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INCLUSION THEOREMS FOR ABSOLUTE SUMMABILITY METHODS

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ABSTRACT. In this paper we have proved two theorems concerning an inclusion between two absolute summability methods by using any absolute summability factor.

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1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n), and $r_n = na_n$. By u_n and t_n we denote the n-th $(C,1)$ means of the sequences (s_n) and (r_n), respectively. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [7]), the condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|C,1;\delta|_k$ $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n|^k < \infty. \tag{1.3}$$

If we take $\delta = 0$, then $|C,1;\delta|_k$ summability is the same as $|C,1|_k$ summability.

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Let (p_n) be a sequence of positive numbers such that

(1.4) $$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

(1.5) $$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (T_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]).

The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

(1.6) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty$$

and it is said to be summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

(1.7) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\Delta T_{n-1}|^k < \infty,$$

where

(1.8) $$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

In the special case when $\delta = 0$ (resp. $p_n = 1$ for all values of n) $\left|\bar{N}, p_n; \delta\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ (resp. $|C, 1; \delta|_k$) summability.

Concerning inclusion relations between $|C, 1|_k$ and $\left|\bar{N}, p_n\right|_k$ summabilities, the following theorems are known.

Theorem 1.1. *([1]).Let $k \geq 1$ and let (p_n) be a sequence of positive numbers such that as $n \to \infty$*

(1.9) $$(i)\ P_n = O(np_n), \quad (ii)\ np_n = O(P_n).$$

If the series $\sum a_n$ is summable $|C, 1|_k$, then it is also summable $\left|\bar{N}, p_n\right|_k$.

Theorem 1.2. *([2]). Let $k \geq 1$ and let (p_n) be a sequence of positive numbers such that condition (1.9) of Theorem 1.1 is satisfied. If the series $\sum a_n$ is summable $\left|\bar{N}, p_n\right|_k$, then it is also summable $|C, 1|_k$.*

2. The Main Result

The aim of this paper is to generalize the above theorems for $|C, 1; \delta|_k$ and $\left|\bar{N}, p_n; \delta\right|_k$ summabilities, by using a summability factors. Now, we shall prove the following theorems.

Theorem 2.1. *Let $k \geq 1$ and $0 \leq \delta k < 1$. Let (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$ and*

(2.1) $$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\}.$$

Let $\sum a_n$ *be summable* $|C,1;\delta|_k$. *Then* $\sum a_n\lambda_n$ *is summable* $\left|\bar{N},p_n;\delta\right|_k$, *if* (λ_n) *satisfies the following conditions:*

$$n\Delta\lambda_n = O\left(\frac{P_n}{np_n}\right)^{\frac{1-\delta k}{k}}, \tag{2.2}$$

$$\lambda_n = O\left(\frac{P_n}{np_n}\right)^{\frac{1}{k}}. \tag{2.3}$$

Theorem 2.2. *Let* $k \geq 1$ *and* $0 \leq \delta k < 1$. *Let* (p_n) *be a sequence of positive numbers such that* $np_n = O(P_n)$ *and satisfies the condition (2.1). Let* $\sum a_n$ *be summable* $\left|\bar{N},p_n;\delta\right|_k$. *Then* $\sum a_n\lambda_n$ *is summable* $|C,1;\delta|_k$, *if* (λ_n) *satisfies the following conditions:*

$$n\Delta\lambda_n = O\left(\frac{np_n}{P_n}\right)^{\frac{1-\delta k}{k}} \tag{2.4}$$

$$\lambda_n = O\left(\frac{np_n}{P_n}\right)^{\frac{1}{k}}. \tag{2.5}$$

Remark 2.3. It may be noted that, if we take $\lambda_n = 1$ and $\delta = 0$ in Theorem 2.1 and Theorem 2.2, then we get Theorem 1.1 and Theorem 1.2, respectively. In this case condition (2.1) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_nP_{n-1}} = \sum_{n=v+1}^{m+1}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O\left(\frac{1}{P_v}\right) \quad \text{as} \quad m \to \infty,$$

which always holds.

