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INCLUSION THEOREMS FOR ABSOLUTE SUMMABILITY METHODS

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Abstract

Contents

◀◀   ▶▶

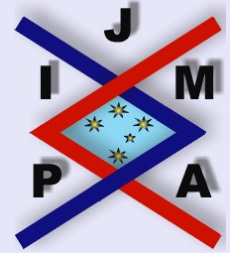◀   ▶

Home Page

Go Back

Close

Quit

Abstract

In this paper we have proved two theorems concerning an inclusion between two absolute summability methods by using any absolute summability factor.

2000 Mathematics Subject Classification: 40D15, 40F05, 40G99
Key words: Absolute summability, Summability factors, Infinite series.

Contents

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n), and $r_n = na_n$. By u_n and t_n we denote the n-th $(C,1)$ means of the sequences (s_n) and (r_n), respectively. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [4])

$$(1.1) \qquad \sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty.$$

But since $t_n = n(u_n - u_{n-1})$ (see [7]), the condition (1.1) can also be written as

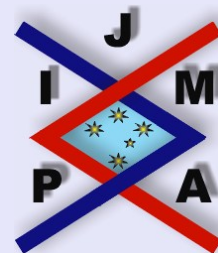$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty.$$

The series $\sum a_n$ is said to be summable $|C,1;\delta|_k$ $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$(1.3) \qquad \sum_{n=1}^{\infty} n^{\delta k-1} |t_n|^k < \infty.$$

If we take $\delta = 0$, then $|C,1;\delta|_k$ summability is the same as $|C,1|_k$ summability.

Let (p_n) be a sequence of positive numbers such that

$$(1.4) \qquad P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

$$(1.5) \qquad T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (T_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]).

The series $\sum a_n$ is said to be summable $\left| \bar{N}, p_n \right|_k$, $k \geq 1$, if (see [1])

$$(1.6) \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\Delta T_{n-1}|^k < \infty$$

and it is said to be summable $\left| \bar{N}, p_n; \delta \right|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$(1.7) \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |\Delta T_{n-1}|^k < \infty,$$

where

$$(1.8) \qquad \Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

In the special case when $\delta = 0$ (resp. $p_n = 1$ for all values of n) $\left| \bar{N}, p_n; \delta \right|_k$ summability is the same as $\left| \bar{N}, p_n \right|_k$ (resp. $|C, 1; \delta|_k$) summability.

Concerning inclusion relations between $|C, 1|_k$ and $\left| \bar{N}, p_n \right|_k$ summabilities, the following theorems are known.
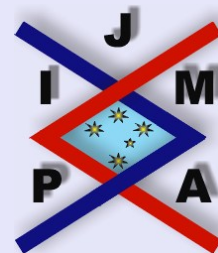
Title Page

Contents

◀◀　　▶▶

◀　　　▶

Go Back

Close

Quit

Theorem 1.1. *([1]).Let $k \geq 1$ and let (p_n) be a sequence of positive numbers such that as $n \to \infty$*

(1.9) $\qquad\qquad (i) \ \ P_n = O(np_n), \quad (ii) \ \ np_n = O(P_n).$

If the series $\sum a_n$ is summable $|C,1|_k$, then it is also summable $\left|\bar{N}, p_n\right|_k$.

Theorem 1.2. *([2]). Let $k \geq 1$ and let (p_n) be a sequence of positive numbers such that condition (1.9) of Theorem 1.1 is satisfied. If the series $\sum a_n$ is summable $\left|\bar{N}, p_n\right|_k$, then it is also summable $|C,1|_k$.*

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

J. Ineq. Pure and Appl. Math. **4**(4) Art. 66, 2003

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2. The Main Result

The aim of this paper is to generalize the above theorems for $|C, 1; \delta|_k$ and $|\bar{N}, p_n; \delta|_k$ summabilities, by using a summability factors. Now, we shall prove the following theorems.

