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A POLYNOMIAL INEQUALITY GENERALISING AN INTEGER INEQUALITY

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Abstract

For any $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in (\mathbb{R}^+)^n$, we establish inequalities between the two homogeneous polynomials $\Delta P_{\mathbf{a}}(x,t) := (x + a_1 t)(x + a_2 t)\cdots(x + a_n t) - x^n$ and $S_{\mathbf{a}}(x,y) := a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1}$ in the positive orthant $x, y, t \in \mathbb{R}^+$. Conditions for $\Delta P_{\mathbf{a}}(x,t) \le t S_{\mathbf{a}}(x,y)$ yield a new proof and broad generalization of the number theoretic inequality that for base $b \ge 2$ the sum of all nonempty products of digits of any $m \in \mathbb{Z}^+$ never exceeds m, and equality holds exactly when all auxiliary digits are $b - 1$. Links with an inequality of Bernoulli are also noted. When $n \ge 2$ and $\mathbf{a}$ is strictly positive, the surface $\Delta P_{\mathbf{a}}(x,t) = t S_{\mathbf{a}}(x,y)$ lies between the planes $y = x + t \max\{a_i : 1 \le i \le n-1\}$ and $y = x + t \min\{a_i : 1 \le i \le n-1\}$. For fixed $t \in \mathbb{R}^+$, we explicitly determine functions $\alpha, \beta, \gamma, \delta$ of $\mathbf{a}$ such that this surface is $y = x + \alpha t + \beta t^2 x^{-1} + O(x^{-2})$ as $x \to \infty$, and $y = \gamma t + \delta x + O(x^2)$ as $x \to 0+$.

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Contents

1. Introduction

For any finite sequence of real numbers $\mathbf{a}$, let $\Pi\mathbf{a}$ be the product of all terms in $\mathbf{a}$, and let $T(\mathbf{a})$, the *total sum of products* of $\mathbf{a}$, be the sum of all products $\Pi\mathbf{x}$ as $\mathbf{x}$ runs through the nonempty subsequences $\mathbf{x} \subseteq \mathbf{a}$. Thus

$$T(\mathbf{a}) := \Sigma\{\Pi\mathbf{x} : \mathbf{x} \subseteq \mathbf{a}, \mathbf{x} \neq \omega\},$$

where ω is the empty sequence. As usual we observe the convention that $\Pi\omega = 1$. There is a rather surprising inequality which $T(\mathbf{a})$ must satisfy in the case of integer sequences. In particular, for given integers $b \geq 2$ and $m \geq 0$, let $\mathbf{a}$ be the sequence of digits in the base b representation of m. Then

$$T(\mathbf{a}) \leq m$$

holds for every such integer m and base b, as shown in [2]. Moreover the inequality is sharp: $T(\mathbf{a}) = m$ holds precisely when the auxiliary digits of m, if any, are all $b - 1$. (The *leading* digit of n is the most significant digit; the less significant digits, if any, are its *auxiliary* digits.) For example

$$T(3, 7, 7) = 255 \leq 377_{(b)},$$

where $377_{(b)}$ is the base b representation of $m = 255, 313, 377, 447, \ldots$ when $b = 8, 9, 10, 11, \ldots$. We also note in passing that if $\mathbf{a}$ is the base b digit sequence of m then $T(\mathbf{a})$ is odd precisely when at least one of the digits of m is odd.

Our main purpose in this paper is to show that the integer inequality just described is an instance of a much more general inequality between polynomials. We shall establish the polynomial inequality and investigate some of its properties.

2. Polynomial Inequality

Let $\mathbf{a}$ be any nonempty finite sequence of real numbers, say

$$\mathbf{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n, \text{ with } n \geq 1.$$

With $\mathbf{a}$ we associate two homogeneous polynomials in two real variables, the *product* polynomial

$$P_{\mathbf{a}}(x,t) := (x + a_1 t)(x + a_2 t) \cdots (x + a_n t) = \prod_{r=1}^{n} (x + a_r t),$$

and the *sum* polynomial

$$S_{\mathbf{a}}(x,y) := a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1} = \sum_{r=1}^{n} a_r x^{n-r} y^{r-1}.$$

Here we shall study these polynomials when $\mathbf{a} \in (\mathbb{R}^+)^n$, where $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$. It turns out that it is natural to compare t times the sum polynomial with the first difference of the product polynomial,

$$\Delta P_{\mathbf{a}}(x,t) := P_{\mathbf{a}}(x,t) - P_{\mathbf{a}}(x,0) = P_{\mathbf{a}}(x,t) - x^n.$$

Note that $tS_{\mathbf{a}}(x,y)$ and $\Delta P_{\mathbf{a}}(x,t)$ are both homogeneous of degree n.

