

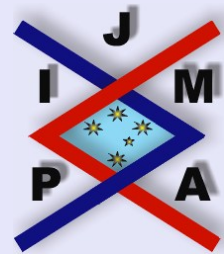
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ADDITIONS TO THE TELYAKOVSKIĬ'S CLASS \mathbb{S}

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Abstract

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Abstract

A sufficient condition of new type is given which implies that certain sequences belong to the Telyakovskii's class \mathcal{S} . Furthermore the relations of two subclasses of the class \mathcal{S} are analyzed.

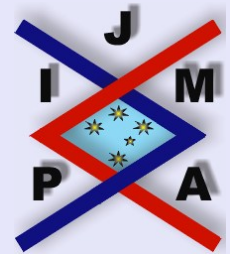
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1. Introduction

In 1973, S.A. Telyakovskii [3] defined the class \mathbb{S} of number sequences which has become a very flourishing definition. Several mathematicians have wanted to extend this definition, but it has turned out that most of them are equivalent to the class \mathbb{S} . For some historical remarks, we refer to [2]. These intentions show that the class \mathbb{S} plays a very important role in many problems.

The definition of the class \mathbb{S} is the following: A null-sequence $\mathbf{a} := \{a_n\}$ belongs to the class \mathbb{S} , or briefly $\mathbf{a} \in \mathbb{S}$, if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$ hold for all n .

We recall only one result of Telyakovskii [3] to illustrate the usability of the class \mathbb{S} .

Theorem 1.1. *Let the coefficients of the series*

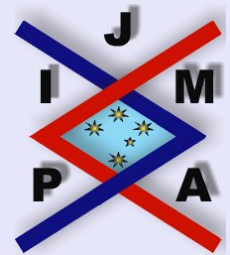
$$(1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

belong to the class \mathbb{S} . Then the series (1.1) is a Fourier series and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right| dx \leq \mathbb{C} \sum_{n=0}^{\infty} a_n,$$

where \mathbb{C} is an absolute constant.

Recently Ž. Tomovski [4] defined certain subclasses of \mathbb{S} , and denoted them by \mathbb{S}_r , $r = 1, 2, \dots$ as follows:



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A null-sequence $\{a_n\}$ belongs to \mathbb{S}_r , if there exists a monotonically decreasing sequence $\{A_n^{(r)}\}$ such that $\sum_{n=1}^{\infty} n^r A_n^{(r)} < \infty$ and $|\Delta a_n| \leq A_n^{(r)}$.

In [5] Tomovski established, among others, a theorem which states that if $\{a_n\} \in \mathbb{S}_r$ then the r -th derivative of the series (1.1) is a Fourier series and the integral of the absolute value its sum function less than equal to $\mathbb{C}(r) \sum_{n=1}^{\infty} n^r A_n^{(r)}$, where $\mathbb{C}(r)$ is a constant.

His proof is a constructive one and follows along similar lines to that of Theorem 1.1.

In [1] we also defined a certain subclass of \mathbb{S} as follows:

Let $\alpha := \{\alpha_n\}$ be a positive monotone sequence tending to infinity. A null-sequence $\{a_n\}$ belongs to the class $\mathbb{S}(\alpha)$, if there exists a monotonically decreasing sequence $\{A_n^{(\alpha)}\}$ such that

$$\sum_{n=1}^{\infty} \alpha_n A_n^{(\alpha)} < \infty \quad \text{and} \quad |\Delta a_n| \leq A_n^{(\alpha)}.$$

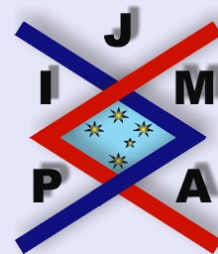
Clearly $\mathbb{S}(\alpha)$ with $\alpha_n = n^r$ includes \mathbb{S}_r .

In [2] we verified that if $\{a_n\} \in \mathbb{S}_r$, then $\{n^r a_n\} \in \mathbb{S}$, with a sequence $\{A_n\}$ that satisfies the inequality

$$(1.2) \quad \sum_{n=1}^{\infty} A_n \leq (r+1) \sum_{n=1}^{\infty} n^r A_n^{(r)}.$$

Thus, this result and Theorem 1.1 immediately imply the theorem of Tomovski mentioned above.

Our theorem which yields (1.2) reads as follows.



