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HYERS-ULAM-RASSIAS STABILITY OF THE K-QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper we obtain the Hyers-Ulam-Rassias stability for the functional equation

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = f(x)+f(y), \quad x,y\in G,$$

where K is a finite cyclic transformation group of the abelian group $(G,+)$, acting by automorphisms of G. As a consequence we can derive the Hyers-Ulam-Rassias stability of the quadratic and the additive functional equations.

Key words and phrases: Group, Additive equation, Quadratic equation, Hyers-Ulam-Rassias stability.

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1. INTRODUCTION

In the book, *A Collection of Mathematical Problems* [32], S.M. Ulam posed the question of the stability of the Cauchy functional equation. Ulam asked: if we replace a given functional equation by a functional inequality, when can we assert that the solutions of the inequality lie near to the solutions of the strict equation?

Originally, he had proposed the following more specific question during a lecture given before the University of Wisconsin's Mathematics Club in 1940.

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> Given a group G_1, a metric group (G_2, d), a number $\varepsilon > 0$ and a mapping $f : G_1 \longrightarrow G_2$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, does there exist an homomorphism $h : G_1 \longrightarrow G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that $d(f(x), h(x)) \leq k\varepsilon$ for all x in G_1?

A partial and significant affirmative answer was given by D.H. Hyers [9] under the condition that G_1 and G_2 are Banach spaces.

In 1978, Th. M. Rassias [18] provided a generalization of Hyers's stability theorem which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. *Let $f : V \longrightarrow X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$), then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed x, then T is linear.

During the 27th International Symposium on functional equations, Th. M. Rassias asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [7] following the same approach as in [18], gave an affirmative answer to Rassias' question for $p > 1$. However, it was showed that a similar result for the case $p = 1$ does not hold.

In 1994, P. Gavrută [8] provided a generalization of the above theorem by replacing the function $(x, y) \longmapsto \theta(\|x\|^p + \|y\|^p)$ with a mapping $\varphi(x, y)$ which satisfies the following condition:

$$\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \qquad \text{or} \qquad \sum_{n=0}^{\infty} 2^n \varphi\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) < \infty$$

for every x, y in a Banach space V.

Since then, a number of stability results have been obtained for functional equations of the forms

$$f(x+y) = g(x) + h(y), \qquad x, y \in G, \tag{1.1}$$

and

$$f(x+y) + f(x-y) = g(x) + h(y), \qquad x, y \in G, \tag{1.2}$$

where G is an abelian group. In particular, for the classical equations of Cauchy and Jensen, the quadratic and the Pexider equations, the reader can be referred to [4] – [22] for a comprehensive account of the subject.

In the papers [24] – [31], H. Stetkær studied functional equations related to the action by automorphisms on a group G of a compact transformation group K. Writing the action of $k \in K$ on $x \in G$ as $k \cdot x$ and letting dk denote the normalized Haar measure on K, the functional equations (1.1) and (1.2) have the form

$$\int_K f(x + k \cdot y)dk = g(x) + h(y), \qquad x, y \in G, \tag{1.3}$$

where $K = \{I\}$ and $K = \{I, -I\}$, respectively, I denoting the identity.

The purpose of this paper is to investigate the Hyers-Ulam-Rassias stability of

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = f(x)+f(y), \qquad x,y\in G, \tag{1.4}$$

where K is a finite cyclic subgroup of $Aut(G)$ (the group of automorphisms of G), $|K|$ denotes the order of K, and G is an abelian group.

The set up allows us to give a unified treatment of the stability of the additive functional equation

$$f(x+y) = f(x)+f(y), \qquad x,y\in G, \tag{1.5}$$

and the quadratic functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y), \qquad x,y\in G. \tag{1.6}$$

In particular, we want to see how the compact subgroup K enters into the solutions formulas.

The stability problem for the quadratic equation (1.6) was first solved by Skof in [23]. In [4] Cholewa extended Skof's result in the following way, where G is an abelian group and E is a Banach space.