With $\mathbf{a}$ we also associate two bounds when $n \geq 2$:

$$M(\mathbf{a}) := \max\{a_r : 1 \leq r \leq n-1\}$$
$$\text{and} \quad m(\mathbf{a}) := \min\{a_r : 1 \leq r \leq n-1\}.$$

Theorem 2.1. *For any finite nonnegative sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \geq 1$*, the inequality*

$$0 \leq \Delta P_{\mathbf{a}}(x,t) \leq tS_{\mathbf{a}}(x,y)$$

holds for all $x, y, t \in \mathbb{R}^+$*, provided* $y \geq x + tM(\mathbf{a})$ *if* $n \geq 2$*. The reverse inequality*

$$\Delta P_{\mathbf{a}}(x,t) \geq tS_{\mathbf{a}}(x,y) \geq 0$$

holds for all $x, y, t \in \mathbb{R}^+$*, provided* $y \leq x + tm(\mathbf{a})$ *if* $n \geq 2$.

Proof. An easy induction on n establishes the identity

$$P_{\mathbf{a}}(x,t) = \prod_{r=1}^{n}(x + a_r t) = x^n + \sum_{r=1}^{n} a_r x^{n-r} t \prod_{s=1}^{r-1}(x + a_s t).$$

For $x, t \in \mathbb{R}^+$ we have $x + a_s t \geq 0$ for each s, so

$$0 \leq \prod_{s=1}^{r-1}(x + a_s t) \leq y^{r-1}$$

holds trivially if $r = 1$, and for $r \geq 2$ it certainly holds if

$$y \geq \max\{x + a_s t : 1 \leq s \leq r-1\} = x + t \cdot \max\{a_s : 1 \leq s \leq r-1\}.$$

Because each $a_r \in \mathbb{R}^+$, it follows for $x, t \in \mathbb{R}^+$ that

$$0 \leq \Delta P_{\mathbf{a}}(x,t) = P_{\mathbf{a}}(x,t) - x^n$$

$$= t\sum_{r=1}^{n} a_r x^{n-r}\prod_{s=1}^{r-1}(x + a_s t)$$
$$\le t\sum_{r=1}^{n} a_r x^{n-r} y^{r-1} = tS_{\mathbf{a}}(x,y)$$

holds trivially if $n = 1$, and for $n \ge 2$ it holds if $y \ge x + tM(\mathbf{a})$. An entirely similar argument establishes the reverse inequality in the theorem. □

Let us define

$$\Sigma(\mathbf{a}) := \sum_{r=1}^{n} a_r.$$

If $\mathbf{a} \in (\mathbb{R}^+)^n$ and $n \ge 2$ then

$$0 \le m(\mathbf{a}) \le M(\mathbf{a}) \le \Sigma(\mathbf{a}).$$

Note that $S_{\mathbf{a}}(1,1) = \Sigma(\mathbf{a})$. This constant plays a natural role in bounding our polynomial inequalities away from zero. Specifically, we have

Corollary 2.2. *Let $\mathbf{a} \in (\mathbb{R}^+)^n$ be a finite nonnegative sequence with $n \ge 3$ and $M(\mathbf{a}) > m(\mathbf{a})$. Then for all strictly positive $x, y, t \in \mathbb{R}^+$ the inequality*

$$0 < t\Sigma(\mathbf{a})x^{n-1} < \Delta P_{\mathbf{a}}(x,t) < tS_{\mathbf{a}}(x,y)$$

holds provided $y \ge x + tM(\mathbf{a})$, and the reverse inequality

$$\Delta P_{\mathbf{a}}(x,t) > tS_{\mathbf{a}}(x,y) \ge t\Sigma(\mathbf{a})z^{n-1} > 0$$

holds provided $y \le x + tm(\mathbf{a})$, with $z := min\{x,y\}$.