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Theorem 1.2. Let $\gamma \geq \beta > 0$ and $\mathbb{S}_\alpha := \mathbb{S}(\alpha)$ if $\alpha_n = n^\alpha$. If $\{a_n\} \in S_\gamma$ then $\{n^\beta a_n\} \in \mathbb{S}_{\gamma-\beta}$ and

$$(1.3) \quad \sum_{n=1}^{\infty} n^{\gamma-\beta} A_n^{(\gamma-\beta)} \leq (\beta + 1) \sum_{n=1}^{\infty} n^\gamma A_n^{(\gamma)}$$

holds.

It is clear that if $\gamma = \beta = r$ then (1.3) gives (1.2) ($A_n^{(0)} = A_n$).

In [2] we also verified that the statement of Theorem 1.2 is not reversible in general.

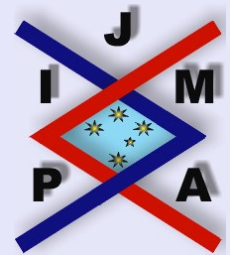
In [3] Telyakovskii realized that in the definition of the class \mathbb{S} we can take $A_n := \max_{k \geq n} |\Delta a_k|$, that is, $\{a_n\} \in \mathbb{S}$ if $a_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \max_{k \geq n} |\Delta a_k| < \infty$.

This definition of \mathbb{S} has not been used often, as I know.

The reason, perhaps, is the appearing of the inconvenient addends $\max_{k \geq n} |\Delta a_k|$.

In the present note first we give a sufficient condition being of similar character as this definition of \mathbb{S} but without $\max_{k \geq n} |\Delta a_k|$, which implies that $\{a_n\} \in \mathbb{S}$.

Second we show that with a certain additional assumption, the assertion of Theorem 1.2 is reversible and the additional condition to be given is necessary in general.



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2. Results

Before formulating the first theorem we recall a definition.

A non-negative sequence $\mathbf{c} := \{c_n\}$ is called locally almost monotone if there exists a constant $K(\mathbf{c})$ depending only on the sequence \mathbf{c} , such that

$$c_n \leq K(\mathbf{c})c_m$$

holds for any m and $m \leq n \leq 2m$. These sequences will be denoted by $\mathbf{c} \in LAMS$.

Theorem 2.1. *If $a := \{a_n\}$ is a null-sequence, $a \in LAMS$ and $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$, then $a \in \mathbb{S}$.*

Theorem 2.2. *Let $\gamma \geq \beta > 0$. If $\{n^\beta a_n\} \in \mathbb{S}_{\gamma-\beta}$, and*

$$(2.1) \quad \sum_{n=1}^{\infty} n^\gamma |\Delta a_n| < \infty,$$

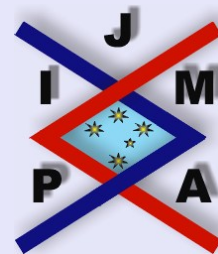
then $\{a_n\} \in \mathbb{S}_\gamma$.

Remark 2.1. *The condition (2.1) is not dispensable, moreover it cannot be weakened in general.*

The following lemma will be required in the proof of Theorem 2.1.

Lemma 2.3. *If $\mathbf{c} := \{c_n\} \in LAMS$ and $\alpha_n := \sup_{k \geq n} c_k$, then for any $\delta > -1$*

$$(2.2) \quad \sum_{n=1}^{\infty} n^\delta \alpha_n \leq K(K(\mathbf{c}), \delta) \sum_{n=1}^{\infty} n^\delta c_n.$$



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Proof. Since $\mathbf{c} \in LAMS$ thus with $K := K(\mathbf{c})$

$$(2.3) \quad \alpha_{2^n} = \sup_{k \geq 2^n} c_k \leq \sup_{m \geq n} K c_{2^m} \leq K \sup_{m \geq n} c_{2^m}.$$

If $\sum n^\delta c_n < \infty$, then $c_n \rightarrow 0$, thus by (2.3) there exists an integer $p = p(n) \geq 0$ such that

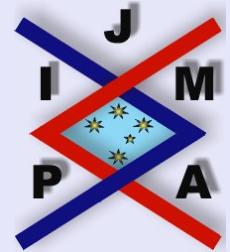
$$\alpha_{2^n} \leq K c_{2^{n+p}}.$$

Then, by the monotonicity of the sequence $\{\alpha_n\}$,

$$\begin{aligned} \sum_{k=n}^{n+p} 2^{k(1+\delta)} \alpha_{2^k} &\leq K c_{2^{n+p}} \sum_{k=n}^{n+p} 2^{k(1+\delta)} \\ &\leq K 2^{(1+\delta)} 2^{(n+p)(1+\delta)} c_{2^{n+p}} \\ &\leq K^2 2^{(1+\delta)2} \sum_{\nu=2^{n+p-1}+1}^{2^{n+p}} \nu^\delta c_\nu \end{aligned}$$

clearly follows. If we start this arguing with $n = 0$, and repeat it with $n + p$ in place of n , if $p \geq 1$; and if $p = 0$ then with $n + 1$ in place of n , and make these blocks repeatedly, furthermore if we add all of these sums, we see that the sum $\sum_{k=3}^{\infty} 2^{k(1+\delta)} \alpha_{2^k}$ will be majorized by the sum $K^2 4^{(1+\delta)} \sum_{n=1}^{\infty} n^\delta c_n$, and this proves (2.2). \square