Theorem 1.2. *Let $\eta > 0$ be a real number and $f : G \longrightarrow E$ satisfies the inequality*

$$|f(x+y)+f(x-y)-2f(x)-2f(y)| \le \eta \quad \textit{for all} \;\; x,y\in G. \tag{1.7}$$

Then for every $x\in G$ the limit $q(x) = \lim_{n\longrightarrow+\infty}\frac{f(2^n x)}{2^{2n}}$ exists and $q : G \longrightarrow E$ is the unique quadratic function satisfying

$$|f(x)-q(x)| \le \frac{\eta}{2}, \quad x\in G. \tag{1.8}$$

In [5] Czerwik obtained a generalization of the Skof-Chelewa result.

Theorem 1.3. *Let $p\neq 2$, $\theta > 0$, $\delta > 0$ be real numbers. Suppose that the function $f : E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta + \theta(\|x\|^p+\|y\|^p) \quad \textit{for all} \;\; x,y\in E_1.$$

Then there exists exactly one quadratic function $q : E_1 \longrightarrow E_2$ such that

$$\|f(x)-q(x)\| \le c + k\theta\|x\|^p$$

for all $x\in E_1$ if $p\ge 0$ and for all $x\in E_1\setminus\{0\}$ if $p\le 0$, where

- $c = \frac{\|f(0)\|}{3}$, $k = \frac{2}{4-2^p}$ *and* $q(x) = \lim_{n\longrightarrow+\infty}\frac{f(2^n x)}{4^n}$, *for* $p<2$.
- $c = 0$, $k = \frac{2}{2^p-4}$ *and* $q(x) = \lim_{n\longrightarrow+\infty}\frac{f(2^n x)}{4^n}$, *for* $p>2$.

Recently, B. Bouikhalene, E. Elqorachi and Th. M. Rassias [1], [2] and [3] proved the Hyers-Ulam-Rassias stability of the functional equation (1.4) with $K=\{I,\sigma\}$ (σ is an automorphism of G such that $\sigma\circ\sigma = I$).

The results obtained in the present paper encompass results from [2] and [18] given in Corollaries 2.5 and 2.6 below.

General Set-Up. Let K be a compact transformation group of an abelian topological group $(G,+)$, acting by automorphisms of G. We let dk denote the normalized Haar measure on K, and the action of $k\in K$ on $x\in G$ is denoted by $k\cdot x$. We assume that the function $k\longmapsto k\cdot y$ is continuous for all $y\in G$.

A continuous mapping $q : G\longrightarrow \mathbb{C}$ is said to be K-quadratical if it satisfies the functional equation

$$\int_K q(x+k\cdot y)dk = q(x)+q(y), \quad x,y\in G. \tag{1.9}$$

When K is finite, the normalized Haar measure dk on K is given by

$$\int_K h(k)dk = \frac{1}{|K|}\sum_{k\in K} h(k)$$

for any $h : K \longrightarrow \mathbb{C}$, where $|K|$ denotes the order of K. So equation (1.9) can in this case be written

$$\frac{1}{|K|}\sum_{k\in K} q(x+k\cdot y) = q(x)+q(y), \quad x,y\in G. \tag{1.10}$$

2. MAIN RESULTS

Let $\varphi : G\times G \longrightarrow \mathbb{R}^+$ be a continuous mapping which satisfies the following condition

$$\psi(x,y) = \sum_{n=1}^{\infty} 2^{-n}\int_K\int_K\cdots\int_K \varphi\left[x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{n-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x,\right.$$
$$\left. y + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{n-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot y\right] dk_1dk_2\dots dk_{n-1} < \infty, \tag{2.1}$$

for all p such that $1\le p\le n-1$, for all $x,y\in G$ (uniform convergence). In what follows, we set $\varphi(x)=\varphi(x,x)$ and $\psi(x)=\psi(x,x)$ for all $x\in G$.

