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OSTROWSKI TYPE INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

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Abstract

Some inequalities of Ostrowski type for isotonic linear functionals defined on a linear class of function $L := \{f : [a, b] \to \mathbb{R}\}$ are established. Applications for integral and discrete inequalities are also given.

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1. Introduction

The following result is known in the literature as Ostrowski's inequality [13].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The following Ostrowski type result for absolutely continuous functions whose derivatives belong to the Lebesgue spaces $L_p[a,b]$ also holds (see [9], [10] and [11]).

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{1.2}$$
$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{(b-a)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q[a,b],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1; & \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r\, dt\right)^{\frac{1}{r}}, \ r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense presented in Theorem 1.1.*

The above inequalities can also be obtained from Fink's result in [12] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see [7]):

Theorem 1.3. *Let* $f : [a,b] \to \mathbb{R}$ *be of* $r - H-$*Hölder type, i.e.,*

$$|f(x) - f(y)| \le H|x-y|^r, \text{ for all } x,y \in [a,b], \tag{1.3}$$

where $r \in (0,1]$ *and* $H > 0$ *are fixed. Then for all* $x \in [a,b]$ *we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1} + \left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ *is also sharp in the above sense.*

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see [3])

$$(1.5) \qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\,L.$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [4]).

Theorem 1.4. *Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then*

$$(1.6) \qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f)$$

for all $x \in [a,b]$.
The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [5] (see also [2]).

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all*

$x \in [a, b]$, we have the inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.7}$$
$$\leq \frac{1}{b-a}\left\{[2x-(a+b)]\,f(x) + \int_a^b sgn(t-x)\,f(t)\,dt\right\}$$
$$\leq \frac{1}{b-a}\left\{(x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)]\right\}$$
$$\leq \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)].$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The version of Ostrowski's inequality for convex functions was obtained in [6] and is incorporated in the following theorem:

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ we have the inequality*

$$\frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x)\right] \tag{1.8}$$
$$\leq \int_a^b f(t)\,dt - (b-a)\,f(x)$$
$$\leq \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right].$$

In both parts of the inequality (1.8) the constant $\frac{1}{2}$ is sharp.

For other Ostrowski type inequalities, see [8].

In this paper we extend Ostrowski's inequality for arbitrary isotonic linear functionals $A : L \to \mathbb{R}$, where L is a linear class of absolutely continuous functions defined on $[a, b]$. Some applications for particular instances of linear functionals A are also provided.

2. Preliminaries

Let L be a *linear class* of real-valued functions, $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$, $t \in E$, then $f \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$;

(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

Usual examples of isotonic linear functional that are normalised are the following ones

$$A(f) := \frac{1}{\mu(X)} \int_X f(x)\, d\mu(x), \quad \text{if } \mu(X) < \infty$$

or

$$A_w(f) := \frac{1}{\int_X w(x)\, d\mu(x)} \int_X w(x) f(x)\, d\mu(x),$$

where $w(x) \geq 0$, $\int_X w(x)\, d\mu(x) > 0$, X is a measurable space and μ is a positive measure on X.

In particular, for $\bar{x} = (x_1, \ldots, x_n)$, $\bar{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n$ with $w_i \geq 0$, $W_n := \sum_{i=1}^{n} w_i > 0$ we have

$$A(\bar{x}) := \frac{1}{n} \sum_{i=1}^{n} x_i$$

and

$$A_{\bar{w}}(\bar{x}) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i,$$

are normalised isotonic linear functionals on $\mathbb{R}^n$.

The following representation result for absolutely continuous functions holds.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ and define $e(t) = t$, $t \in [a, b]$, $g(t, x) = \int_0^1 f'[(1-\lambda)x + \lambda t]\, d\lambda$, $t \in [a, b]$ and $x \in [a, b]$. If $A : L \to \mathbb{R}$ is a normalised linear functional on a linear class L of absolutely continuous functions defined on $[a, b]$ and $(x - e) \cdot g(\cdot, x) \in L$, then we have the representation*

$$f(x) = A(f) + A[(x - e) \cdot g(\cdot, x)], \tag{2.1}$$

for $x \in [a, b]$.

