journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 3, Article 74, 8 pp.

INEQUALITIES OF JENSEN-PEČARIĆ-SVRTAN-FAN TYPE

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Received 22 January, 2007; accepted 06 July, 2008
Communicated by S.S. Dragomir

ABSTRACT. By using the theory of majorization, the following inequalities of Jensen-Pečarić-Svrtan-Fan type are established: Let I be an interval, $f : I \to \mathbb{R}$ and $t \in I, x, a, b \in I^n$. If $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1, a_1+b_1 \leq \cdots \leq a_n+b_n; f(t) > 0, f'(t) > 0, f''(t) > 0, f'''(t) < 0$ for any $t \in I$, then

$$\frac{f(A(a))}{f(A(b))} = \frac{f_{n,n}(a)}{f_{n,n}(b)} \leq \cdots \leq \frac{f_{k+1,n}(a)}{f_{k+1,n}(b)} \leq \frac{f_{k,n}(a)}{f_{k,n}(b)} \leq \cdots \leq \frac{f_{1,n}(a)}{f_{1,n}(b)} = \frac{A(f(a))}{A(f(b))},$$

the inequalities are reversed for $f''(t) < 0, f'''(t) > 0, \forall t \in I$, where $A(\cdot)$ is the arithmetic mean and

$$f_{k,n}(x) := \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

Key words and phrases: Jensen's inequality, Pečarić-Svrtan's inequality, Fan's inequality, Theory of majorization, Hermite matrix.

2000 *Mathematics Subject Classification.* 26D15, 26E60.

1. INTRODUCTION

In what follows, we shall use the following symbols:

$$x := (x_1, \ldots, x_n); \quad f(x) := (f(x_1), \ldots, f(x_n)); \quad G(x) := (x_1x_2\cdots x_n)^{1/n};$$

$$A(x) := \frac{x_1 + x_2 + \cdots + x_n}{n}; \quad \mathbb{R}^n_+ := [0, +\infty)^n; \quad \mathbb{R}^n_{++} := (0, +\infty)^n;$$

$$I^n := \{x|x_i \in I, i = 1, \ldots, n, I \text{ is an interval}\};$$

This work is supported by the Natural Science Foundation of China (10671136) and the Natural Science Foundation of Sichuan Province Education Department (07ZA207).

The authors are deeply indebted to anonymous referees for many useful comments and keen observations which led to the present improved version of the paper as it stands, and also are grateful to their friend Professor Wan-lan Wang for numerous discussions and helpful suggestions in preparation of this paper.

033-07

$$f_{k,n}(x) := \frac{1}{\binom{n}{k}} \sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right), \quad k=1,\dots,n.$$

Jensen's inequality states that: Let $f : I \to \mathbb{R}$ be a convex function and $x \in I^n$. Then

$$f(A(x)) \le A(f(x)). \tag{1.1}$$

This well-known inequality has a great number of generalizations in the literature (see [1] – [6]). An interesting generalization of (1.1) due to Pečarić and Svrtan [5] is:

$$f(A(x)) = f_{n,n}(x) \le \cdots \le f_{k+1,n}(x) \le f_{k,n}(x) \le \cdots \le f_{1,n}(x) = A(f(x)). \tag{1.2}$$

In 2003, Tang and Wen [6] obtained the following generalization of (1.2):

$$f_{r,s,n} \ge \cdots \ge f_{r,s,i} \ge \cdots \ge f_{r,s,s} \ge \cdots \ge f_{r,j,j} \ge \cdots \ge f_{r,r,r} = 0, \tag{1.3}$$

where

$$f_{r,s,n} := \binom{n}{r}\binom{n}{s}(f_{r,n} - f_{s,n}), \quad f_{k,n} := f_{k,n}(x), \quad 1 \le r \le s \le n.$$