Proof of Theorem 2.1. Since

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v$$

we have that

$$a_n = \frac{n+1}{n}t_n - t_{n-1}.$$

Let (T_n) denote the $(\bar{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_v\frac{v+1}{v}t_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} p_v\lambda_v\frac{v+1}{v}t_v \\ &\quad + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\lambda_{v+1}t_v + \frac{p_n}{P_n}\lambda_n\frac{n+1}{n}t_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.} \end{aligned}$$

Since

$$|T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}|^k \leq 4^k(|T_{n,1}|^k+|T_{n,2}|^k+|T_{n,3}|^k+|T_{n,4}|^k),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k<\infty \quad \text{for} \quad r=1,2,3,4. \tag{2.6}$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,1}|^k\\
&\quad\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\frac{v+1}{v}|t_v||\Delta\lambda_v|\right)^k\\
&\quad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}|t_v||v\Delta\lambda_v|\frac{P_v}{vp_v}p_v\right)^k\\
&\quad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}|t_v||v\Delta\lambda_v|p_v\right)^k\\
&\quad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}|t_v|^k|v\Delta\lambda_v|^kp_v\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&\quad= O(1)\sum_{v=1}^{m}|t_v|^k|v\Delta\lambda_v|^kp_v\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\\
&\quad= O(1)\sum_{v=1}^{m}|t_v|^k|v\Delta\lambda_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k-1}\\
&\quad= O(1)\sum_{v=1}^{m}v^{\delta k-1}|t_v|^k=O(1) \quad \text{as} \quad m\to\infty.
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1.

Again using Hölder's inequality,

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k\\
&\qquad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&\qquad= O(1)\sum_{v=1}^{m}p_v|t_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k|\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m} v^{\delta k-1}|t_v|^k|\lambda_v|^k\\
&= O(1)\sum_{v=1}^{m} v^{\delta k-1}|t_v|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1.

Also

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k\\
&\quad\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v\frac{|\lambda_{v+1}|}{v}|t_v|\right)^k\\
&\quad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left(\sum_{v=1}^{n-1}vp_v\frac{|\lambda_{v+1}|}{v}|t_v|\right)^k\\
&\quad= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|\lambda_{v+1}|^k|t_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n}p_v\right)^{k-1}\\
&\quad= O(1)\sum_{v=1}^{m}p_v|\lambda_{v+1}|^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\\
&\quad= O(1)\sum_{v=1}^{m}|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_{v+1}|^k\\
&\quad= O(1)\sum_{v=1}^{m}v^{\delta k-1}|t_v|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1.

Lastly

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{np_n}\right)^{\delta k-1}|\lambda_n|^k|t_n|^k n^{\delta k-1}\\
&= O(1)\sum_{n=1}^{m}n^{\delta k-1}|t_n|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2.1.

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n$. We have

$$
T_n = \frac{1}{P_n}\sum_{v=0}^{n}p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v.
$$

Since

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=0}^{n} P_{v-1} a_v,$$

we have that

$$a_n = \frac{-P_n}{p_n}\Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta T_{n-2}. \tag{2.7}$$

Let

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v \lambda_v.$$

By using (2.7) we get

$$\begin{aligned}
t_n &= \frac{1}{n+1}\sum_{v=1}^{n} v\left(\frac{-P_v}{p_v}\Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2}\right)\lambda_v \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}(-v)\frac{P_v}{p_v}\Delta T_{v-1}\lambda_v - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} + \frac{1}{n+1}\sum_{v=1}^{n} v\frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2}\lambda_v \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}(-v)\frac{P_v}{p_v}\Delta T_{v-1}\lambda_v + \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\frac{P_{v-1}}{p_v}\Delta T_{v-1}\lambda_{v+1} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}P_{v-1}\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}(P_v - p_v)\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-v\lambda_v P_v + (v+1)\lambda_{v+1}P_v - (v+1)\lambda_{v+1}p_v\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\{-(\Delta v\lambda_v)P_v - (v+1)\lambda_{v+1}p_v\} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{-P_v}{p_v}\Delta T_{v-1}\{v\Delta\lambda_v - \lambda_{v+1}\} - \frac{1}{n+1}\sum_{v=1}^{n-1}\Delta T_{v-1}(v+1)\lambda_{v+1} \\
&\qquad - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= -\frac{1}{n+1}\sum_{v=1}^{n-1}\frac{vP_v}{p_v}\Delta\lambda_v\Delta T_{v-1} + \frac{1}{n+1}\sum_{v=1}^{n-1}\frac{P_v}{p_v}\Delta T_{v-1}\lambda_{v+1} \\
&\qquad - \frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)\lambda_{v+1}\Delta T_{v-1} - \frac{nP_n\lambda_n}{(n+1)p_n}\Delta T_{n-1} \\
&= t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}, \quad say.
\end{aligned}$$

