Journal of Inequalities in Pure and Applied Mathematics

AN ERROR ESTIMATE FOR FINITE VOLUME METHODS FOR THE STOKES EQUATIONS

A. ALAMI-IDRISSI AND M. ATOUNTI

Université Mohammed V-Agdal
Faculté des Sciences
Dépt de Mathématiques & Informatique
Avenue Ibn Batouta
BP 1014 Rabat 10000, Morocco
EMail: alidal@fsr.ac.ma

EMail: atounti@hotmail.com

ISSN (electronic): 1443-5756
045-01

volume 3, issue 2, article 17, 2002.

Received 18 May, 2001; accepted 8 November, 2001.

Communicated by: R.N. Mohapatra

Abstract

In the present paper, we study an error estimate for finite volume methods for the stokes equations. The error is proven to be of order h, in H_0^1-norm discrete and in L^2-norm, where h represents the size of the mesh. The result is new even for the finite volume method.

2000 Mathematics Subject Classification: 35Q30, 76D07.
Key words: Finite volume method, Stokes equations, Discrete Poincaré inequality, Error estimate.

Contents

1. Introduction

The numerical solution of the Navier-Stokes equations for incompressible viscous fluids has motivated many authors, so much so that giving a complete bibliography has become an impossible task. Therefore, we restrict our attention only to crucial contributions making use of finite element approximations and mixed finite element methods, among them we mention [2, 3, 6, 8, 12, 13, 14, 15, 16, 17, 18] (see also the references therein).

The finite volume element method is used in [9], the basic idea is based on the Box method. From the Crouzeix-Raviart element, the authors constructed the mesh of this method since every triangulation is associated to the spaces of finite elements. Later on, they applied the Babuska theorem to the Stokes problem, thus they obtained an analysis of error.

The finite volume projection method for the numerical approximation of two-dimensional incompressible flows on triangular unstructured grids is presented in [4]. The authors considered the unsteady Navier-Stokes equations, the velocity field is approximated by either piecewise constant or piecewise linear functions on the triangles, and the pressure field is approximated by piecewise linear functions. For the discretization of the diffusive flows, a dual grid connecting the centers of the triangles of the primary grid is introduced there. Using this grid, a stable and accurate discrete Laplacian is obtained.

The finite volume scheme for the Stokes problem is obtained from a mixed finite element method with a well chosen numerical integration diagonalizing the mass matrix which is used in [1]. The analysis of the corresponding finite volume scheme is directly deduced from general results of mixed finite element theory and the authors gave an optimal a priori error estimate.

The finite volume method on unstructured staggered grids for the Stokes problem is presented in [10]. The authors used an admissible mesh of triangles satisfying the properties required for the finite element method. In the case of acute angles, they proved the existence and the uniqueness of the solution, therefore, if the mesh consist of equilateral triangles, the authors obtained the convergence result.

In this paper, we are interested in the study of an error estimate for finite volume method for the Stokes equations in dimension $d = 2$ or 3, on unstructured staggered grids. The main difficulty of this problem is due to the coupling of the velocity with the pressure. For this reason, we use the Galerkin expansion for the approximation of the pressure such that the pressure unknowns are located at the vertices. The existence and the uniqueness of the solution results are proved by Eymard, Gallouet and Herbin in [10]. We prove here that the error estimate is of order one.

This paper is organized as follows: In Section 2, we introduce the continuous Stokes equations under some assumptions. In Section 3, we get the numerical scheme and the main results of the existence and the uniqueness of the numerical solution. Finally, in Section 4, we present the error estimate for the velocity.

2. The Continuous Equations

We consider here the Stokes problem:

$$(2.1)\qquad -\nu\Delta u^i(x)+\frac{\partial p}{\partial x_i}(x) = f^i(x) \qquad \forall x\in\Omega\ ,\forall i=1,\dots,d,$$

$$(2.2)\qquad \sum_{i=1}^{d}\frac{\partial u^i}{\partial x_i} = 0 \qquad \forall x\in\Omega,$$

with Dirichlet boundary condition:

$$(2.3)\qquad u^i(x)=0 \qquad \forall x\in\partial\Omega\ \ ,\forall i=1,\dots,d,$$

under the following assumption.