Ky Fan's arithmetic-geometric mean inequality is (see [7]): Let $x \in (0, 1/2]^n$. Then

$$\frac{A(x)}{A(1-x)} \ge \frac{G(x)}{G(1-x)}. \tag{1.4}$$

In this paper, we shall establish further extensions of (1.2) and (1.4) as follows:

Theorem 1.1. *Let I be an interval. If $f : I \to \mathbb{R}$, $a, b \in I^n$ $(n \ge 2)$ and*

(i) $a_1 \le \cdots \le a_n \le b_n \le \cdots \le b_1$, $a_1 + b_1 \le \cdots \le a_n + b_n$;
(ii) $f(t) > 0$, $f'(t) > 0$, $f''(t) > 0$, $f'''(t) < 0$ *for any* $t \in I$,

then

$$\frac{f(A(a))}{f(A(b))} = \frac{f_{n,n}(a)}{f_{n,n}(b)} \le \cdots \le \frac{f_{k+1,n}(a)}{f_{k+1,n}(b)} \le \frac{f_{k,n}(a)}{f_{k,n}(b)} \le \cdots \le \frac{f_{1,n}(a)}{f_{1,n}(b)} = \frac{A(f(a))}{A(f(b))}. \tag{1.5}$$

The inequalities are reversed for $f''(t) < 0$, $f'''(t) > 0$, $\forall t \in I$. The equality signs hold if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

In Section 3, several interesting results of Ky Fan shall be deduced. In Section 4, the matrix variant of (1.5) will be established.

2. PROOF OF THEOREM 1.1

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be a function whose second derivative exists and $x \in I^n$,*

$$\alpha \in \Omega_n = \{\alpha \in \mathbb{R}^n_+ : \alpha_1 + \cdots + \alpha_n = 1\}.$$

Writing

$$S(\alpha, x) := \frac{1}{n!} \sum_{i_1\cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}),$$

where $\sum_{i_1\cdots i_n}$ denotes summation over all permutations of $\{1, 2, \dots, n\}$,

$$F(\alpha) := \log\left[\frac{S(\alpha, a)}{S(\alpha, b)}\right], \quad a, b \in I^n,$$

$$u_i(x) := \alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \sum_{j=3}^{n} \alpha_j x_{i_j},$$
$$v_i(x) := \alpha_1 x_{i_2} + \alpha_2 x_{i_1} + \sum_{j=3}^{n} \alpha_j x_{i_j}, \qquad i = (i_1, i_2, \dots, i_n).$$

Then there exist $\xi_i(a)$ between $u_i(a)$ and $v_i(a)$, and $\xi_i(b)$ between $u_i(b)$ and $v_i(b)$ such that

$$(2.1)\quad (\alpha_1 - \alpha_2)\left(\frac{\partial F}{\partial \alpha_1} - \frac{\partial F}{\partial \alpha_2}\right) = \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le i_n}\left[\frac{f''(\xi_i(a))(u_i(a)-v_i(a))^2}{S(\alpha,a)} - \frac{f''(\xi_i(b))(u_i(b)-v_i(b))^2}{S(\alpha,b)}\right],$$

where $\sum_{i_3\cdots i_n}$ denotes the summation over all permutations of $\{1, 2, \dots, n\} \setminus \{i_1, i_2\}$.

Proof. Note the following identities:

$$S(\alpha, x) = \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1\ne i_2\le n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})$$
$$= \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n} [f(u_i(x)) + f(v_i(x))];$$
$$\frac{\partial}{\partial \alpha_1}[f(u_i) + f(v_i)] - \frac{\partial}{\partial \alpha_2}[f(u_i) + f(v_i)] = [f'(u_i) - f'(v_i)](x_{i_1} - x_{i_2});$$
$$(\alpha_1 - \alpha_2)\left(\frac{\partial S}{\partial \alpha_1} - \frac{\partial S}{\partial \alpha_2}\right) = \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}[f'(u_i) - f'(v_i)](\alpha_1-\alpha_2)(x_{i_1} - x_{i_2})$$
$$(2.2)\quad = \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}[f'(u_i) - f'(v_i)](u_i - v_i).$$