Proof. We sharpen the details of the proof of Theorem 2.1. The condition $M(\mathbf{a}) > m(\mathbf{a})$ ensures that $M(\mathbf{a}) > 0$, so if x, t are strictly positive reals then $x + a_s t > x$ for at least one $s \le n-1$, and

$$\prod_{s=1}^{r-1}(x + a_s t) > x^{r-1}$$

holds for some $r \le n$. Then

$$\begin{aligned}\Delta P_{\mathbf{a}}(x,t) &= P_{\mathbf{a}}(x,t) - x^n \\ &= \sum_{r=1}^{n} a_r x^{n-r} t \prod_{s=1}^{r-1}(x + a_s t) \\ &> \sum_{r=1}^{n} a_r x^{n-1} t = t\Sigma(\mathbf{a})x^{n-1} > 0.\end{aligned}$$

If $y \ge x + tM(\mathbf{a})$, then $M(\mathbf{a}) > m(\mathbf{a})$ ensures that

$$\prod_{s=1}^{r-1}(x + a_s t) < y^{r-1}$$

holds for at least one $r \le n$, so

$$\Delta P_{\mathbf{a}}(x,t) = P_{\mathbf{a}}(x,t) - x^n < t \sum_{r=1}^{n} a_r x^{n-r} y^{r-1} = tS_{\mathbf{a}}(x,y).$$

For the second inequality, if $0 < y \le x + tm(\mathbf{a})$ then $M(\mathbf{a}) > m(\mathbf{a})$ ensures that

$$\prod_{s=1}^{r-1}(x + a_s t) > y^{r-1}$$

holds for at least one $r \le n$, so

$$\begin{aligned}\Delta P_{\mathbf{a}}(x,t) &= P_{\mathbf{a}}(x,t) - x^n \\ &> t\sum_{r=1}^{n} a_r x^{n-r} y^{r-1} \\ &= tS_{\mathbf{a}}(x,y) \\ &\ge t\sum_{r=1}^{n} a_r z^{n-1} = t\Sigma(\mathbf{a})z^{n-1} > 0,\end{aligned}$$

where $z := \min\{x, y\}$. □

Corollary 2.3. *For any real c and given finite sequence $\mathbf{a} \in \mathbb{R}^n$, if $n = 1$ or if $n \ge 2$ and $M(\mathbf{a}) = m(\mathbf{a}) = c$, then*

$$\Delta P_{\mathbf{a}}(x,t) = tS_{\mathbf{a}}(x, x + ct)$$

is an identity for all $x, t \in \mathbb{R}$.

Proof. First suppose $\mathbf{a} \in (\mathbb{R}^+)^n$ and $c, x, t \in \mathbb{R}^+$. If $n = 1$ both inequalities in Theorem 2.1 hold, so $\Delta P_{\mathbf{a}}(x,t) = tS_{\mathbf{a}}(x, x + ct)$. The same result holds if $n \ge 2$ when $M(\mathbf{a}) = m(\mathbf{a}) = c$ and $y = x + ct$. Since we have a degree n polynomial equality which holds for more than n values of x and more than n

values of t, it must in fact be a polynomial identity, and therefore holds for all $x, t \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$ with $M(\mathbf{a}) = m(\mathbf{a})$. □

We shall now show that the integer inequality proved in [2], and the conditions under which it is an equality, are directly deducible from the above results. Thus Theorem 2.1 provides a new proof of the results in [2] as well as placing them in a much more general context.

Corollary 2.4. *For any integers $b \geq 2$ and $m \geq 0$, let $\mathbf{a} \in (\mathbb{Z}^+)^n$ be the sequence of base b digits of m. Then the total sum of products of these digits satisfies $T(\mathbf{a}) \leq m$, with equality precisely when every auxiliary digit of m is $b-1$.*