The main results of the present paper are based on the following proposition.

Proposition 2.1. *Let G be an abelian group and let $\varphi : G\times G\longrightarrow \mathbb{R}^+$ be a continuous control mapping which satisfies (2.1). Suppose that $f : G\longrightarrow \mathbb{C}$ is continuous and satisfies the inequality*

$$\left|\int_K f(x+k\cdot y)dk - f(x) - f(y)\right| \le \varphi(x,y) \tag{2.2}$$

for all $x,y\in G$. Then, the formula $q(x) = \lim_{n\longrightarrow\infty}\frac{f_n(x)}{2^n}$, with

$$f_0(x)=f(x) \text{ and } f_n(x)=\int_K f_{n-1}(x+k\cdot x)dk \text{ for all } n\ge 1, \tag{2.3}$$

defines a continuous function which satisfies

$$|f(x)-q(x)|\le\psi(x) \text{ and } \int_K q(x+k\cdot x)dk = 2q(x) \text{ for all } x\in G. \tag{2.4}$$

Furthermore, the continuous function q with the condition (2.4) is unique.

Proof. Replacing y by x in (2.2) gives

$$|f_1(x)-2f(x)| = \left|\int_K f(x+k\cdot x)dk - 2f(x)\right| \le \varphi(x) \tag{2.5}$$

and consequently

$$\begin{aligned} |f_2(x)-2f_1(x)| &= \left|\int_K f_1(x+k_1\cdot x)dk_1 - 2\int_K f(x+k_1\cdot x)dk_1\right| \\ &\le \int_K |f_1(x+k_1\cdot x) - 2f(x+k_1\cdot x)|dk_1 \\ &\le \int_K \varphi(x+k_1\cdot x)dk_1. \end{aligned} \tag{2.6}$$

Next, we prove that

$$\left|\frac{f_n(x)}{2^n} - \frac{f_{n-1}(x)}{2^{n-1}}\right| \le 2^{-n}\int_K\int_K\cdots\int_K \varphi\left(x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{n-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right) dk_1dk_2\dots dk_{n-1} \tag{2.7}$$

for all $n \in \mathbb{N}\setminus\{0\}$. Clearly (2.7) is true for the case $n = 1$, since setting $n = 1$ in (2.7) gives (2.5). Now, assume that the induction assumption is true for $n \in \mathbb{N}\setminus\{0\}$, and consider

$$\begin{aligned} |f_{n+1}(x) - 2f_n(x)| &= \left|\int_K f_n(x + k_n\cdot x)dk_n - 2\int_K f_{n-1}(x + k_n\cdot x)dk_n\right| \\ &\le \int_K |f_n(x + k_n\cdot x) - 2f_{n-1}(x + k_n\cdot x)|dk_n. \end{aligned} \tag{2.8}$$

Then

$$\begin{aligned} &\left|\frac{f_{n+1}(x)}{2^{n+1}} - \frac{f_n(x)}{2^n}\right| \\ &\le \frac{1}{2}\int_K\left|\frac{f_n(x + k_n\cdot x)}{2^n} - \frac{f_{n-1}(x + k_n\cdot x)}{2^{n-1}}\right| dk_n \\ &\le 2^{-(n+1)}\int_K\int_K\cdots\int_K \varphi\left(x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right) dk_1dk_2\dots dk_n, \end{aligned} \tag{2.9}$$

so that the inductive assumption (2.7) is indeed true for all positive integers. Hence, for $r > s$ we get

$$\begin{aligned} &\left|\frac{f_r(x)}{2^r} - \frac{f_s(x)}{2^s}\right| \\ &\le \sum_{n=s}^{r-1}\left|\frac{f_{n+1}(x)}{2^{n+1}} - \frac{f_n(x)}{2^n}\right| \\ &\le \sum_{n=s}^{r-1} 2^{-(n+1)}\int_K\cdots\int_K \varphi\left(x + \sum_{i_j<i_{j+1}\,,\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_r})\cdot x\right) dk_1\dots dk_n, \end{aligned} \tag{2.10}$$