By $F(\alpha) = \log S(\alpha, a) - \log S(\alpha, b)$ and (2.2), we have

$$(\alpha_1 - \alpha_2)\left(\frac{\partial F}{\partial \alpha_1} - \frac{\partial F}{\partial \alpha_2}\right)$$
$$= (\alpha_1 - \alpha_2)\left\{[S(\alpha,a)]^{-1}\left[\frac{\partial S(\alpha,a)}{\partial \alpha_1} - \frac{\partial S(\alpha,a)}{\partial \alpha_2}\right] - [S(\alpha,b)]^{-1}\left[\frac{\partial S(\alpha,b)}{\partial \alpha_1} - \frac{\partial S(\alpha,b)}{\partial \alpha_2}\right]\right\}$$
$$= \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left\{\frac{[f'(u_i(a)) - f'(v_i(a))][u_i(a) - v_i(a)]}{S(\alpha,a)} - \frac{[f'(u_i(b)) - f'(v_i(b))][u_i(b) - v_i(b)]}{S(\alpha,b)}\right\}$$
$$= \frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left\{\frac{f''(\xi_i(a))[u_i(a)-v_i(a)]^2}{S(\alpha,a)} - \frac{f''(\xi_i(b))[u_i(b)-v_i(b)]^2}{S(\alpha,b)}\right\}.$$

Here we used the Mean Value Theorem for $f'(t)$. This completes the proof. □

Lemma 2.2. *Under the hypotheses of Theorem 1.1, F is a Schur-convex function or a Schur-concave function on Ω_n, where F is defined by Lemma 2.1.*

Proof. It is easy to see that Ω_n is a symmetric convex set and F is a differentiable symmetric function on Ω_n. To prove that F is a Schur-convex function on Ω_n, it is enough from [8, p .57] to prove that

$$(\alpha_1 - \alpha_2)\left(\frac{\partial F}{\partial \alpha_1} - \frac{\partial F}{\partial \alpha_2}\right) \geq 0, \quad \forall \alpha \in \Omega_n. \tag{2.3}$$

To prove (2.3), it is enough from Lemma 2.1 to prove

$$\frac{f''(\xi_i(a))[u_i(a) - v_i(a)]^2}{S(\alpha, a)} \geq \frac{f''(\xi_i(b))[u_i(b) - v_i(b)]^2}{S(\alpha, b)}. \tag{2.4}$$

Using the given conditions $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$, $f(t) > 0$ and $f'(t) > 0$, we obtain that $a_j \leq b_j$ $(j = 1, 2, \ldots, n)$ and the inequalities:

$$\frac{1}{S(\alpha, a)} \geq \frac{1}{S(\alpha, b)} > 0. \tag{2.5}$$

By the given condition (i) of Theorem 1.1 and $1 \leq i_1 < i_2 \leq n$, we have

$$a_{i_2} - a_{i_1} \geq b_{i_1} - b_{i_2} \geq 0$$

and

$$[u_i(a) - v_i(a)]^2 \geq [u_i(b) - v_i(b)]^2 \geq 0. \tag{2.6}$$

From (2.5) and (2.6), we get

$$\frac{[u_i(a) - v_i(a)]^2}{S(\alpha, a)} \geq \frac{[u_i(b) - v_i(b)]^2}{S(\alpha, b)} \geq 0. \tag{2.7}$$

Note that $a, b \in I^n$, $u_i(a)$, $v_i(a)$, $u_i(b)$, $v_i(b) \in I$, and

$$\begin{aligned}
\min\{u_i(a), v_i(a)\} &\leq \xi_i(a) \\
&\leq \max\{u_i(a), v_i(a)\} \\
&\leq \min\{u_i(b), v_i(b)\} \\
&\leq \xi_i(b) \\
&\leq \max\{u_i(b), v_i(b)\}.
\end{aligned}$$

It follows that

$$\xi_i(a) \leq \xi_i(b) \quad (\xi_i(a), \xi_i(b) \in I). \tag{2.8}$$

If $f''(t) > 0, f'''(t) < 0$ for any $t \in I$, from these and (2.8) we get

$$f''(\xi_i(a)) \geq f''(\xi_i(b)) > 0. \tag{2.9}$$

Combining with (2.7) and (2.9), we have proven that (2.4) holds, hence, F is a Schur-convex function on Ω_n.