Proof. Assume that the base b digits of m are arranged in $\mathbf{a}$ in order of increasing significance, so a_n is the leading digit. Then $S_{\mathbf{a}}(1, b) = m$. Furthermore $M(\mathbf{a}) \leq b - 1$. Put $x = 1, t = 1$ and $y = b$. Then $y \geq x + tM(\mathbf{a})$, so the first inequality in Theorem 2.1 yields

$$T(\mathbf{a}) = P_{\mathbf{a}}(1,1) - 1 = \Delta P_{\mathbf{a}}(1,1) \leq S_{\mathbf{a}}(1,b) = m,$$

as required. Now consider when equality holds. By Corollary 2.2, the strict inequality $T(\mathbf{a}) < m$ holds if $n \geq 3$ and the auxiliary digits are not all equal, so suppose $n \geq 2$ and all auxiliary digits are equal to $M(\mathbf{a})$. Corollary 2.3 shows that $T(\mathbf{a}) = m^*$, where $m^* = S_{\mathbf{a}}(1, M(\mathbf{a}) + 1)$ is the integer with base $M(\mathbf{a}) + 1$ digit sequence $\mathbf{a}$ if we permit the slightly nonstandard possibility that the leading digit may exceed $M(\mathbf{a})$. Thus $m^* = m$ if $M(\mathbf{a}) = b - 1$, and $m^* < m$ if $M(\mathbf{a}) < b - 1$. If $n = 1$, Corollary 2.3 confirms the already obvious $T(\mathbf{a}) = m$. □

We now note some examples of Theorem 2.1.

Example 2.1. *With* $t = 1$, $\mathbf{a} = (a, b, c, d) \in (\mathbb{R}^+)^4$*, and the change of variables* $x \leftarrow t, y \leftarrow x$ *with* $x, t \in \mathbb{R}^+$*, we have*

$$(t + a)(t + b)(t + c)(t + d) - t^4 \le at^3 + bt^2x + ctx^2 + dx^3$$

when $x \ge t + \max\{a, b, c\}$*. The reverse inequality holds when* $x \le t + \min\{a, b, c\}$.

Example 2.2. *With* $t = 1$, $\mathbf{a} = (d, c, b, a) \in (\mathbb{R}^+)^4$*, and the change of variables* $x \leftarrow t, y \leftarrow x$ *with* $x, t \in \mathbb{R}^+$*, we have*

$$(t + a)(t + b)(t + c)(t + d) - t^4 \le ax^3 + btx^2 + ct^2x + dt^3$$

when $x \ge t + \max\{b, c, d\}$*. The reverse inequality holds when* $x \le t + \min\{b, c, d\}$.

Example 2.3. *In Example 2.2, let* $t = 1$ *and replace* (a, b, c, d) *in that example with* (a, bt, ct^2, dt^3)*, where* a, b, c, d, t *are strictly positive. Then*

$$(1 + a)(1 + bt)(1 + ct^2)(1 + dt^3) - 1 \le ax^3 + btx^2 + ct^2x + dt^3$$

when $x \ge 1 + \max\{bt, ct^2, dt^3\}$.

Example 2.4. *Replace* (a, b, c, d) *in Example 2.2 by* $(a, bt^{-1}, ct^{-2}, dt^{-3})$*, so*

$$(t + a)(t^2 + b)(t^3 + c)(t^4 + d) - t^{10} \le t^6(ax^3 + bx^2 + cx + d)$$

when $x \ge t + \max\{bt^{-1}, ct^{-2}, dt^{-3}\}$.

Example 2.5. *Evidently*

$$\Delta P_{\mathbf{a}}(1,1) \geq S_{\mathbf{a}}(1,1) = \Sigma(\mathbf{a})$$

holds for any $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \geq 1$*, and holds with strict inequality if* $n \geq 2$ *and* $\mathbf{a}$ *has at least two strictly positive terms. However, it is interesting to note that it also holds with strict inequality for any* $\mathbf{a} \in (-1,0)^n$ *with* $n \geq 2$*, a result which goes back to Jacques [= James = Jakob] Bernoulli (1654-1705) in the case where the sequence* $\mathbf{a}$ *is constant (see [1, Theorem 58]). Our focus in the present paper is on cases in which* $\mathbf{a} \in (\mathbb{R}^+)^n$.