Theorem 2.1. *Let $k \geq 1$ and $0 \leq \delta k < 1$. Let (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$ and*

$$(2.1) \qquad \sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{ \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v} \right\}.$$

Let $\sum a_n$ be summable $|C, 1; \delta|_k$. Then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$, if (λ_n) satisfies the following conditions:

$$(2.2) \qquad n\Delta\lambda_n = O\left(\frac{P_n}{np_n}\right)^{\frac{1-\delta k}{k}},$$

$$(2.3) \qquad \lambda_n = O\left(\frac{P_n}{np_n}\right)^{\frac{1}{k}}.$$

Theorem 2.2. *Let $k \geq 1$ and $0 \leq \delta k < 1$. Let (p_n) be a sequence of positive numbers such that $np_n = O(P_n)$ and satisfies the condition (2.1). Let $\sum a_n$ be summable $|\bar{N}, p_n; \delta|_k$. Then $\sum a_n \lambda_n$ is summable $|C, 1; \delta|_k$, if (λ_n) satisfies the following conditions:*

$$(2.4) \qquad n\Delta\lambda_n = O\left(\frac{np_n}{P_n}\right)^{\frac{1-\delta k}{k}}$$

J. Ineq. Pure and Appl. Math. **4**(4) Art. 66, 2003

http://jipam.vu.edu.au

$$(2.5) \qquad \lambda_n = O\left(\frac{np_n}{P_n}\right)^{\frac{1}{k}}.$$

Remark 2.1. *It may be noted that, if we take $\lambda_n = 1$ and $\delta = 0$ in Theorem 2.1 and Theorem 2.2, then we get Theorem 1.1 and Theorem 1.2, respectively. In this case condition (2.1) reduces to*

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O\left(\frac{1}{P_v}\right) \quad as \quad m \to \infty,$$

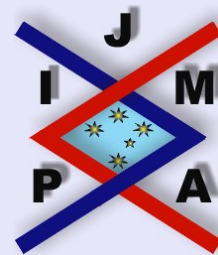*which always holds.*

Proof of Theorem 2.1. Since

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v$$

we have that

$$a_n = \frac{n+1}{n}t_n - t_{n-1}.$$

Let (T_n) denote the $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v \lambda_v.$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Close

Quit

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v.$$

By Abel's transformation, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v \frac{v+1}{v} t_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v \frac{v+1}{v} t_v$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{v} \lambda_{v+1} t_v + \frac{p_n}{P_n} \lambda_n \frac{n+1}{n} t_n$$

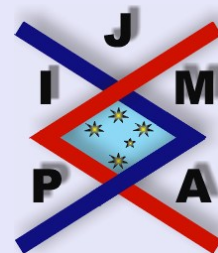$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say}.$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k),$$

to complete the proof of the theorem, it is enough to show that

$$(2.6) \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,r}|^k < \infty \quad \text{for} \quad r = 1, 2, 3, 4.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close
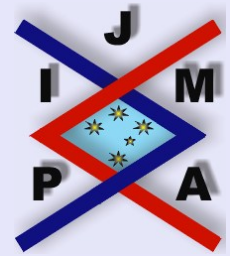
Quit

$\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,1}|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \frac{v+1}{v} |t_v| |\Delta \lambda_v|\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} |t_v| |v\Delta \lambda_v| \frac{P_v}{v p_v} p_v\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} |t_v| |v\Delta \lambda_v| p_v\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} |t_v|^k |v\Delta \lambda_v|^k p_v \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} |t_v|^k |v\Delta \lambda_v|^k p_v \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |t_v|^k |v\Delta \lambda_v|^k \left(\frac{P_v}{p_v}\right)^{\delta k - 1}$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k - 1} |t_v|^k = O(1) \quad \text{as} \quad m \to \infty.$$

by virtue of the hypotheses of Theorem 2.1.
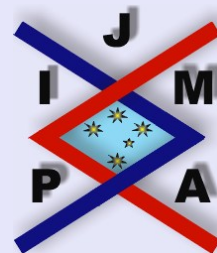
Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Again using Hölder's inequality,

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,2}|^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k - 1} |t_v|^k |\lambda_v|^k$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k - 1} |t_v|^k |\lambda_v|^k$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k - 1} |t_v|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.1.