Similarly, if $f''(t) < 0, f'''(t) > 0$ for any $t \in I$, we obtain

$$-f''(\xi_i(a)) \geq -f''(\xi_i(b)) > 0. \tag{2.10}$$

Combining with (2.7) and (2.10), we know that the inequalities are reversed in (2.4) and (2.3). Therefore, F is a Schur-concave function on Ω_n. This ends the proof of Lemma 2.2. □

Remark 1. When $\alpha_1 \neq \alpha_2$, there is equality in (2.3) if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$. In fact, there is equality in (2.3) if and only if there is equality in (2.5), (2.8), (2.9) and the first inequality in (2.6) or all the equality signs hold in (2.6). For the first case, by $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$, we get $a_1 = \cdots = a_n, b_1 = \cdots = b_n$. For the second case, we have $a_{i_1} - a_{i_2} = 0 = b_{i_1} - b_{i_2}$. Since $1 \leq i_1 < i_2 \leq n$ and i_1, i_2 are arbitrary, we get $a_1 = \cdots = a_n$, $b_1 = \cdots = b_n$. Clearly, if $a_1 = \cdots = a_n$, $b_1 = \cdots = b_n$, then (2.3) reduces to an equality.

Proof of Theorem 1.1. First we note that if

$$\alpha = \alpha_k := \left(\underbrace{k^{-1}, k^{-1}, \ldots, k^{-1}}_{k}, 0, \ldots, 0 \right),$$

we obtain that

$$S(\alpha_k, x) = f_{k,n}(x)$$

and

$$F(\alpha_k) = \log \frac{f_{k,n}(a)}{f_{k,n}(b)}. \tag{2.11}$$

By Lemma 2.2, we observe that $F(\alpha)$ is a Schur-convex(concave) function on Ω_n. Using $\alpha_{k+1} \prec \alpha_k$ for $\alpha_k, \alpha_{k+1} \in \Omega_n$ and the definition of Schur-convex(concave) functions, we have [8]

$$F(\alpha_{k+1}) \leq (\geq) F(\alpha_k), \qquad k = 1, \ldots, n-1. \tag{2.12}$$

It follows from (2.11) and (2.12) that (1.5) holds. Since $\alpha_{k+1} \neq \alpha_k$, combining this fact with Remark 1, we observe that the equality signs hold in (1.5) if and only if $a_1 = \cdots = a_n$, $b_1 = \cdots = b_n$. This completes the proof of Theorem 1.1. □

3. COROLLARY OF THEOREM 1.1

Corollary 3.1. *Let* $0 < r < 1$, $s \geq 1$, $0 < a_i \leq 2^{-1/s}$, $b_i = (1 - a_i^s)^{1/s}$, $i = 1, \ldots, n$, $f(t) = t^r$, $t \in (0,1)$. *Then the inequalities in* (1.5) *are reversed.*

Proof. Without loss of generality, we can assume that $0 < a_1 \leq \cdots \leq a_n$. By $b_i = (1 - a_i^s)^{1/s}$ and $0 < a_i \leq 2^{-1/s}$ $(i = 1, \ldots, n)$, we have

$$0 < a_1 \leq \cdots \leq a_n \leq 2^{-1/s} \leq b_n \leq \cdots \leq b_1 < 1.$$