The *reverse* of a given finite sequence $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with $n \geq 1$ is the sequence $\mathbf{a}^R := (a_n, \ldots, a_2, a_1) \in \mathbb{R}^n$. Then

$$P_{\mathbf{a}^R}(x,t) = P_{\mathbf{a}}(x,t) \quad \text{and} \quad S_{\mathbf{a}^R}(x,y) = S_{\mathbf{a}}(y,x).$$

Let $\max(\mathbf{a}) := \max\{a_r : 1 \leq r \leq n\}$ and $\min(\mathbf{a}) := \min\{a_r : 1 \leq r \leq n\}$. If $n \geq 2$ we have

$$\max\{M(\mathbf{a}), M(\mathbf{a}^R)\} = \max(\mathbf{a}) \quad \text{and} \quad \min\{m(\mathbf{a}), m(\mathbf{a}^R)\} = \min(\mathbf{a}).$$

With these observations, combining the principles used in Examples 2.1 and 2.2 readily yields

Corollary 2.5. *For any finite nonnegative sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \geq 1$*, the inequality*

$$0 \leq \Delta P_{\mathbf{a}}(t,1) \leq \min\{S_{\mathbf{a}}(t,x), S_{\mathbf{a}}(x,t)\}$$

holds for all $x, t \in \mathbb{R}^+$*, provided* $x \geq t + \max(\mathbf{a})$ *if* $n \geq 2$*. The reverse inequality*

$$\Delta P_{\mathbf{a}}(t, 1) \geq \max\{S_{\mathbf{a}}(t, x), S_{\mathbf{a}}(x, t)\} \geq 0$$

holds for all $x, t \in \mathbb{R}^+$*, provided* $x \leq t + \min(\mathbf{a})$ *if* $n \geq 2$.

3. Conditions for Equality to Hold

When does the inequality studied in Theorem 2.1 become an equality? To reduce this to a problem in two variables, let us examine the $t = 1$ cross-section. Suppose $n \geq 2$ and $\mathbf{a} \in (\mathbb{R}^+)^n$ is strictly positive, that is, $a_r > 0$ for $1 \leq r \leq n$. We have from Theorem 2.1:

$$\Delta P_{\mathbf{a}}(x,1) \begin{cases} \leq S_{\mathbf{a}}(x,y) & \text{when } y \geq x + M(\mathbf{a}), \\ \geq S_{\mathbf{a}}(x,y) & \text{when } y \leq x + m(\mathbf{a}). \end{cases}$$

If x, y are strictly positive, then

$$\frac{\partial}{\partial y} S_{\mathbf{a}}(x,y) > 0,$$

and continuity of $S_{\mathbf{a}}(x,y)$ as a function of y ensures the following result:

Lemma 3.1. *For any strictly positive $x \in \mathbb{R}^+$ and any strictly positive sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$, there is a unique $y_0 > 0$ such that*

$$\Delta P_{\mathbf{a}}(x,1) \begin{cases} < & S_{\mathbf{a}}(x,y) & \text{if } y > y_0, \\ = & S_{\mathbf{a}}(x,y_0) & \\ > & S_{\mathbf{a}}(x,y) & \text{if } 0 < y < y_0. \end{cases}$$

Furthermore

$$x + m(\mathbf{a}) \leq y_0 \leq x + M(\mathbf{a}).$$

In what follows we shall determine y_0 more explicitly. It is convenient to introduce some notation. Let $\Sigma_k(\mathbf{a})$ be the kth *elementary symmetric function* of the sequence $\mathbf{a}$, defined to be the sum of products $\Pi\mathbf{x}$ as $\mathbf{x}$ runs through all the k-term subsequences $\mathbf{x} \subseteq \mathbf{a}$. Thus

$$\Sigma_k(\mathbf{a}) := \Sigma\{\Pi\mathbf{x} : \mathbf{x} \subseteq \mathbf{a}, |\mathbf{x}| = k\}.$$

In particular $\Sigma_1(\mathbf{a}) = \Sigma(\mathbf{a}) = \Sigma_{r=1}^{n} a_r$ and $\Sigma_2(\mathbf{a}) = \Sigma_{r=1}^{n-1}\Sigma_{s=r+1}^{n} a_r a_s$. Again let

$$W_k(\mathbf{a}) := \sum_{r=1}^{n} \binom{r-1}{k-1} a_r.$$

We call $W_k(\mathbf{a})$ the kth *binomially-weighted sum* of the sequence $\mathbf{a}$. Note that $W_1(\mathbf{a}) = \Sigma_1(\mathbf{a})$.