Assumption 1. *(i)* Ω *is an open bounded connected polygonal subset of* $\mathbb{R}^d$,$d=2,3.$

(ii) $\nu>0.$

(iii) $f^i\in L^2(\Omega)$; $\forall i=1,...,d.$
In the above equation, u^i *represents the* i^{th} *component of the velocity of a fluid,* ν *the kinematic viscosity and* p *the pressure. There exist several convenient mathematical formulations of (2.1) – (2.3).*

3. A Finite Volume Scheme on Unstructured Staggered Grids

The finite volume scheme is found by integrating equation (2.1) on a control volume of a discretization mesh and finding an approximation of the fluxes on the control volume boundary in terms of the discrete unknowns. Let us first give the assumptions which are needed on the mesh.

Definition 3.1. *Admissible mesh.*
Let Ω, be an open bounded polygonal subset of $\mathbb{R}^d$, ($d = 2$ or 3). An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of control volumes, which are open polygonal convex subsets of $\bar{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$, (they are the edges (2D), or sides (3D) of the control volumes), with strictly positive $(d-1)$-dimensional measure and a family of points of Ω denoted by $\mathcal{P}$ satisfying the following properties:

(i) The closure of the union of all the control volumes is $\bar{\Omega}$.

(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \backslash K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$, let $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.

(iii) For any $(K, L) \in \mathcal{T}^2$, with $K \neq L$, either the d-dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$.

(iv) The family $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$ is such that $x_K \in \bar{K}$ and if $\sigma = K|L$ it is assumed that $x_k \neq x_L$ and that the straight line $\mathcal{D}_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.

(v) *For any $\sigma \in \mathcal{E}$ such that $\sigma \in \partial\Omega$, let K be the control volume such that $\sigma \in \mathcal{E}_K$, if $x_K \notin \sigma$, let $\mathcal{D}_{K,\sigma}$ be the straight line going through x_K and orthogonal to σ. Then the condition $\mathcal{D}_{K,\sigma} \cap \sigma \neq \emptyset$ is assumed, let $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma$.*

In the sequel, the following notations are used:

- $size(\mathcal{T})$ =sup $\{diam(K), K \in \mathcal{T}\}$.
- $m(K)$ the $d-$dimensional Lebesgue of K, for any $K \in \mathcal{T}$.
- $m(\sigma)$ the $(d-1)$-dimensional Lebesgue of σ, for any $\sigma \in \mathcal{E}$.
- $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}, \sigma \not\subset \partial\Omega\}$ and $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$.
- If $\sigma \in \mathcal{E}_{int}$, $\sigma = K|L$ then $d_\sigma = d_{K|L} = d(x_K, x_L)$ and if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}$ then $d_\sigma = d_{K,\sigma} = d(x_K, y_\sigma)$.
- For any $\sigma \in \mathcal{E}$ the transmissibility through σ is defined by $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$ if $d_\sigma \neq 0$ and $\tau_\sigma = 0$ if $d_\sigma = 0$.

In some results and proofs given below, there are summations over $\sigma \in \mathcal{E}_0$ with $\mathcal{E}_0 = \{\sigma \in \mathcal{E}; d_\sigma \neq 0\}$. For simplicity $\mathcal{E}_0 = \mathcal{E}$ is assumed.

Let us now introduce the space of piecewise constant functions associated with an admissible mesh and discrete H_0^1-norm for this space. This discrete norm will be used to obtain an estimate of the approximate solution given by a finite volume scheme.

Definition 3.2. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $(d = 2, 3)$ and $\mathcal{T}$ be an admissible mesh. Define $X(\mathcal{T})$ to be the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.*

Definition 3.3. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $(d = 2, 3)$ and $\mathcal{T}$ be an admissible mesh. For $u \in X(\mathcal{T})$, define the discrete H_0^1-norm by:*

$$\|u\|_{1,\mathcal{T}} = \left(\sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u)^2 \right)^{\frac{1}{2}},$$

where:

$$\begin{aligned} D_\sigma u &= |u_K - u_L| if \sigma \in \mathcal{E}_{int}, \sigma = K|L. \\ D_\sigma u &= |u_K| if \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K \end{aligned}$$

and u_K denotes the value taken by u on the control volume K.