Now we take $g(t) := t + (1 - t^s)^{1/s} (0 < t \leq 2^{-1/s})$, so $g'(t) = 1 - (1 - t^s)^{(1/s)-1} t^{s-1} \geq 0$, i.e., g is an increasing function. Thus

$$a_1 + b_1 \leq \cdots \leq a_n + b_n.$$

It is easy to see that $f(t) = t^r > 0$, $f'(t) = rt^{r-1} > 0$, $f''(t) = r(r-1)t^{r-2} < 0$, $f'''(t) = r(r-1)(r-2)t^{r-3} > 0$ for any $t \in (0,1)$. By Theorem 1.1, Corollary 3.1 can be deduced. This completes the proof. □

Corollary 3.2. *Let* $a \in (0, 1/2]^n$. *Writing*

$$[AG; x]_{k,n} := \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right)^{\frac{1}{\binom{n}{k}}},$$

we have

$$
\begin{aligned}
\frac{A(a)}{A(1-a)} &= \frac{[AG;a]_{n,n}}{[AG;1-a]_{n,n}} \\
&\geq \cdots \geq \frac{[AG;a]_{k+1,n}}{[AG;1-a]_{k+1,n}} \geq \frac{[AG;a]_{k,n}}{[AG;1-a]_{k,n}} \\
&\geq \cdots \geq \frac{[AG;a]_{1,n}}{[AG;1-a]_{1,n}} = \frac{G(a)}{G(1-a)}. \qquad (3.1)
\end{aligned}
$$

Equalities hold throughout if and only if $a_1 = \cdots = a_n$. *(Compare* (3.1) *with* [7, 10, 11]*)*

Proof. We choose $s = 1$ in Corollary 3.1. Raising each term to the power of $1/r$ and letting $r \to 0$ in (1.5), (3.1) can be deduced. This ends the proof. □

Corollary 3.3. *Let* $f : I \to \mathbb{R}$ *be such that* $f(t) > 0$, $f'(t) > 0$, $f''(t) > 0$, $f'''(t) < 0$ *for any* $t \in I$. *Let* $\Phi : I_0 \to I$ *be increasing and* $\Psi : I_0 \to I$ *be decreasing, and suppose that* $\Phi + \Psi$ *is increasing and* $\sup \Phi \leq \inf \Psi$. *Then*

$$
\frac{f\left(|I_0|^{-1}\int_{I_0}\Phi dt\right)}{f\left(|I_0|^{-1}\int_{I_0}\Psi dt\right)} \leq \frac{\int_{I_0} f(\Phi)dt}{\int_{I_0} f(\Psi)dt}, \qquad (3.2)
$$

where $|I_0|$ *is the length of the interval* I_0. *The inequality is reversed for* $f''(t) < 0$, $f'''(t) > 0$, $\forall t \in I$.

In fact, since (3.2) is an integral version of the inequality $\frac{f(A(a))}{f(A(b))} \leq \frac{A(f(a))}{A(f(b))}$, therefore (3.2) holds by Theorem 1.1.

According to Theorem 1.1, (1.5) implies inequalities (1.1), (1.2) and (3.1), and the implication (3.1) to (1.4) is obvious. Consequently, Theorem 1.1 is a generalization of Jensen's inequality (1.1), Pečarić-Svrtan's inequalities (1.2) and Fan's inequality (1.4). Note that Theorem 1.1 contains a great number of inequalities as special cases. To save space we omit the details.

4. A MATRIX VARIANT

Let $A = (a_{ij})_{n\times n} (n \geq 2)$ be a Hermite matrix of order n. Then $\operatorname{tr} A = \sum_{i=1}^n a_{ii}$ is the trace of A. As is well-known, there exists a unitary matrix U such that $A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U^*$, where U^* is the transpose conjugate matrix of U and the components of $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A. Thus $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$. Let $\lambda \in I^n$. Then, for $f : I \to \mathbb{R}$, we define $f(A) := U \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n))U^*$ (see [9]). Note that $\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = U^*AU$. Based on the above, we may use the following symbols: If, for A, we keep the elements on the cross points of the $i_1, \ldots, i_k$th rows and the $i_1, \ldots, i_k$th columns; replacing the other elements by nulls, then we denote this new matrix by $A_{i_1\cdots i_k}$. Clearly, we have $\operatorname{tr}[U^*AU]_{i_1\cdots i_k} = \lambda_{i_1} + \cdots + \lambda_{i_k}$. Thus we also define that