Proof. For any $x, t \in [a, b]$ with $t \neq x$, one has

$$\frac{f(x) - f(t)}{x - t} = \frac{\int_t^x f'(u)}{x - t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\, d\lambda = g(t, x),$$

giving the equality

$$f(x) = f(t) + (x-t)\, g(t,x) \tag{2.2}$$

for any $t, x \in [a,b]$.

Applying the functional A, we get

$$A(f(x)\cdot \mathbf{1}) = A(f + (x-e)\, g(\cdot, x)),$$

for any $x \in [a,b]$.

Since

$$A(f(x)\cdot \mathbf{1}) = f(x)\, A(\mathbf{1}) = f(x)$$

and

$$A(f + (x-e)\cdot g(\cdot,x)) = A(f) + A((x-e)\cdot g(\cdot,x)),$$

the equality (2.1) is obtained. $\square$

The following particular cases are of interest:

Corollary 2.2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the representation:*

$$\begin{aligned} f(x) &= \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt \\ &\quad + \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \end{aligned} \tag{2.3}$$

for any $x \in [a,b]$, *where* $p : [a,b] \to \mathbb{R}$ *is a Lebesgue integrable function with* $\int_a^b w(t)\,dt \neq 0.$

In particular, we have

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right)dt \tag{2.4}$$

for each $x \in [a,b]$.

The proof is obvious by Lemma 2.1 applied for the normalised linear functionals

$$A_w(f) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt, \quad A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$$

defined on

$$L := \{f : [a,b] \to \mathbb{R},\ f \text{ is absolutely continuous on } [a,b]\}.$$

The following discrete case also holds.

Corollary 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then we have the representation:*

$$f(x) = \frac{1}{W_n}\sum_{i=1}^n w_i f(x_i) + \frac{1}{W_n}\sum_{i=1}^n w_i (x-x_i)\left(\int_0^1 f'[(1-\lambda)x + \lambda x_i]\,d\lambda\right) \tag{2.5}$$

for any $x \in [a,b]$*, where* $x_i \in [a,b]$, $w_i \in \mathbb{R}$ $(i = \{1,\dots,n\})$ *with* $W_n := \sum_{i=1}^{n} w_i \neq 0.$

In particular, we have

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) + \frac{1}{n}\sum_{i=1}^{n}(x - x_i)\left(\int_0^1 f'[(1-\lambda)x + \lambda x_i]\,d\lambda\right) \tag{2.6}$$

for any $x \in [a,b]$.

3. Ostrowski Type Inequalities

The following theorem holds.

Theorem 3.1. *With the assumptions of Lemma 2.1, and assuming that $A : L \to \mathbb{R}$ is isotonic, then we have the inequalities*

$$|f(x) - A(f)| \leq \begin{cases} A\left(|x-e| \|f'\|_{[x,\cdot],\infty}\right) & \text{if } |x-e| \|f'\|_{[x,\cdot],\infty} \in L,\ f' \in L_\infty[a,b]; \\ A\left(|x-e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p}\right) & \text{if } |x-e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p} \in L,\ f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ A\left(\|f'\|_{[x,\cdot],1}\right) & \text{if } \|f'\|_{[x,\cdot],1} \in L, \end{cases} \tag{3.1}$$

where

$$\|h\|_{[m,n],\infty} := ess \sup_{\substack{t\in[m,n] \\ (t\in[n,m])}} |h(t)| \quad \textit{and}$$

$$\|h\|_{[m,n],p} := \left|\int_m^n |h(t)|^p \, dt\right|^{\frac{1}{p}}, \ p \geq 1.$$

If we denote

$$M_\infty(x) := A\left(|x-e| \|f'\|_{[x,\cdot],\infty}\right),$$

$$M_p(x) := A\left(|x-e|^{\frac{1}{q}} \|f'\|_{[x,\cdot],p}\right), \ \textit{and} \ M_1(x) := A\left(\|f'\|_{[x,\cdot],1}\right),$$

then we have the inequalities:

$$M_{\infty}(x) \tag{3.2}$$
$$\leq \begin{cases} \|f'\|_{[a,b],\infty} A(|x-e|) \text{ if } |x-e| \in L,\ f' \in L_{\infty}[a,b]; \\ \left[A\left(\|f'\|^{\beta}_{[x,\cdot],\infty}\right)\right]^{\frac{1}{\beta}} \left[A\left(|x-e|^{\alpha}\right)\right]^{\frac{1}{\alpha}} \\ \qquad \text{if } \|f'\|^{\beta}_{[x,\cdot],\infty}, |x-e|^{\alpha} \in L,\ f' \in L_{\infty}[a,b],\ \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] A\left(\|f'\|_{[x,\cdot],\infty}\right) \text{ if } \|f'\|_{[x,\cdot],\infty} \in L,\ f' \in L_{\infty}[a,b]. \end{cases}$$