Lemma 3.1 (Discrete Poincaré inequality). *Let Ω be an open bounded polygonal subset $\mathbb{R}^d$, $(d = 2, 3)$, $\mathcal{T}$ be an admissible mesh and $u \in X(\mathcal{T})$, then:*

$$\|u\|_{L^2} \leq diam(\Omega) \|u\|_{1,\mathcal{T}},$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete H_0^1-norm.

Proof. See [10, p. 38, 11]. □

Assume K and L to be two neighboring control volumes of the mesh. A consistent discretization of the normal flux $- \bigtriangledown u.n$ over the interface of two control volumes K and L may be performed with differential quotient involving values of the unknown located on the orthogonal line to the interface between K and L, on either side of this interface.

In [10], the authors consider the mesh of Ω, denoted by $\mathcal{T}$, consisting of triangles, satisfying the properties required for the finite element method, see [7], with acute angles only, and defining, for all $K \in \mathcal{T}$, the point x_K as the intersection of the orthogonal bisectors of the sides of the triangles K yields that $\mathcal{T}$ is an admissible mesh. For $s \in S_{\mathcal{T}}$, let ϕ_s be the shape function associated to s in P_1. A possible finite volume scheme using a Galerkin expansion for the pressure is defined by the following equations:

$$\nu \sum_{\sigma \in \mathcal{E}_K} F^i_{K,\sigma} + \sum_{s \in S_K} p_s \int_K \frac{\partial \phi_s}{\partial x_i}(x)dx = m(K) f^i_K \quad \forall K \in \mathcal{T} \ , \forall i = 1, ..., d, \tag{3.1}$$

$$F^i_{K,\sigma} = \tau_\sigma (u^i_K - u^i_L), \qquad \text{if } \sigma \in \mathcal{E}_{int} \ , \sigma = K|L \ , i = 1, ..., d, \tag{3.2}$$

$$F^i_{K,\sigma} = \tau_\sigma u^i_K, \qquad \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K \ , i = 1, ..., d, \tag{3.3}$$

$$\sum_{K \in \mathcal{T}} \sum_{i}^{d} u^i_K \int_K \frac{\partial \phi_s}{\partial x_i}(x)dx = 0 \quad \forall s \in S_{\mathcal{T}}, \tag{3.4}$$

$$\int_\Omega \sum_{s \in S_{\mathcal{T}}} p_s \phi_s(x)dx = 0, \quad \text{and} \tag{3.5}$$

$$f^i_K = \frac{1}{m(K)} \int_K f^i(x)dx \quad , \forall K \in \mathcal{T}. \tag{3.6}$$

The discrete unknowns of (3.1) – (3.6) are $u_K^i, K \in \mathcal{T}\ , \forall i = 1, \ldots, d$, and $p_s,\ s \in S_{\mathcal{T}}$.
The approximate solutions are defined by:

$$u_K^i(x) = u_K^i \qquad a.e\ x \in K\ , \forall K \in \mathcal{T}\ , \forall i = 1, ..., d \tag{3.7}$$

and

$$p_{\mathcal{T}} = \sum_{s \in S_{\mathcal{T}}} p_s \phi_s. \tag{3.8}$$

The existence and the uniqueness of the solution of the discrete problem (3.1) – (3.6) are proved by Eymard, Gallouet and Herbin in [10]. Moreover, if the element of $\mathcal{T}$ are equilateral triangles then they obtained the following convergence result.

Proposition 3.2. *Under Assumption 1, there exists an unique solution to (3.1) – (3.6), denoted by $\{u_K^i,\ K \in \mathcal{T},\ i = 1, ..., d\}$ and $\{p_s,\ s \in S_{\mathcal{T}}\}$. Furthermore, if the elements of $\mathcal{T}$ are equilateral triangle, then $u_{\mathcal{T}} \longrightarrow u$, as $size(\mathcal{T}) \to 0$, where u is the unique solution to (2.1) – (2.3) and $u_{\mathcal{T}} = (u_{\mathcal{T}}^1, ..., u_{\mathcal{T}}^d)$ is defined by (3.7).*

Proof. See [10, p. 205]. ∎

4. Error Estimate

In this section, we present the error estimate theorem that is of order one.