$$
\begin{aligned}
f_{k,n}(A) &:= \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{\lambda_{i_1} + \cdots + \lambda_{i_k}}{k}\right) \\
&= \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{1}{k}\operatorname{tr}[U^*AU]_{i_1\cdots i_k}\right).
\end{aligned}
$$

In particular, we have

$$
f_{1,n}(A) = \frac{1}{n}\sum_{i=1}^n f(\lambda_i) = \frac{1}{n}\operatorname{tr}(f(A));
$$

$$f_{n,n}(A) = f\left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right) = f\left(\frac{1}{n}\mathrm{tr}A\right);$$

$$f_{n-1,n}(A) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{\mathrm{tr}A - \lambda_i}{n-1}\right) = \frac{1}{n}\mathrm{tr}f\left(\frac{E\cdot\mathrm{tr}A - A}{n-1}\right),$$

where E is a unit matrix. In fact, from

$$U^*\left(\frac{E\cdot\mathrm{tr}A - A}{n-1}\right)U = \mathrm{diag}\left(\frac{\mathrm{tr}A-\lambda_1}{n-1},\ldots,\frac{\mathrm{tr}A-\lambda_n}{n-1}\right),$$

we get

$$\mathrm{tr}f\left(\frac{E\cdot\mathrm{tr}A - A}{n-1}\right) = \sum_{i=1}^{n} f\left(\frac{\mathrm{tr}A-\lambda_i}{n-1}\right).$$

Based on the above facts and Theorem 1.1, we observe the following.

Theorem 4.1. *Let I be an interval and let $\lambda, \mu \in I^n$. Suppose the components of λ,μ are the eigenvalues of Hermitian matrices A and B. If*

(i) $\lambda_1 \le \cdots \le \lambda_n \le \mu_n \le \cdots \le \mu_1$, $\lambda_1 + \mu_1 \le \cdots \le \lambda_n + \mu_n$;

(ii) *the function $f : I \to \mathbb{R}$ satisfies $f(t) > 0$, $f'(t) > 0$, $f''(t) > 0$, $f'''(t) < 0$ for any $t \in I$, and we have*

$$\frac{f\left(\frac{1}{n}\mathrm{tr}A\right)}{f\left(\frac{1}{n}\mathrm{tr}B\right)} \le \frac{\mathrm{tr}f\left(\frac{E\cdot\mathrm{tr}A-A}{n-1}\right)}{\mathrm{tr}f\left(\frac{E\cdot\mathrm{tr}B-B}{n-1}\right)} \le \cdots \le \frac{f_{k+1,n}(A)}{f_{k+1,n}(B)} \le \frac{f_{k,n}(A)}{f_{k,n}(B)} \le \cdots \le \frac{\mathrm{tr}f(A)}{\mathrm{tr}f(B)}.$$

The inequalities are reversed for $f''(t) < 0$, $f'''(t) > 0$, $\forall t \in I$. Equalities hold throughout if and only if $\lambda_1 = \cdots = \lambda_n$ and $\mu_1 = \cdots = \mu_n$.

Remark 2. If $I = (0, 1/2]$, $0 < \lambda_1 \le \cdots \le \lambda_n \le 1/2$, $B = E - A$, then the precondition (i) of Theorem 4.1 can be satisfied.

Remark 3. Lemma 2.2 possesses a general and meaningful result that should be an important theorem. Theorem 1.1 is only an application of Lemma 2.2.

Remark 4. If $f(t) < 0$, $f'(t) < 0$ for any $t \in I$, then we can apply Theorem 1.1 to $-f$.

Remark 5. In [12, 13], several applications on Jensen's inequalities are displayed.

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