which by assumption (2.1) converges to zero (uniformly) as r and s tend to infinity. Thus the sequence of complex functions $\frac{f_n(x)}{2^n}$ is a Cauchy sequence for each fixed $x \in G$ and then this sequence converges for each fixed $x \in G$ to some limit in $\mathbb{C}$, which is continuous on G. We call this limit $q(x)$. Next, we prove that

$$\left|\frac{f_n(x)}{2^n} - f(x)\right| \le \sum_{l=1}^{n} 2^{-l}\int_K\cdots\int_K \varphi\left(x + \sum_{i_j<i_{j+1}\,,\, k_{i_j}\in\{k_1,k_2,\dots,k_{l-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right) dk_1\dots dk_{l-1} \tag{2.11}$$

for all $n \in \mathbb{N}\setminus\{0\}$. We get the case of $n = 1$ by (2.5)

$$|f_1(x) - 2f(x)| = \left|\int_K f(x + k\cdot x)dk - 2f(x)\right| \le \varphi(x), \tag{2.12}$$

so the induction assumption (2.11) is true for $n = 1$. Assume that (2.11) is true for $n \in \mathbb{N} \setminus \{0\}$. By using (2.9), we obtain

$$\begin{aligned}
(2.13)\quad & |f_{n+1}(x) - 2^{n+1} f(x)| \\
& \leq |f_{n+1}(x) - 2f_n(x)| + 2|f_n(x) - 2^n f(x)| \\
& \leq \int_K \cdots \int_K \varphi \left(x + \sum_{i_j < i_{j+1},\, k_{i_j} \in \{k_1, k_2, \ldots, k_n\}} (k_{i_1} k_{i_2} \cdots k_{i_p}) \cdot x \right) dk_1 dk_2 \ldots dk_n \\
& \quad + 2^{n+1} \sum_{l=1}^{n} 2^{-l} \\
& \quad \times \int_K \cdots \int_K \varphi \left(x + \sum_{i_j < i_{j+1},\, k_{i_j} \in \{k_1, k_2, \ldots, k_{l-1}\}} (k_{i_1} k_{i_2} \cdots k_{i_p}) \cdot x \right) dk_1 dk_2 \ldots dk_{l-1}
\end{aligned}$$

so (2.11) is true for all $n \in \mathbb{N} \setminus \{0\}$. By letting $n \longrightarrow +\infty$, we obtain the first assertion of (2.4). We now shall show that q satisfies the second assertion of (2.4). By using (2.2) we get

$$\begin{aligned}
(2.14)\quad & \left| \int_K f_1(x + k_1 \cdot x) dk_1 - f_1(x) - f_1(x) \right| \\
& = \left| \int_K \int_K f(x + k_1 \cdot x + k_2 \cdot (x + k_1 \cdot x)) dk_1 dk_2 \right. \\
& \qquad \left. - \int_K f(x + k_1 \cdot x) dk_1 - \int_K f(x + k_1 \cdot x) dk_1 \right| \\
& \leq \int_K \left| \int_K f(x + k_1 \cdot x + k_2 \cdot (x + k_1 \cdot x)) dk_2 - f(x + k_1 \cdot x) - f(x + k_1 \cdot x) \right| dk_1 \\
& \leq \int_K \varphi(x + k_1 \cdot x) dk_1.
\end{aligned}$$