Remark 2.2. *Following the steps of the proof it is easy to see that with φ_n in place of n^δ , (2.2) also holds if $\{\varphi_n\} \in LAMS$ and $2^n \varphi_{2^n}$ is quasi geometrically increasing.*



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3. Proofs

Proof of Theorem 2.1. Using Lemma 2.3 with $c_n = a_n$ and $\delta = 0$, we immediately get that

$$(3.1) \quad \sum_{n=1}^{\infty} \max_{k \geq n} |\Delta a_k| < \infty,$$

namely the assumption $a_n \rightarrow 0$ yields that $\sup |\Delta a_k| = \max |\Delta a_k|$, and thus (3.1) implies that $\{a_n\} \in \mathbb{S}$. \square

Proof of Theorem 2.2. With respect to the equality

$$|\Delta(n^\beta a_n)| = |n^\beta(a_n - a_{n+1}) - a_{n+1}((n+1)^\beta - n^\beta)|$$

it is clear that

$$n^\beta |\Delta a_n| \leq A_n^{(\gamma-\beta)} + K n^{\beta-1} |a_{n+1}|,$$

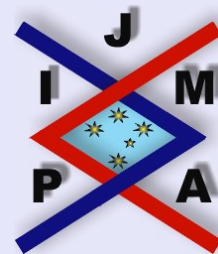
where K is a constant $K = K(\beta) > 0$ independent of n .

Hence, multiplying with $n^{-\beta}$, we get that

$$(3.2) \quad |\Delta a_n| \leq n^{-\beta} A_n^{(\gamma-\beta)} + K n^{-1} \sum_{k=n+1}^{\infty} |\Delta a_k|,$$

thus if we define

$$A_n^{(\gamma)} := n^{-\beta} A_n^{(\gamma-\beta)} + K n^{-1} \sum_{k=n+1}^{\infty} |\Delta a_k|,$$



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then this sequence $A_n^{(\gamma)}$ is clearly monotonically decreasing, and $A_n^{(\gamma)} \geq |\Delta a_n|$, furthermore by the assumptions of Theorem 1.2 and (3.2)

$$\sum_{n=1}^{\infty} n^{\gamma} A_n^{(\gamma)} < \infty,$$

since

$$\sum_{n=1}^{\infty} n^{\gamma-1} \sum_{k=n+1}^{\infty} |\Delta a_k| \leq K(\gamma) \sum_{k=1}^{\infty} k^{\gamma} |\Delta a_k| < \infty.$$

Thus $\{a_n\} \in \mathbb{S}_{\gamma}$ is proved. The proof is complete. \square

Proof of Remark 2.1. Let $a_n = n^{-\beta}$, then $|\Delta n^{\beta} a_n| = 0$, therefore $\{n^{\beta} a_n\} \in \mathbb{S}_{\gamma-\beta}$ holds e.g. with $A_n^{(\gamma-\beta)} = n^{\beta-\gamma-2}$. On the other hand $|\Delta a_n| \geq (n+1)^{-\beta-1}$, thus, by $\gamma \geq \beta$,

$$(3.3) \quad \sum_{n=1}^{\infty} n^{\gamma} |\Delta a_n| = \infty,$$

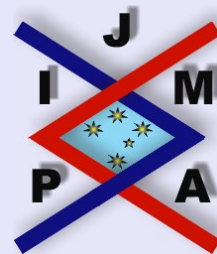
consequently, if $A_n^{(\gamma)} \geq |\Delta a_n|$, then

$$\sum_{n=1}^{\infty} n^{\gamma} A_n^{(\gamma)} = \infty$$

also holds, therefore $\{a_n\} \notin \mathbb{S}_{\gamma}$.

In this case, by (3.3), the additional condition (2.1) does not maintain.

Herewith, Remark 2.1 is verified, namely we can also see that the condition (2.1) cannot be weakened in general. \square



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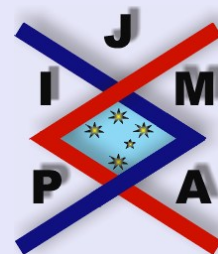
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