Also

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,3}|^k$$

$$\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \frac{|\lambda_{v+1}|}{v} |t_v| \right)^k$$

J. Ineq. Pure and Appl. Math. **4**(4) Art. 66, 2003

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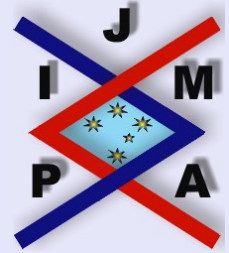
$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} v p_v \frac{|\lambda_{v+1}|}{v} |t_v|\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v |\lambda_{v+1}|^k |t_v|^k \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n} p_v\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} p_v |\lambda_{v+1}|^k |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |t_v|^k \left(\frac{P_v}{p_v}\right)^{\delta k - 1} |\lambda_{v+1}|^k$$

$$= O(1) \sum_{v=1}^{m} v^{\delta k - 1} |t_v|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.1.

Lastly

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |T_{n,4}|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n|^k |t_n|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{n p_n}\right)^{\delta k - 1} |\lambda_n|^k |t_n|^k n^{\delta k - 1}$$

$$= O(1) \sum_{n=1}^{m} n^{\delta k - 1} |t_n|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.1.

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n$. We have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v.$$

Since

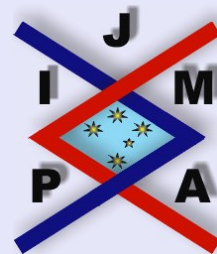$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n} P_{v-1} a_v,$$

we have that

(2.7) $$a_n = \frac{-P_n}{p_n} \Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}} \Delta T_{n-2}.$$

Let

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v.$$

By using (2.7) we get

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v \left(\frac{-P_v}{p_v} \Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}} \Delta T_{v-2} \right) \lambda_v$$

$$= \frac{1}{n+1} \sum_{v=1}^{n-1} (-v) \frac{P_v}{p_v} \Delta T_{v-1} \lambda_v - \frac{n P_n \lambda_n}{(n+1) p_n} \Delta T_{n-1}$$

$$+ \frac{1}{n+1} \sum_{v=1}^{n} v \frac{P_{v-2}}{p_{v-1}} \Delta T_{v-2} \lambda_v$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}(-v)\frac{P_v}{p_v}\Delta T_{v-1}\lambda_v$$

$$+ \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\frac{P_{v-1}}{p_v}\Delta T_{v-1}\lambda_{v+1} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}P_{v-1}\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}(P_v - p_v)\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}P_v - (v+1)\lambda_{v+1}p_v\}$$

$$- \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-(\Delta v\lambda_v)P_v - (v+1)\lambda_{v+1}p_v\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$

$$= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{-P_v}{p_v}\Delta T_{v-1}\{v\Delta\lambda_v - \lambda_{v+1}\} - \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta T_{v-1}(v+1)\lambda_{v+1}$$

$$- \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1}$$
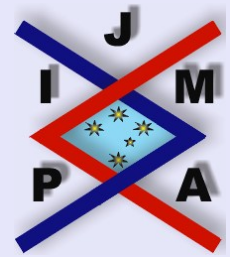
$$= -\frac{1}{n+1} \sum_{v=1}^{n-1} \frac{vP_v}{p_v} \Delta\lambda_v \Delta T_{v-1} + \frac{1}{n+1} \sum_{v=1}^{n-1} \frac{P_v}{p_v} \Delta T_{v-1} \lambda_{v+1}$$

$$- \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1)\lambda_{v+1} \Delta T_{v-1} - \frac{nP_n\lambda_n}{(n+1)p_n} \Delta T_{n-1}$$

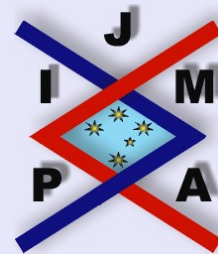$$= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}, \quad say.$$

Since

$$\left| t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4} \right|^k \le 4^k \left(\left| t_{n,1} \right|^k + \left| t_{n,2} \right|^k + |\ \left| t_{n,3} \right|^k + |t4|^k \right),$$

to complete the proof of Theorem 2.2, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k -1} \left| t_{n,r} \right|^k < \infty \quad for \quad r = 1,2,3,4.$$