Lemma 3.2. *For any finite strictly positive sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *and any positive integer* $k \leq n$, *we have*

$$\frac{\min(\mathbf{a})^k}{\max(\mathbf{a})} \leq \frac{\Sigma_k(\mathbf{a})}{W_k(\mathbf{a})} \leq \frac{\max(\mathbf{a})^k}{\min(\mathbf{a})},$$

with strict inequalities when $\mathbf{a}$ *is not constant.*

Proof. Let $\mathbf{a}^* \in (\mathbb{R}^+)^n$ be the constant sequence with every term equal to $\max(\mathbf{a})$. Then

$$\Sigma_k(\mathbf{a}) \leq \Sigma_k(\mathbf{a}^*) = \binom{n}{k} \max(\mathbf{a})^k,$$

and the inequality is strict when $\mathbf{a}$ is not constant. Also

$$W_k(\mathbf{a}) = \sum_{r=1}^{n} \binom{r-1}{k-1} a_r \geq \sum_{r=1}^{n} \binom{r-1}{k-1} \min(\mathbf{a}) = \binom{n}{k} \min(\mathbf{a}) > 0,$$

so

$$\frac{\Sigma_k(\mathbf{a})}{W_k(\mathbf{a})} \le \frac{\max(\mathbf{a})^k}{\min(\mathbf{a})},$$

with strict inequality when $\mathbf{a}$ is not constant. An entirely similar argument establishes the other inequality in the lemma. □

For any real $c > 0$, if $\mathbf{c} \in (\mathbb{R}^+)^n$ is the constant sequence with every term equal to c, then Lemma 3.2 shows that $\Sigma_k(\mathbf{c})/W_k(\mathbf{c}) = c^{k-1}$. Hence $(\Sigma_k(\mathbf{a})/W_k(\mathbf{a}))^{\frac{1}{k-1}}$ is a measure of central tendency for the terms of the sequence $\mathbf{a} \in (\mathbb{R}^+)^n$, for each integer k in the interval $2 \le k \le n$. The case $k = 2$ enters into the asymptotic behaviour of y_0, as we now show.

Theorem 3.3. *For strictly positive $x, y \in \mathbb{R}^+$ and any strictly positive sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \ge 2$, the equality $\Delta P_{\mathbf{a}}(x, 1) = S_{\mathbf{a}}(x, y)$ holds for large x when*

$$y = x + \alpha + O(x^{-1}) \qquad (x \to \infty),$$

where

$$\alpha := \frac{\Sigma_2(\mathbf{a})}{W_2(\mathbf{a})}.$$

Proof. Let $y_0 = x + f_0(x)$, so $\Delta P_{\mathbf{a}}(x, 1) = S_{\mathbf{a}}(x, x + f_0(x))$. Then $m(\mathbf{a}) \le f_0(x) \le M(\mathbf{a})$ by Lemma 3.1, so $O(f_0(x)) = O(1)$ as $x \to \infty$. Hence

$$\begin{aligned} S_{\mathbf{a}}(x, x + f_0(x)) &= \sum_{r=1}^{n} a_r x^{n-r}(x + f_0(x))^{r-1} \\ &= \left(\sum_{r=1}^{n} a_r\right) x^{n-1} + \left(\sum_{r=1}^{n} (r-1) a_r\right) f_0(x) x^{n-2} + O(x^{n-3}) \end{aligned}$$

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$$= \Sigma_1(\mathbf{a})x^{n-1} + W_2(\mathbf{a})f_0(x)x^{n-2} + O(x^{n-3}).$$

Also

$$\begin{aligned}\Delta P_{\mathbf{a}}(x,1) &= (x+a_1)(x+a_2)\cdots(x+a_n) - x^n \\ &= \Sigma_1(\mathbf{a})x^{n-1} + \Sigma_2(\mathbf{a})x^{n-2} + O(x^{n-3}).\end{aligned}$$

But these two expressions are equal, so for large x it follows that

$$f_0(x) = \frac{\Sigma_2(\mathbf{a})}{W_2(\mathbf{a})} + O(x^{-1}).$$

□

By Theorem 3.3, if we put $y_0 = x + \alpha + f_1(x)$ then $O(f_1(x)) = O(x^{-1})$ as $x \to \infty$. Explicit expansion of $\Delta P_{\mathbf{a}}(x,1)$ and $S_{\mathbf{a}}(x, x + \alpha + f_1(x))$ as far as terms in x^{n-3} yields