Theorem 4.1. *Under Assumption 1, let $\mathcal{T}$ be an admissible mesh as given by Definition 3.1 and $u^i_{\mathcal{T}} \in X(\mathcal{T})$, $\forall i = 1, ..., d$, such that $u^i_{\mathcal{T}} = u^i_K$, $\forall i = 1, ..., d$ for a.e. $x \in K$, for all $K \in \mathcal{T}$ where $(u^i_K)_{K \in \mathcal{T}}$ is the solution to (3.1) – (3.6). Let $u = (u^i)$ be the unique variational solution of problem (2.1) – (2.3) and for each $K \in \mathcal{T}$, $e^i_K = u^i(x_K) - u^i_K$, and $e^i_{\mathcal{T}} \in \mathcal{T}$ defined by $e^i_{\mathcal{T}}(x) = e^i_K$ for a.e. $x \in K$, for all $K \in \mathcal{T}$. Then there exists $C > 0$ depending only on u, Ω and d such that:*

$$\|e^i_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C size(\mathcal{T}) \tag{4.1}$$

and

$$\|e^i_{\mathcal{T}}\|_{L^2} \leq diam(\Omega) C size(\mathcal{T}), \tag{4.2}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete H^1_0-norm.

Proof. Integrating over K the equation (2.1), then:

$$-\nu \int_{\partial K} \nabla u^i \cdot \overrightarrow{n}_{\partial K} d\sigma_{\partial K} + \int_K \frac{\partial p}{\partial x_i}(x) dx = \int_K f^i(x) \quad \forall i = 1, \dots, d. \tag{4.3}$$

As

$$\int_{\partial K} \nabla u^i \cdot \overrightarrow{n}_{\partial K} d\sigma_{\partial K} = \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u^i \cdot \overrightarrow{n}_\sigma d\sigma \quad \forall i = 1, \dots, d.$$

We denote by:

$$\overline{F}^{i}_{K,\sigma} = -\int_{\sigma} \nabla u^{i} \cdot \overrightarrow{n}_{\sigma} d\sigma \quad \forall i = 1, \dots, d,$$

then:

$$\nu \sum_{\sigma \in \mathcal{E}_K} \overline{F}^{i}_{K,\sigma} + \int_K \frac{\partial p}{\partial x_i}(x)dx = \int_K f^{i}(x) \quad \forall i = 1, \dots, d. \tag{4.4}$$

Let $F^{*,i}_{K,\sigma}$ be defined by:

$$\begin{aligned} F^{*,i}_{K,\sigma} &= \tau_\sigma(u^i(x_K) - u^i(x_L)) \quad \text{if } \sigma \in \mathcal{E}_{int}\ , \sigma = K|L; i = 1, \dots, d, \\ F^{*,i}_{K,\sigma} &= \tau_\sigma u^i(x_K) \quad \text{if } \sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K\ , i = 1, \dots, d, \end{aligned}$$

then the consistency error may be defined as:

$$\overline{F}^{i}_{K,\sigma} - F^{*,i}_{K,\sigma} = m(\sigma) R^{i}_{K,\sigma} \quad \forall i = 1, \dots, d.$$

Thanks to the regularity of u, there exists $C_1 \in \mathbb{R}$, only depending on u, such that:

$$\left| R^{i}_{K,\sigma} \right| \le C_1 size(\mathcal{T}) \quad \forall K \in \mathcal{T} \text{ and } \sigma \in \mathcal{E}_K \quad \forall i = 1, \dots, d. \tag{4.5}$$

If $\sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K\ , \sigma = K|L$, then, we have:

$$\begin{aligned} \overline{F}^{i}_{K,\sigma} - F^{i}_{K,\sigma} &= \overline{F}^{i}_{K,\sigma} - F^{*,i}_{K,\sigma} + F^{*,i}_{K,\sigma} - F^{i}_{K,\sigma} \\ &= m(\sigma) R^{i}_{K,\sigma} + F^{*,i}_{K,\sigma} - F^{i}_{K,\sigma} \\ &= m(\sigma) R^{i}_{K,\sigma} + \tau_\sigma(e^{i}_{K} - e^{i}_{L}), \end{aligned} \tag{4.6}$$

and if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$, then, we have:

$$\overline{F}^i_{K,\sigma} - F^i_{K,\sigma} = m(\sigma) R^i_{K,\sigma} + \tau_\sigma e^i_K. \tag{4.7}$$