Make the induction assumption

$$\begin{aligned}
(2.15)\quad & \left| \int_K f_n(x + k \cdot x) dk - 2f_n(x) \right| \\
& \qquad \leq \int_K \cdots \int_K \varphi \left(x + \sum_{i_j < i_{j+1},\, k_{i_j} \in \{k_1, k_2, \ldots, k_n\}} (k_{i_1} k_{i_2} \cdots k_{i_p}) \cdot x \right) dk_1 \ldots dk_n,
\end{aligned}$$

which is true for $n = 1$ by (2.14). For $n + 1$ we have

$$\begin{aligned}
& \left| \int_K f_{n+1}(x + k_{n+1} \cdot x) dk_{n+1} - 2f_{n+1}(x) \right| \\
& = \left| \int_K \int_K f_n(x + k_{n+1} \cdot x + k \cdot (x + k_{n+1} \cdot x) dk_{n+1} dk - 2 \int_K f_n(x + k_{n+1} \cdot x)) dk_{n+1} \right| \\
& \leq \int_K \left| \int_K f_n(x + k_{n+1} \cdot x + k \cdot (x + k_{n+1} \cdot x)) dk - 2f_n(x + k_{n+1} \cdot x) \right| dk_{n+1}
\end{aligned}$$

$$\le \int_K \left\{ \int_K \cdots \int_K \varphi \Bigg(x + k_{n+1}\cdot x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot(x+k_{n+1}\cdot x) \Bigg) dk_1 \dots dk_n \right\} dk_{n+1}$$

$$= \int_K \cdots \int_K \varphi \left(x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{n+1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x \right) dk_1 \dots dk_{n+1}.$$

Thus (2.15) is true for all $n \in \mathbb{N}\setminus\{0\}$. Now, in view of the condition (2.1), q satisfies the second assertion of (2.4).

To demonstrate the uniqueness of the mapping q subject to (2.4), let us assume on the contrary that there is another mapping $q' : G \longrightarrow \mathbb{C}$ such that

$$|f(x) - q'(x)| \le \psi(x) \text{ and } \int_K q'(x + k\cdot x)dk = 2q'(x) \text{ for all } x \in G.$$

First, we prove by induction the following relation

$$\left|\frac{f_n(x)}{2^n} - q'(x)\right| \le \frac{1}{2^n}\int_K \cdots \int_K \psi \left(x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x \right) dk_1 \dots dk_n. \tag{2.16}$$

For $n = 1$, we have

$$|f_1(x) - 2q'(x)| = \left|\int_K f(x+k\cdot x)dk - \int_K q'(x+k\cdot x)dk\right| \le \int_K \psi(x+k\cdot x)dk \tag{2.17}$$

so (2.16) is true for $n = 1$. By using the following

$$|f_{n+1}(x) - 2^{n+1}q'(x)| = \left|\int_K f_n(x+k\cdot x)dk - 2^n\int_K q'(x+k\cdot x)dk\right| \le \int_K |f_n(x+k\cdot x) - 2^n q'(x+k\cdot x)|dk \tag{2.18}$$

we get the rest of the proof by proving that

$$\frac{1}{2^n}\int_K \cdots \int_K \psi \left(x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x \right) dk_1 \dots dk_n$$

converges to zero. In fact by setting

$$X = x + \sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x$$

it follows that

$$\begin{aligned}
&\frac{1}{2^n}\int_K\cdots\int_K \psi\left(x+\sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_n\}}(k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right)dk_1\dots dk_n\\
&=\frac{1}{2^n}\int_K\cdots\int_K\left\{\sum_{r=1}^{+\infty}2^{-r}\int_K\cdots\int_K\varphi\left(X+\sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_{n+1},k_{n+2},\dots,k_{n+r-1}\}}(k_{i_1}k_{i_2}\cdots k_{i_q})\cdot X\right)\right.\\
&\qquad\qquad\left. dk_{n+1}\dots dk_{n+r-1}\right\}dk_1\dots .dk_n\\
&=\sum_{r=1}^{+\infty}2^{-(n+r)}\int_K\cdots\int_K\varphi\left(x+\sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{n+r-1}\}}(k_{i_1}k_{i_2}\cdots k_{i_l})\cdot x\right)dk_1\dots dk_{n+r-1}\\
&=\sum_{m=n+1}^{+\infty}2^{-m}\int_K\cdots\int_K\varphi\left(x+\sum_{i_j<i_{j+1},\, k_{i_j}\in\{k_1,k_2,\dots,k_{m-1}\}}(k_{i_1}k_{i_2}\cdots k_{i_l})\cdot x\right)dk_1\dots dk_{m-1}.
\end{aligned}$$

In view of (2.1), this converges to zero, so $q=q'$. This ends the proof. □

Our main result reads as follows.