Corollary 3.4. *For any finite strictly positive sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \geq 3$, *the equality* $\Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x,y)$ *holds for large* $x, y \in \mathbb{R}^+$ *when*

$$y = x + \alpha + \beta x^{-1} + O(x^{-2}) \qquad (x \to \infty),$$

where

$$\alpha := \frac{\Sigma_2(\mathbf{a})}{W_2(\mathbf{a})} \quad \textit{and} \quad \beta := \frac{\Sigma_3(\mathbf{a}) - \alpha^2 W_3(\mathbf{a})}{W_2(\mathbf{a})}.$$

From Lemma 3.1 we immediately deduce

Corollary 3.5. *If* $M(\mathbf{a}) = m(\mathbf{a}) = c$, *then* $\alpha = c$ *and* $\beta = 0$.

Next we shall consider y_0 when x is small but positive. It will be convenient to use $G(\mathbf{a})$ to denote the geometric mean of $\{a_r : 1 \leq r \leq n-1\}$, so $G(\mathbf{a}) := (a_1 a_2 \cdots a_{n-1})^{\frac{1}{n-1}}$. For any finite strictly positive sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ we define $\mathbf{a}^{-1} := (a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$, so $\Sigma_1(\mathbf{a}^{-1})$ is the sum of reciprocals of the terms of $\mathbf{a}$. Of course, $\Sigma_1(\mathbf{a}^{-1}) = \Sigma_{n-1}(\mathbf{a})/\Sigma_n(\mathbf{a})$. This sum enters into the small scale behaviour of y_0, as we now show.

Theorem 3.6. *For strictly positive $x, y \in \mathbb{R}^+$ and any strictly positive sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$, the equality $\Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x,y)$ holds for small x when*

$$y = \gamma + \delta x + O(x^2) \qquad (x \to 0+),$$

where

$$\gamma := G(\mathbf{a}) \quad \text{and} \quad \delta := \frac{\gamma a_n \Sigma_1(\mathbf{a}^{-1}) - a_{n-1}}{(n-1)a_n}.$$

Proof. For $0 < x < M(\mathbf{a})$ let $y_0 = g_0(x)$, so $\Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x, g_0(x))$. Lemma 3.1 ensures that $m(\mathbf{a}) < g_0(x) < 2M(\mathbf{a})$, so $O(g_0(x)) = O(1)$ as $x \to 0+$. Then

$$S_{\mathbf{a}}(x, g_0(x)) = \sum_{r=1}^{n} a_{n-r+1} x^{r-1} g_0(x)^{n-r} = a_n g_0(x)^{n-1} + O(x)$$

and

$$\Delta P_{\mathbf{a}}(x,1) = (a_1 a_2 \cdots a_n) + O(x),$$

so equality of these expressions implies that

$$g_0(x) = G(\mathbf{a}) + O(x).$$

Now let $y_0 = G(\mathbf{a}) + g_1(x)$, so $O(g_1(x)) = O(x)$ as $x \to 0+$. Then

$$\begin{aligned} S_{\mathbf{a}}(x, G(\mathbf{a}) + g_1(x)) \\ &= \sum_{r=1}^{n} a_{n-r+1} x^{r-1} (G(\mathbf{a}) + g_1(x))^{n-r} \\ &= a_n G(\mathbf{a})^{n-1} + (n-1) a_n G(\mathbf{a})^{n-2} g_1(x) + a_{n-1} x G(\mathbf{a})^{n-2} + O(x^2) \end{aligned}$$

and

$$\Delta P_{\mathbf{a}}(x, 1) = (a_1 a_2 \cdots a_n) \left\{ 1 + \left(\sum_{r=1}^{n} a_r^{-1} \right) x + O(x^2) \right\}.$$

Equality of these two expressions implies that

$$g_1(x) = \frac{(a_n G(\mathbf{a}) \Sigma_1(\mathbf{a}^{-1}) - a_{n-1})\, x}{(n-1) a_n} + O(x^2),$$

and the theorem follows. □

From Lemma 3.1 we deduce

Corollary 3.7. *If $M(\mathbf{a}) = m(\mathbf{a}) = c$, then $\gamma = c$ and $\delta = 1$.*