$$M_p(x) \tag{3.3}$$
$$\leq \begin{cases} \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} A\left(|x-e|^{\frac{1}{q}}\right) \text{ if } |x-e|^{\frac{1}{q}} \in L,\ f' \in L_p[a,b]; \\ \left[A\left(\|f'\|^{\beta}_{[x,\cdot],p}\right)\right]^{\frac{1}{\beta}} \left[A\left(|x-e|^{\frac{\alpha}{q}}\right)\right]^{\frac{1}{\alpha}} \\ \qquad \text{if } \|f'\|^{\beta}_{[x,\cdot],p}, |x-e|^{\frac{\alpha}{q}} \in L,\ f' \in L_p[a,b],\ \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\frac{1}{q}} A\left(\|f'\|_{[x,\cdot],p}\right) \text{ if } \|f'\|_{[x,\cdot],p} \in L,\ f' \in L_p[a,b] \end{cases}$$

and

$$M_1(x) \leq \begin{cases} \frac{1}{2}\|f'\|_{[a,b],1} + \frac{1}{2}\left|\|f'\|_{[a,x],1} - \|f'\|_{[x,b],1}\right|, \\ \left[A\left(\|f'\|^{\beta}_{[x,\cdot],1}\right)\right]^{\frac{1}{\beta}}, \quad \beta > 1. \end{cases} \tag{3.4}$$

Proof. Using (2.1) and taking the modulus, we have

$$\begin{aligned}|f(x)-A(f)| &= |A((x-e)\cdot g(\cdot,x))| \\ &\leq A(|(x-e)\cdot g(\cdot,x)|) \\ &= A(|x-e|\,|g(\cdot,x)|)\,. \end{aligned} \tag{3.5}$$

For $t \neq x$ $(t, x \in [a,b])$ we may state

$$\begin{aligned}|g(t,x)| &\leq \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda \\ &\leq ess \sup_{\lambda\in[0,1]} |f'((1-\lambda)x+\lambda t)| \\ &= \|f'\|_{[x,t],\infty}\,. \end{aligned}$$

Hölder's inequality will produce

$$\begin{aligned}|g(t,x)| &\leq \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda \\ &\leq \left[\int_0^1 |f'((1-\lambda)x+\lambda t)|^p\,d\lambda\right]^{\frac{1}{p}} \\ &= \left(\frac{1}{x-t}\int_t^x |f'(u)|^p\,du\right)^{\frac{1}{p}} \\ &= |x-t|^{-\frac{1}{p}}\,\|f'\|_{[x,t],p}\,, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{aligned}$$

and finally

$$|g(t,x)| \leq \int_0^1 |f'((1-\lambda)x+\lambda t)|\,d\lambda = \frac{1}{t-x}\,\|f'\|_{[x,t],1}\,.$$

Consequently

$$(3.6)\qquad |(x-e)|\,|g(\cdot,x)| \le \begin{cases} |x-e|\,\|f'\|_{[x,\cdot],\infty} & \text{if } f'\in L_\infty[a,b]\,; \\ |x-e|^{\frac{1}{q}}\,\|f'\|_{[x,\cdot],p} & \text{if } f'\in L_p[a,b]\,, \\ \|f'\|_{[x,\cdot],1} \end{cases}$$

for any $x\in[a,b]$.

Applying the functional A to (3.6) and using (3.5) we deduce the inequality (3.1).

We have

$$\begin{aligned} M_\infty(x) &\le \sup_{t\in[a,b]}\left\{\|f'\|_{[x,t],\infty}\right\} A(|x-e|) \\ &= \max\left\{\|f'\|_{[a,x],\infty}\,,\|f'\|_{[x,b],\infty}\right\} A(|x-e|) \\ &= \|f'\|_{[a,b],\infty}\,A(|x-e|) \end{aligned}$$

and the first inequality in (3.2) is proved.