Theorem 2.2. *Let K be a finite cyclic subgroup of the group of automorphisms of the abelian group $(G,+)$. Let $\varphi : G\times G\longrightarrow\mathbb{R}^+$ be a mapping such that*

$$\begin{aligned}
\psi(x,y)=\sum_{n=1}^{\infty}\frac{|K|}{(2|K|)^n}\sum_{k_1,\dots,k_{n-1}\in K}\varphi\Bigg(&x+\sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_{n-1}\}}(k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x,\\
&y+\sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_{n-1}\}}(k_{i_1}k_{i_2}\cdots k_{i_p})\cdot y\Bigg)<\infty,
\end{aligned}\tag{2.19}$$

for all $x,y\in G$. Suppose that $f : G\longrightarrow\mathbb{C}$ satisfies the inequality

$$\left|\frac{1}{|K|}\sum_{k\in K}f(x+k\cdot y)-f(x)-f(y)\right|\le\varphi(x,y)\tag{2.20}$$

for all $x,y\in G$. Then, the limit $q(x)=\lim_{n\longrightarrow\infty}\frac{f_n(x)}{2^n}$, with

$$f_0(x)=f(x)\ \text{ and }\ f_n(x)=\frac{1}{|K|}\sum_{k\in K}f_{n-1}(x+k\cdot x)\ \text{ for all }\ n\ge 1,\tag{2.21}$$

exists for all $x\in G$, and $q : G\longrightarrow\mathbb{C}$ is the unique K-quadratical mapping which satisfies

$$|f(x)-q(x)|\le\psi(x)\ \text{ for all } x\in G.\tag{2.22}$$

Proof. In this case, the induction relations corresponding to (2.7) and (2.11) can be written as follows

$$\left|\frac{f_n(x)}{2^n} - \frac{f_{n-1}(x)}{2^{n-1}}\right| \leq \frac{|K|}{(2|K|)^n} \sum_{k_1,\dots,k_{n-1}\in K} \varphi\left(x + \sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_{n-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right) \tag{2.23}$$

for any $n \in \mathbb{N} \setminus \{0\}$.

$$\left|\frac{f_n(x)}{2^n} - f(x)\right| \leq \sum_{l=1}^{n} \frac{|K|}{(2|K|)^l} \sum_{k_1,\dots,k_{l-1}\in K} \varphi\left(x + \sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_{l-1}\}} (k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x\right), \tag{2.24}$$

for all integers $n \in \mathbb{N} \setminus \{0\}$. So, we can easily deduce that $q(x) = \lim_{n\longrightarrow+\infty} \frac{f_n(x)}{2^n}$ exists for all $x \in G$ and q satisfies the inequality (2.22). Now, we will show that q is a K-quadratical function. For all $x, y \in G$, we have

$$\begin{aligned}
&\left|\frac{1}{|K|}\sum_{k\in K} f_1(x+k\cdot y) - f_1(x) - f_1(y)\right| \\
&= \left|\frac{1}{|K|}\sum_{k\in K}\frac{1}{|K|}\sum_{k_1\in K} f((x+k\cdot y)+k_1\cdot(x+k\cdot y)) \right.\\
&\qquad \left. - \frac{1}{|K|}\sum_{k_1\in K} f(x+k_1\cdot x) - \frac{1}{|K|}\sum_{k_1\in K} f(y+k_1\cdot y)\right| \\
&= \left|\frac{1}{|K|}\sum_{k\in K}\frac{1}{|K|}\sum_{k_1\in K} f((x+k_1\cdot x)+k\cdot(y+k_1\cdot y)) \right.\\
&\qquad \left. - \frac{1}{|K|}\sum_{k_1\in K} f(x+k_1\cdot x) - \frac{1}{|K|}\sum_{k_1\in K} f(y+k_1\cdot y)\right| \\
&\leq \frac{1}{|K|}\sum_{k_1\in K}\left|\frac{1}{|K|}\sum_{k\in K} f((x+k_1\cdot x)+k\cdot(y+k_1\cdot y)) \right.\\
&\qquad \left. - f(x+k_1\cdot x) - f(y+k_1\cdot y)\right| \\
&\leq \frac{1}{|K|}\sum_{k_1\in K}\varphi(x+k_1\cdot x, y+k_1\cdot y).
\end{aligned} \tag{2.25}$$