Let us now consider the geometry underlying Theorems 3.3 and 3.6. The positive quadrant $x, y \in \mathbb{R}^+$ is divided into an "S-region", where

$$\Delta P_{\mathbf{a}}(x, 1) < S_{\mathbf{a}}(x, y),$$

and a "ΔP-region", where

$$\Delta P_{\mathbf{a}}(x, 1) > S_{\mathbf{a}}(x, y).$$

The boundary between these two regions is

$$E_1(\mathbf{a}) := \{(x, y) \in (\mathbb{R}^+)^2 : \Delta P_{\mathbf{a}}(x, 1) = S_{\mathbf{a}}(x, y)\}.$$

On this boundary curve the polynomials $\Delta P_{\mathbf{a}}(x, 1)$ and $S_{\mathbf{a}}(x, y)$ are equal, so we call $E_1(\mathbf{a})$ the *equipoise curve* for $\mathbf{a}$. Lemma 3.1 ensures that $E_1(\mathbf{a})$ lies in the strip between the parallel lines $y = x + M(\mathbf{a})$ and $y = x + m(\mathbf{a})$. By Theorem 3.3 the equipoise curve is asymptotic to $y = x + \alpha$, and by Theorem 3.6 it cuts the y-axis at $y = G(\mathbf{a})$, with slope δ.

When $n = 2$, we have $M(\mathbf{a}) = m(\mathbf{a}) = \alpha = G(\mathbf{a}) = a_1$ and $E_1(\mathbf{a})$ is the line $y = x + a_1$. When $n \geq 3$, as $x \to \infty$ the equipoise curve approaches the asymptote from the S-region side if $\beta > 0$, and from the ΔP-region side if $\beta < 0$.

It appears likely that the equipoise curve never crosses the asymptote, though we were not able to demonstrate this in general. The condition for such a crossing to occur is a polynomial of degree $n - 3$ in x, so such crossings are possible only when $n \geq 4$. However it seems unlikely that there are ever any solutions with $x > 0$. When $n = 3$, it is clear that $E_1(\mathbf{a})$ must be entirely on one side of the asymptote unless $\alpha^2 = a_1 a_2$. In the latter case, $\beta = 0$ and $E_1(\mathbf{a})$ actually coincides with the asymptote; this behaviour is demonstrated by $E_1(1, 4, 4)$ for example.

Throughout the preceding discussion in this section we have been comparing $\Delta P_{\mathbf{a}}(x, 1)$ with $S_{\mathbf{a}}(x, y)$ in the positive x, y-quadrant. A simple observation enables us to deduce the corresponding information comparing $\Delta P_{\mathbf{a}}(x, t)$ with $tS_{\mathbf{a}}(x, y)$ in the positive x, y, t-orthant. For any $t \in \mathbb{R}^+$ and $\mathbf{a} \in (\mathbb{R}^+)^n$, let $t\mathbf{a} := (ta_1, ta_2, \ldots, ta_n) \in (\mathbb{R}^+)^n$. Then

$$P_{t\mathbf{a}}(x, 1) = P_{\mathbf{a}}(x, t) \quad \text{and} \quad S_{t\mathbf{a}}(x, y) = tS_{\mathbf{a}}(x, y),$$

so all the relevant facts about $\Delta P_{\mathbf{a}}(x,t) = tS_{\mathbf{a}}(x,y)$ follow from our earlier results in this section by replacing $\mathbf{a}$ by $t\mathbf{a}$. In particular, the *equipoise surface*

$$E_2(\mathbf{a}) := \{(x,y,t) \in (\mathbb{R}^+)^3 : \Delta P_{\mathbf{a}}(x,t) = tS_{\mathbf{a}}(x,y)\}$$

lies in the region between the planes $y = x + tM(\mathbf{a})$ and $y = x + tm(\mathbf{a})$, which coincide if $M(\mathbf{a}) = m(\mathbf{a})$, and otherwise intersect in the line $y = x, t = 0$. For any fixed $t > 0$ the equipoise surface satisfies

$$y = x + \alpha t + \beta t^2 x^{-1} + O(x^{-2}) \qquad (x \to \infty)$$

and

$$y = \gamma t + \delta x + O(x^2) \qquad (x \to 0+).$$

However, the device of replacing $\mathbf{a}$ by $t\mathbf{a}$ does not provide any information about the comparison of the product and sum polynomials for a general finite sequence $\mathbf{a} \in \mathbb{R}^n$. As hinted at by Bernoulli's Inequality, mentioned in Example 2.5, there is potentially much of interest in this more general case.

References

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