Since

$$|t_{n,1} + t_{n,2} + t_{n,3} + t_{n,4}|^k \le 4^k(|t_{n,1}|^k + |t_{n,2}|^k + |\ |t_{n,3}|^k + |t4|^k),$$

to complete the proof of Theorem 2.2, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_{n,r}|^k < \infty \quad for \quad r = 1,2,3,4.$$

Now, when $k > 1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} |t_{n,1}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1} \frac{1}{n^k} \left(\sum_{v=1}^{n-1} \frac{P_v}{p_v} v \, |\Delta \lambda_v| \, |\Delta T_{v-1}| \right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k |\Delta \lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \left(\frac{1}{n} \sum_{v=1}^{n-1} 1 \right)^{k-1} \\
&\leq \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k |\Delta \lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k |\Delta \lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2-\delta k}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k |\Delta \lambda_v|^k \, v^k \, |\Delta T_{v-1}|^k \frac{1}{v^{1-\delta k}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{\delta k+k-1} |\Delta T_{v-1}|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.2.
Again using Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} |t_{n,2}|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1-k} \left(\sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{P_v}{p_v} |\Delta T_{v-1}| \right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{1}{n^{2-\delta k}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k |\lambda_{v+1}|^k \, |\Delta T_{v-1}|^k \left(\frac{1}{n} \sum_{v=1}^{n-1} 1 \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k |\lambda_{v+1}|^k \, |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{2-\delta k}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k |\lambda_{v+1}|^k \, |\Delta T_{v-1}|^k \frac{1}{v^{1-\delta k}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right))^{\delta k+k-1} |\Delta T_{v-1}|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

Also, we have that

$$\begin{aligned}\sum_{n=2}^{m+1} n^{\delta k-1}\left|t_{n,3}\right|^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1-k}\left(\sum_{v=1}^{n-1}(v+1)\left|\lambda_{v+1}\right|\left|\Delta T_{v-1}\right|\right)^k \\ &= O(1)\sum_{n=2}^{m+1} n^{\delta k-1-k}\left(\sum_{v=1}^{n-1} v\left|\lambda_{v+1}\right|\left|\Delta T_{v-1}\right|\right)^k \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2-\delta k}}\sum_{v=1}^{n-1} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k\left(\frac{1}{n}\sum_{v=1}^{n-1}1\right)^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{2-\delta k}}\sum_{v=1}^{n-1} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k \\ &= O(1)\sum_{v=1}^{m} v^k\left|\lambda_{v+1}\right|^k\left|\Delta T_{v-1}\right|^k\sum_{n=v+1}^{m+1}\frac{1}{n^{2-\delta k}} \\ &= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}\left|\Delta T_{v-1}\right|^k \\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k+k-1}\left|\Delta T_{v-1}\right|^k \\ &= O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2.2.
Finally, we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{\delta k-1}\left|t_{n,4}\right|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^k n^{\delta k-1}\left|\lambda_n\right|^k\left|\Delta T_{n-1}\right|^k \\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\Delta T_{n-1}\right|^k \\ &= O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 2.2.

Therefore, we get that

$$\sum_{n=1}^{m} n^{\delta k-1}\left|t_{n,r}\right|^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r=1,2,3,4.$$

This completes the proof of Theorem 2.2. □

References

[1] H. BOR, On two summability methods, *Math. Proc. Cambridge Phil. Soc.*, **97** (1985), 147–149.

[2] H. BOR, A note on two summability methods, *Proc. Amer. Math. Soc.*, **98** (1986), 81–84.

[3] H. BOR, On the local property of $\left|\bar{N}, p_n; \delta\right|_k$ summability of factored Fourier series, *J. Math. Anal. Appl.*, **179** (1993), 646–649.

[4] T.M. FLETT, On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc.*, **7** (1957), 113–141.

[5] T.M. FLETT, Some more theorems concerning the absolute summability of Fourier series, *Proc. London Math. Soc.*, **8** (1958), 357–387.

[6] G.H. HARDY, *Divergent Series*, Oxford University Press., Oxford, (1949).

[7] E. KOGBETLIANTZ, Sur les series absolument sommables par la methode des moyannes aritmetiques, *Bull. Sci. Math.*, **49**(2) (1925), 234–256.