Make the induction assumption

$$\left|\frac{1}{|K|}\sum_{k\in K}\frac{f_n(x+k\cdot y)}{2^n}-\frac{f_n(x)}{2^n}-\frac{f_n(y)}{2^n}\right| \tag{2.26}$$
$$\leq \frac{1}{(2|K|)^n}\sum_{k_1,\dots,k_n\in K}\varphi\left[x+\sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_n\}}(k_{i_1}k_{i_2}\cdots k_{i_p})\cdot x, y\right.$$
$$\left.+\sum_{i_j<i_{j+1};\, k_{i_j}\in\{k_1,\dots,k_n\}}(k_{i_1}k_{i_2}\cdots k_{i_p})\cdot y\right]$$

which is true for $n=1$ by (2.26). By using

$$\left|\frac{1}{|K|}\sum_{k\in K}f_n(x+k\cdot y)-f_n(x)-f_n(y)\right|$$
$$=\left|\frac{1}{|K|}\sum_{k'\in K}\frac{1}{|K|}\sum_{k'\in K}f_{n-1}(x+k\cdot y+k'\cdot(x+k\cdot y))\right.$$
$$\left.-\frac{1}{|K|}\sum_{k'\in K}f_{n-1}(x+k'\cdot x)-\frac{1}{|K|}\sum_{k\in K}f_{n-1}(y+k'\cdot y)\right|$$
$$=\left|\frac{1}{|K|}\sum_{k'\in K}\frac{1}{|K|}\sum_{k\in K}f_{n-1}(x+k'\cdot x+k\cdot(y+k'\cdot y))\right.$$
$$\left.-\frac{1}{|K|}\sum_{k'\in K}f_{n-1}(x+k'\cdot x)-\frac{1}{|K|}\sum_{k'\in K}f_{n-1}(y+k'\cdot y)\right|$$
$$\leq\frac{1}{|K|}\sum_{k'\in K}\left|\frac{1}{|K|}\sum_{k\in K}f_{n-1}(x+k'\cdot x+k\cdot(y+k'\cdot y))-f_{n-1}(x+k'\cdot x)-f_{n-1}(y+k'\cdot y)\right|$$

we get the result (2.26) for all $n\in\mathbb{N}\setminus\{0\}$. Now, in view of the condition (2.19), q is a K-quadratical function. This completes the proof. □

Corollary 2.3. *Let K be a finite cyclic subgroup of the group of automorphisms of G, let $\delta>0$. Suppose that $f: G\longrightarrow\mathbb{C}$ satisfies the inequality*

$$\left|\sum_{k\in K}f(x+k\cdot y)-|K|f(x)-|K|f(y)\right|\leq\delta \tag{2.27}$$

for all $x,y\in G$. Then, the limit $q(x)=\lim_{n\longrightarrow\infty}\frac{f_n(x)}{2^n}$, with

$$f_0(x)=f(x)\ \text{ and }\ f_n(x)=\frac{1}{|K|}\sum_{k\in K}f_{n-1}(x+k\cdot x)\ \text{ for }\ n\geq 1 \tag{2.28}$$

exists for all $x\in G$, and $q: G\longrightarrow\mathbb{C}$ is the unique K-quadratical mapping which satisfies