Subtracting (3.1) from (4.3) then:

$$\nu \sum_{\sigma \in \mathcal{E}_K} \left(\overline{F}^i_{K,\sigma} - F^i_{K,\sigma} \right) + \int_K \frac{\partial p}{\partial x_i}(x) dx - \sum_{s \in S_K} p_s \int_K \frac{\partial \phi_s}{\partial x_i}(x) dx = \int_K f^i(x) - m(K) f^i_{K,\sigma}. \tag{4.8}$$

Multiplying (4.8) by e^i_K, summing for $K \in \mathcal{T}$ and i, then we obtain:

$$\begin{aligned} &\sum_i^d \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \left(\overline{F}^i_{K,\sigma} - F^i_{K,\sigma} \right) e^i_K \\ &\quad = \sum_i^d \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (e^i_K - e^i_L) e^i_K + \sum_i^d \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) R^i_{K,\sigma} e^i_K \\ &\quad = \sum_i^d \sum_{\sigma \in \mathcal{E}} \tau_\sigma |D_\sigma e^i_{\mathcal{T}}|^2 + \sum_i^d \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) R^i_{K,\sigma} e^i_K. \end{aligned} \tag{4.9}$$

Using $div(u) = 0$ and the relation (3.4), we deduce that:

$$\sum_i^d \sum_{K \in \mathcal{T}} \left(\int_K \frac{\partial p}{\partial x_i}(x) dx - \sum_{s \in S_K} p_s \int_K \frac{\partial \phi_s}{\partial x_i}(x) dx \right) e^i_K = 0. \tag{4.10}$$

From the relation (3.6), then:

$$\sum_{i}^{d}\sum_{K\in\mathcal{T}}\left(\int_K f^i(x)-m(K)f^i_{K,\sigma}\right)e^i_K=0. \tag{4.11}$$

Replacing (4.9), (4.10), (4.11) in (4.8), hence:

$$\sum_{i}^{d}\sum_{\sigma\in\mathcal{E}}\tau_\sigma|D_\sigma e^i_K|^2=-\sum_{i}^{d}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)R^i_{K,\sigma}e^i_K,$$

then:

$$\sum_{i}^{d}\|e^i_{\mathcal{T}}\|^2_{1,\mathcal{T}}=-\sum_{i}^{d}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)R^i_{K,\sigma}e^i_K. \tag{4.12}$$

Thanks to the propriety of conservativity, one has $R^i_{K,\sigma}=-R^i_{L,\sigma}$ for $\sigma\in\mathcal{E}_{int}$, such that $\sigma=K|L$, let $R^i_\sigma=|R^i_{K,\sigma}|$.
Reordering the summation over the edges and using the Cauchy-Schwarz inequality, one obtains:

$$\begin{aligned}\left|\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)R^i_{K,\sigma}e^i_K\right| &\leq \sum_{\sigma\in\mathcal{E}_K}m(\sigma)|R^i_\sigma||D_\sigma e^i_{\mathcal{T}}| \\ &\leq \left(\sum_{\sigma\in\mathcal{E}_K}\frac{m(\sigma)}{d_\sigma}|D_\sigma e^i_{\mathcal{T}}|^2\right)^{\frac{1}{2}}\left(\sum_{\sigma\in\mathcal{E}}m(\sigma)d_\sigma|R^i_\sigma|^2\right)^{\frac{1}{2}}.\end{aligned}$$

From the relation (4.5), we have $|R^i_\sigma| \le C_1 size(\mathcal{T})$ and we remark that $\sum\limits_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma = m(\Omega)$, then we deduce the existence of C_2, only depending on u and Ω, such that:

$$\left| \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m(\sigma) R^i_{K,\sigma} e^i_K \right| \le C_2 \|e^i_{\mathcal{T}}\|_{1,\mathcal{T}} size(\mathcal{T}). \tag{4.13}$$

Then:

$$\sum_{i=1}^{d} \|e^i_{\mathcal{T}}\|^2_{1,\mathcal{T}} \le C_2 \left(\sum_{i=1}^{d} \|e^i_{\mathcal{T}}\|_{1,\mathcal{T}} \right) size(\mathcal{T}). \tag{4.14}$$