Using Hölder's inequality for the functional A, i.e.,

$$(3.7)\qquad |A(hg)| \le [A(|h|^\alpha)]^{\frac{1}{\alpha}}\left[A\left(|g|^\beta\right)\right]^{\frac{1}{\beta}},\quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,$$

where hg, $|h|^\alpha$, $|g|^\beta\in L$, we have

$$M_\infty(x) \le [A(|x-e|^\alpha)]^{\frac{1}{\alpha}}\left[A\left(\|f'\|^\beta_{[x,\cdot],\infty}\right)\right]^{\frac{1}{\beta}}$$

and the second part of (3.2) is proved.

In addition,

$$\begin{aligned} M_\infty(x) &\le \sup_{t\in[a,b]} |x-t|\, A\left(\|f'\|_{[x,\cdot],\infty}\right) \\ &= \max\{x-a, b-x\}\, A\left(\|f'\|_{[x,\cdot],\infty}\right) \\ &= \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right] A\left(\|f'\|_{[x,\cdot],\infty}\right) \end{aligned}$$

and the inequality (3.2) is completely proved.

We also have

$$\begin{aligned} M_p(x) &\le \sup_{t\in[a,b]} \left\{\|f'\|_{[x,t],p}\right\} A\left(|x-e|^{\frac{1}{q}}\right) \\ &= \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} A\left(|x-e|^{\frac{1}{q}}\right). \end{aligned}$$

Using Hölder's inequality (3.7) one has

$$M_p(x) \le \left[A\left(|x-e|^{\frac{\alpha}{q}}\right)\right]^{\frac{1}{\alpha}} \left[A\left(\|f'\|^{\beta}_{[x,\cdot],p}\right)\right]^{\frac{1}{\beta}}, \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$$

and

$$\begin{aligned} M_p(x) &\le \sup_{t\in[a,b]} \left\{|x-t|^{\frac{1}{q}}\right\} A\left(\|f'\|_{[x,\cdot],p}\right) \\ &= \max\left\{(x-a)^{\frac{1}{q}}, (b-x)^{\frac{1}{q}}\right\} A\left(\|f'\|_{[x,\cdot],p}\right) \\ &= \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right]^{\frac{1}{q}} A\left(\|f'\|_{[x,\cdot],p}\right), \end{aligned}$$

proving the inequality (3.3).

Finally,

$$\begin{aligned} A\left(\|f'\|_{[x,\cdot],1}\right) &\leq \sup_{t\in[a,b]}\left\{\|f'\|_{[x,t],1}\right\} A(\mathbf{1}) \\ &= \max\left\{\|f'\|_{[a,x],1}, \|f'\|_{[x,b],1}\right\} \\ &= \frac{1}{2}\|f'\|_{[a,b],1} + \frac{1}{2}\left|\|f'\|_{[a,x],1} - \|f'\|_{[x,b],1}\right|. \end{aligned}$$

By Hölder's inequality, we have

$$A\left(\|f'\|_{[x,\cdot],1}\right) \leq \left[A\left(\|f'\|^{\beta}_{[x,\cdot],1}\right)\right]^{\frac{1}{\beta}}, \quad \beta > 1,$$

and the last part of (3.4) is also proved. □

4. The Case where $|f'|$ is Convex

The following theorem also holds.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function such that $f' : (a,b) \to \mathbb{R}$ is convex in absolute value, i.e., $|f'|$ is convex on (a,b). If $A : L \to \mathbb{R}$ is a normalised isotonic linear functional and $|x-e|, |x-e|\,|f'| \in L$, then*

$$|f(x) - A(f)| \leq \frac{1}{2}\left[|f'(x)|\, A(|x-e|) + A(|x-e|\,|f'|)\right] \tag{4.1}$$

$$\leq \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],\infty} + |f'(x)|\right] A(|x-e|)\,, & \text{if } f' \in L_\infty[a,b]\,; \\ \frac{1}{2}\left[|f'(x)|\, A(|x-e|) + \left[A(|x-e|^\alpha)\right]^{\frac{1}{\alpha}} \left[A\left(|f'|^\beta\right)\right]^{\frac{1}{\beta}}\right] & \\ \qquad \text{if } |x-e|^\alpha, |f'|^\beta \in L,\ \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; & \\ \frac{1}{2}\left[|f'(x)|\, A(|x-e|) + \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] A(|f'|)\right] & \text{if } |f'| \in L. \end{cases}$$