$$|f(x)-q(x)|\leq\frac{\delta}{|K|}\ \text{ for all } x\in G. \tag{2.29}$$

Corollary 2.4. *Let K be a finite cyclic subgroup of the group of automorphisms of the normed space $(G, \|\cdot\|)$, let $\theta \geq 0$ and $p < 1$. Suppose that $f : G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$\left| \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - f(x) - f(y) \right| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.30}$$

for all $x, y \in G$. Then, the limit $q(x) = \lim_{n \longrightarrow \infty} \frac{f_n(x)}{2^n}$, with

$$f_0(x) = f(x) \;\; \text{and} \;\; f_n(x) = \frac{1}{|K|} \sum_{k \in K} f_{n-1}(x + k \cdot x) \;\; \text{for} \;\; n \geq 1 \tag{2.31}$$

exists for all $x \in G$, and $q : G \longrightarrow \mathbb{C}$ is the unique K-quadratical mapping which satisfies

$$\begin{aligned} &|f(x) - q(x)| \\ &\leq \sum_{n=1}^{\infty} \frac{|K|}{(2|K|)^n} \sum_{k_1, \ldots, k_{n-1} \in K} 2\theta \left\| x + \sum_{i_j < i_{j+1};\, k_{i_j} \in \{k_1, \ldots, k_{n-1}\}} (k_{i_1} k_{i_2} \cdots k_{i_p}) \cdot x \right\|^p \end{aligned} \tag{2.32}$$

for all $x \in G$.

Corollary 2.5 ([18])**.** *Let $K = \{I\}$, $\theta \geq 0$ and $p < 1$. Suppose that $f : G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|f(x + y) - f(x) - f(y)| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.33}$$

for all $x, y \in G$. Then, the limit $q(x) = \lim_{n \longrightarrow \infty} \frac{f_n(x)}{2^n}$, with

$$f_n(x) = f(2^n x) \;\; \text{for} \;\; n \in \mathbb{N} \setminus \{0\} \tag{2.34}$$

exists for all $x \in G$, and $q : G \longrightarrow \mathbb{C}$ is the unique additive mapping which satisfies

$$|f(x) - q(x)| \leq \frac{2\theta \|x\|^p}{2 - 2^p} \;\; \text{for all} \; x \in G. \tag{2.35}$$

Corollary 2.6 ([2])**.** *Let $K = \{I, \sigma\}$, where $\sigma : G \longrightarrow G$ is an involution of G, and let $\varphi : G \times G \longrightarrow [0, \infty)$ be a mapping satisfying the condition*

$$\begin{aligned} \psi(x, y) = \sum_{n=0}^{\infty} & 2^{-2n-1} [\varphi(2^n x, 2^n y) \\ & + (2^n - 1)\varphi(2^{n-1}x + 2^{n-1}\sigma(x), 2^{n-1}y + 2^{n-1}\sigma(y))] < \infty \end{aligned} \tag{2.36}$$

for all $x, y \in G$. Let $f : G \longrightarrow \mathbb{C}$ satisfy

$$|f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y)| \leq \varphi(x, y) \tag{2.37}$$

for all $x, y \in G$. Then, there exists a unique solution $q : G \longrightarrow \mathbb{C}$ of the equation

$$q(x + y) + q(x + \sigma(y)) = 2q(x) + 2q(y) \;\; x, y \in G \tag{2.38}$$

given by

$$q(x) = \lim_{n \longrightarrow +\infty} 2^{-2n} \{f(2^n x) + (2^n - 1) f(2^{n-1}x + 2^{n-1}\sigma(x))\} \tag{2.39}$$

which satisfies the inequality

$$|f(x) - q(x)| \leq \psi(x, x) \tag{2.40}$$

for all $x \in G$.

Remark 2.7. We can replace in Theorem 2.2 the condition that K is a finite cyclic subgroup by the condition that K is a compact commutative subgroup of $Aut(G)$.

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