Using Young's inequality, there exists C_3 only depending on u, Ω and d, such that:

$$\left(\sum_{i=1}^{d} \|e^i_{\mathcal{T}}\|^2_{1,\mathcal{T}} \right)^{\frac{1}{2}} \le C_3 size(\mathcal{T}). \tag{4.15}$$

We have:

$$\|e^i_{\mathcal{T}}\|_{1,\mathcal{T}} \le \left(\sum_{i=1}^{d} \|e^i_{\mathcal{T}}\|^2_{1,\mathcal{T}} \right)^{\frac{1}{2}} \le C_3 size(\mathcal{T}) \quad \forall i = 1,\dots,d. \tag{4.16}$$

Applying the discrete Poincaré inequality, we obtain the relation (4.2). □

References

[1] A. AGOUZAL AND F. OUDIN, Finite volume scheme for Stokes problem, Publication interne, *Equipe d'analyse numérique*, Lyon Saint-Etienne, **262** (1997).

[2] J.P. BENQUE, B. IBLER AND G. LABADIE, A finite element method for Navier-Stokes equations, Numerical Methods for nonlinear problem, *Proceedings of the international conference held at University College Swansca*, **1** (1980), 709–720.

[3] P.B. BOCHEV, Analysis of least-squares finite element methods for the Navier-Stokes equations, *SIAM J. Numer. Anal.*, **34**(5) (1997), 1817–1844.

[4] N. BOTTA AND D. HEMPEL, A finite volume projection method for the numerical solution of the incompressible Navier-Stokes equations on triangular grids, in F. Benkhaldoun and R.Vilsmeier eds, *Finite Volumes for Complex Applications, Problems and Perspectives*, Hermes, Paris, (1996), 355–363.

[5] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, (1991).

[6] M.O. BRISTEAN, R. GLOWINSKY, B. MANTEL, J. PERIAUX, P. PERRIER AND O. PIRONNEAU, A finite element approximation of Navier-Stokes for incompressible viscous fluids. Iterative methods of solution, Lecture Notes in Mathematics **771**, *Approximation Methods for Navier-Stokes*, Springer-Verlag, (1980), 78–128.

[7] P.G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, (1978).

[8] C. CONCA, Approximation de quelques problèmes de type Stokes par une méthode d'éléments finis mixtes, *Cours of University Paris VI and National Centre of Scientific Search*.

[9] P. EMONOT, Méthodes de volumes éléments finis: Applications aux équations de Navier-Stokes et résultats de convergence, Thèse, (1992), Université Claude Bernard Lyon I.

[10] R. EYMARD, T. GALLOUET AND R. HERBIN, Finite Volume Methods, *Handbook of Numerical Analysis*, Ph. Ciarlet et Lions eds, Prepublication 1997-19.

[11] T. GALLOUET, *An Introduction to Finite Volume Methods*, Course CEA/EDF/INRIA, (1992).

[12] V. GIRAULT AND P.A. RAVIART, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, (1986)

[13] J.L. GUERMOND AND L. QUARTAPELLE, On the approximation of the unsteady Navier-Stokes equations by finite element projection methods, *Numer. Math.*, **80** (1998), 207–238.

[14] J.G. HEYWOOD, Classical solutions of the Navier-Stokes equations, Lecture Notes in Mathematics **771**, *Approximation methods for Navier-Stokes*, Springer-Verlag, (1980), 235–248.

[15] O. PIRONNEAU, *Méthodes des Éléments Finis Pour les Fluides*, Masson, (1988).

[16] R. RANNACHER, On the finite element approximation of the nonstationary Navier-Stokes problem, Lecture Notes in Mathematics **771**, *Approximation Methods for Navier-Stokes*, Springer-Verlag, (1980), 408–424.

[17] R. RAUTMANN, On the convergence rate of nonstationary Navier-Stokes approximations, Lecture Notes in Mathematics **771**, *Approximation Methods for Navier-Stokes*, Springer-Verlag, (1980), 425–449.

[18] R. TEMAM, *Navier-Stokes Equations*, North-Holland, (1977).