Proof. Since $|f'|$ is convex, we have

$$\begin{aligned} |g(t,x)| &\leq \int_0^1 |f'((1-\lambda)x + \lambda t)|\, d\lambda \\ &= |f'(x)| \int_0^1 (1-\lambda)\, d\lambda + |f'(t)| \int_0^1 \lambda\, d\lambda \\ &= \frac{|f'(x)| + |f'(t)|}{2}. \end{aligned}$$

Thus,

$$\begin{aligned}|f(x) - A(f)| &\leq A\left(|x-e| \cdot \frac{|f'(x)| + |f'(t)|}{2}\right) \\ &= \frac{1}{2}\left[|f'(x)| A(|x-e|) + A(|x-e||f'|)\right]\end{aligned}$$

and the first part of (4.1) is proved.

We have

$$\begin{aligned}A(|x-e||f'|) &\leq ess \sup_{t\in[a,b]} \{|f'(t)|\} \cdot A(|x-e|) \\ &= \|f'\|_{[a,b],\infty} A(|x-e|).\end{aligned}$$

By Hölder's inequality for isotonic linear functionals, we have

$$A(|x-e||f'|) \leq [A(|x-e|^{\alpha})]^{\frac{1}{\alpha}} \left[A\left(|f'|^{\beta}\right)\right]^{\frac{1}{\beta}}, \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and finally,

$$\begin{aligned}A(|x-e||f'|) &\leq \sup_{t\in[a,b]} |x-t| \cdot A(|f'|) \\ &= \max(x-a, b-x) \cdot A(|f'|) \\ &= \left(\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right) A(|f'|).\end{aligned}$$

The theorem is thus proved. $\square$

5. Some Integral Inequalities

If we consider the normalised isotonic linear functional $A(f) = \frac{1}{b-a}\int_a^b f$, then by Theorem 3.1 for $f : [a,b] \to \mathbb{R}$ an absolutely continuous function, we may state the following integral inequalities

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{5.1}$$
$$\leq \frac{1}{b-a}\int_a^b |x-t| \, \|f'\|_{[x,t],\infty}\,dt$$
$$\leq \begin{cases} \|f'\|_{[a,b],\infty}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a) & \text{(Ostrowski's inequality)} \\ & \text{provided } f' \in L_\infty[a,b]; \\ \left[\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],\infty}^{\beta}\,dt\right]^{\frac{1}{\beta}}\left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(\alpha+1)(b-a)}\right]^{\frac{1}{\alpha}} \\ \text{if } f' \in L_\infty[a,b],\ \|f'\|_{[x,\cdot],\infty} \in L_\beta[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2} + \frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\int_a^b \|f'\|_{[x,t],\infty}\,dt \\ \qquad \text{if } f' \in L_\infty[a,b], \text{ and if } \|f'\|_{[x,\cdot],\infty} \in L_1[a,b], \end{cases}$$

for each $x \in [a,b]$;

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{b-a}\int_a^b |x-t|^{\frac{1}{q}} \, \|f'\|_{[x,t],p}\,dt \tag{5.2}$$

$$\leq \begin{cases} q\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\}\left[\frac{(b-x)^{\frac{1}{q}+1}+(x-a)^{\frac{1}{q}+1}}{(b-a)(q+1)}\right], \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ and } f'\in L_p[a,b]; \\ q^{\frac{1}{\alpha}}\left(\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],p}^{\beta}\,dt\right)^{\frac{1}{\beta}}\left[\frac{(b-x)^{\frac{\alpha}{q}+1}+(x-a)^{\frac{\alpha}{q}+1}}{(b-a)(q+\alpha)}\right]^{\frac{1}{\alpha}} \text{ if } f'\in L_p[a,b], \\ \qquad\qquad \text{and } \|f'\|_{[x,\cdot],p}\in L_\beta[a,b], \text{ where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]^{\frac{1}{q}}\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],p}\,dt \\ \qquad\qquad \text{if } f'\in L_p[a,b], \text{ and } \|f'\|_{[x,\cdot],p}\in L_1[a,b], \end{cases}$$