Now, when $k > 1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1} n^{\delta k -1} \left| t_{n,1} \right|^k \le \sum_{n=2}^{m+1} n^{\delta k -1} \frac{1}{n^k} \left(\sum_{v=1}^{n-1} \frac{P_v}{p_v} v \left| \Delta\lambda_v \right| \left| \Delta T_{v-1} \right| \right)^k$$

$$\le \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k \left| \Delta\lambda_v \right|^k v^k \left| \Delta T_{v-1} \right|^k \left(\frac{1}{n} \sum_{v=1}^{n-1} 1 \right)^{k-1}$$

$$\le \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k \left| \Delta\lambda_v \right|^k v^k \left| \Delta T_{v-1} \right|^k$$

J. Ineq. Pure and Appl. Math. 4(4) Art. 66, 2003

http://jipam.vu.edu.au

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k |\Delta\lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k |\Delta\lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \frac{1}{v^{1-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k + k - 1} |\Delta T_{v-1}|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.2.
Again using Hölder's inequality,

$$\sum_{n=2}^{m+1} n^{\delta k - 1} |t_{n,2}|^k \le \sum_{n=2}^{m+1} n^{\delta k - 1 - k} \left(\sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{P_v}{p_v} |\Delta T_{v-1}|\right)^k$$

$$\le \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k |\lambda_{v+1}|^k |\Delta T_{v-1}|^k \left(\frac{1}{n} \sum_{v=1}^{n-1} 1\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k |\lambda_{v+1}|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k |\lambda_{v+1}|^k |\Delta T_{v-1}|^k \frac{1}{v^{1-\delta k}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k + k - 1} |\Delta T_{v-1}|^k$$
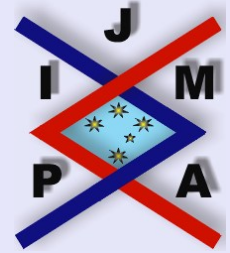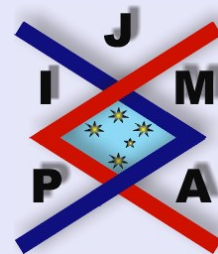
$$= O(1) \quad \text{as} \quad m \to \infty,$$

Also, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} \left|t_{n,3}\right|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1-k} \left(\sum_{v=1}^{n-1}(v+1)\left|\lambda_{v+1}\right|\left|\Delta T_{v-1}\right|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k-1-k} \left(\sum_{v=1}^{n-1} v\left|\lambda_{v+1}\right|\left|\Delta T_{v-1}\right|\right)^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k \left(\frac{1}{n}\sum_{v=1}^{n-1}1\right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k \\
&= O(1) \sum_{v=1}^{m} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k \sum_{n=v+1}^{m+1}\frac{1}{n^{2-\delta k}} \\
&= O(1) \sum_{v=1}^{m} v^{\delta k+k-1}\left|\Delta T_{v-1}\right|^k \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k+k-1}\left|\Delta T_{v-1}\right|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.2.

Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

Finally, we have that

$$\sum_{n=1}^{m} n^{\delta k-1} \left| t_{n,4} \right|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^k n^{\delta k-1} \left| \lambda_n \right|^k \left| \Delta T_{n-1} \right|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{\delta k+k-1} \left| \Delta T_{n-1} \right|^k$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 2.2.

Therefore, we get that

$$\sum_{n=1}^{m} n^{\delta k-1} \left| t_{n,r} \right|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2.2. $\qquad \square$
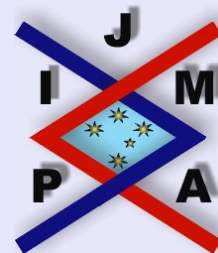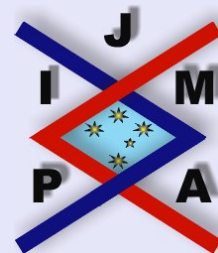
Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit

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Title Page

Contents

◀◀   ▶▶

◀   ▶

Go Back

Close

Quit