a worse situation since the algebraic relation among $\sin A$, $\sin B$, and $\sin C$ is more complicated; see [6, Theorem 5]. It is degree 4 and it is not linear in any of the variables. Things are even worse for inequalities that involve both the sines and cosines of the angles of a triangle. Here, one may need the theory of multisymmetric functions.

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