for each $x\in[a,b]$ and

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.3}$$
$$\leq \frac{1}{b-a}\int_a^b \|f'\|_{[x,t],1}\,dt$$
$$\leq \begin{cases} \frac{1}{2}\|f'\|_{[a,b],1}+\frac{1}{2}\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right| \text{ if } f'\in L_1[a,b]; \\ \left(\frac{1}{b-a}\int_a^b \|f'\|_{[x,t],1}^{\beta}\,dt\right)^{\frac{1}{\beta}} \\ \text{if } f'\in L_1[a,b],\ \|f'\|_{[x,.],1}\in L_\beta[a,b], \text{ where } \beta>1, \end{cases}$$

for each $x\in[a,b]$.

If we assume now that $f:[a,b]\to\mathbb{R}$ is absolutely continuous and such that $|f'|$ is convex on (a,b), then by Theorem 4.1 we obtain the following integral inequalities established in [1]

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{5.4}$$

$$\leq \frac{1}{2}\left[|f'(x)| \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a) + \frac{1}{b-a}\int_a^b |x-t|\,|f'(t)|\,dt \right]$$

$$\leq \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],\infty} + |f'(x)| \right]\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a) \text{ if } f' \in L_\infty[a,b]; \\ \frac{1}{2}\left\{ |f'(x)| \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a) \right. \\ \qquad \left. + \left[\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(\alpha+1)(b-a)} \right]^{\frac{1}{\alpha}} \left[\frac{1}{b-a}\int_a^b |f'(t)|^\beta\,dt \right]^{\frac{1}{\beta}} \right\} \\ \qquad \text{if } f' \in L_\beta[a,b],\ \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ \frac{1}{2}\left\{ |f'(x)| \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a) + \left[\frac{1}{2} + \frac{\left|x-\frac{a+b}{2}\right|}{b-a} \right] \int_a^b |f'(t)|\,dt \right\} \\ \qquad \text{if } f' \in L_1[a,b], \end{cases}$$

for each $x \in [a,b]$.

6. Some Discrete Inequalities

For a given interval $[a,b]$, consider the division

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and the intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. If $h_i := x_{i+1} - x_i > 0$ $\left(i = \overline{0, n-1}\right)$ we may define the following functionals

$$A(f; I_n, \xi) := \frac{1}{b-a}\sum_{i=0}^{n-1} f(\xi_i)\, h_i \qquad \text{(Riemann Rule)}$$

$$A_T(f; I_n) := \frac{1}{b-a}\sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}\cdot h_i \qquad \text{(Trapezoid Rule)}$$

$$A_M(f; I_n) := \frac{1}{b-a}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\cdot h_i \qquad \text{(Mid-point Rule)}$$

$$A_S(f; I_n) := \frac{1}{3}A_T(f; I_n) + \frac{2}{3}A_M(f; I_n)\,. \qquad \text{(Simpson Rule)}$$

We observe that, all the above functionals are obviously linear, isotonic and normalised.

Consequently, all the inequalities obtained in Sections 2 and 3 may be applied for these functionals.

If, for example, we use the following inequality (see Theorem 3.1)

$$|f(x) - A(f)| \leq \|f'\|_{[a,b]} A(|x-e|)\,, \quad x \in [a,b]\,, \tag{6.1}$$

provided $f:[a,b]\to\mathbb{R}$ is absolutely continuous and $f'\in L_{\infty}[a,b]$, then we get the inequalities

$$\left|f(x)-\frac{1}{b-a}\sum_{i=0}^{n-1}f(\xi_i)h_i\right|\le\|f'\|_{[a,b],\infty}\frac{1}{b-a}\sum_{i=0}^{n-1}|x-\xi_i|h_i, \tag{6.2}$$

$$\left|f(x)-\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i\right| \le\|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\sum_{i=0}^{n-1}\frac{|x-x_i|+|x-x_{i+1}|}{2}h_i, \tag{6.3}$$

$$\left|f(x)-\frac{1}{b-a}\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)\cdot h_i\right| \le\|f'\|_{[a,b],\infty}\frac{1}{b-a}\sum_{i=0}^{n-1}\left|x-\frac{x_i+x_{i+1}}{2}\right|h_i, \tag{6.4}$$

for each $x\in[a,b]$.

Similar results may be stated if one uses for example Theorem 4.1. We omit the details.

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