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SOME GENERALIZED CONVOLUTION PROPERTIES ASSOCIATED WITH CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

SHIGEYOSHI OWA AND H.M. SRIVASTAVA

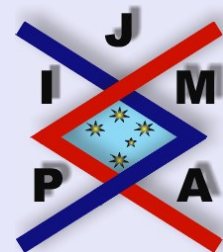
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Abstract

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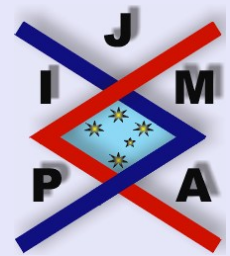


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Abstract

For functions belonging to each of the subclasses $\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$ of normalized analytic functions in open unit disk \mathbb{U} , which are introduced and investigated in this paper, the authors derive several properties involving their generalized convolution by applying certain techniques based especially upon the Cauchy-Schwarz and Hölder inequalities. A number of interesting consequences of these generalized convolution properties are also considered.

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1. Introduction and Definitions

Let \mathcal{A}_n denote the class of functions $f(z)$ *normalized* in the form:

$$(1.1) \quad f(z) = z + \sum_{k=n}^{\infty} a_k z^k \quad (n \in \mathbb{N} \setminus \{1\}; \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{M}_n(\alpha)$ the subclass of \mathcal{A}_n consisting of functions $f(z)$ which satisfy the inequality:

$$(1.2) \quad \Re \left(\frac{zf'(z)}{f(z)} \right) < \alpha \quad (\alpha > 1; z \in \mathbb{U}).$$

Also let $\mathcal{N}_n(\alpha)$ be the subclass of \mathcal{A}_n consisting of functions $f(z)$ which satisfy the inequality:

$$(1.3) \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) < \alpha \quad (\alpha > 1; z \in \mathbb{U}).$$

For $n = 2$ and $1 < \alpha < \frac{4}{3}$, the classes $M_2(\alpha)$ and $N_2(\alpha)$ were investigated earlier by Uralegaddi *et al.* (cf. [5]; see also [4] and [6]). In fact, following these earlier works in conjunction with those by Nishiwaki and Owa [1] (see also [3]), it is easy to derive Lemma 1.1 and Lemma 1.2 below, which provide the sufficient conditions for functions $f \in \mathcal{A}_n$ to be in the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}_n(\alpha)$, respectively.



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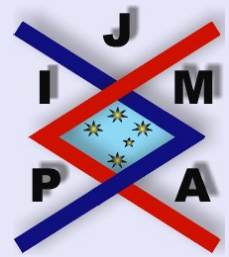


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Lemma 1.1. If $f \in \mathcal{A}_n$ given by (1.1) satisfies the condition:

$$(1.4) \quad \sum_{k=n}^{\infty} (k-n) |a_k| \leq \alpha - 1 \quad \left(1 < \alpha < \frac{n+1}{2}\right),$$

then $f \in \mathcal{M}_n(\alpha)$.

Lemma 1.2. If $f \in \mathcal{A}_n$ given by (1.1) satisfies the condition:

$$(1.5) \quad \sum_{k=n}^{\infty} k(k-\alpha) |a_k| \leq \alpha - 1 \quad \left(1 < \alpha < \frac{n+1}{2}\right),$$

then $f \in \mathcal{N}'_n(\alpha)$.

For examples of functions in the classes $\mathcal{M}_n(\alpha)$ and $\mathcal{N}'_n(\alpha)$, let us first consider the function $\varphi(z)$ defined by

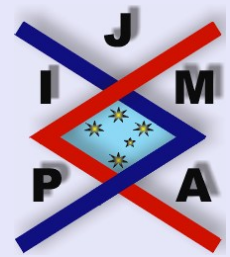
$$(1.6) \quad \varphi(z) := z + \sum_{k=n}^{\infty} \left(\frac{n(\alpha-1)}{k(k+1)(k-\alpha)} \right) z^k,$$

which is of the form (1.1) with

$$(1.7) \quad a_k = \frac{n(\alpha-1)}{k(k+1)(k-\alpha)} \quad (k = n, n+1, n+2, \dots),$$

so that we readily have

$$(1.8) \quad \sum_{k=n}^{\infty} \left(\frac{k-\alpha}{\alpha-1} \right) |a_k| = n \sum_{k=n}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1.$$



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Thus, by Lemma 1.1, $\varphi \in \mathcal{M}_n(\alpha)$. Furthermore, since

$$(1.9) \quad f(z) \in \mathcal{N}_n(\alpha) \iff zf'(z) \in \mathcal{M}_n(\alpha),$$

we observe that the function $\psi(z)$ defined by

$$(1.10) \quad \psi(z) := z + \sum_{k=n}^{\infty} \left(\frac{n(\alpha-1)}{k^2(k+1)(k-\alpha)} \right) z^k$$

belongs to the class $\mathcal{N}_n(\alpha)$.

In view of Lemma 1.1 and Lemma 1.2, we now define the subclasses

$$\mathcal{M}_n^*(\alpha) \subset \mathcal{M}_n(\alpha) \quad \text{and} \quad \mathcal{N}_n^*(\alpha) \subset \mathcal{N}_n(\alpha),$$

which consist of functions $f(z)$ satisfying the conditions (1.4) and (1.5), respectively.

Finally, for functions $f_j \in A_n$ ($j = 1, \dots, m$) given by

$$(1.11) \quad f_j(z) = z + \sum_{k=n}^{\infty} a_{k,j} z^k \quad (j = 1, \dots, m),$$

the Hadamard product (or convolution) is defined by

$$(1.12) \quad (f_1 * \dots * f_m)(z) := z + \sum_{k=n}^{\infty} \left(\prod_{j=1}^m a_{k,j} \right) z^k.$$

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2. Convolution Properties of Functions in the Classes

$\mathcal{M}_n^*(\alpha)$ and $\mathcal{N}_n^*(\alpha)$

For the Hadamard product (or convolution) defined by (1.12), we first prove

Theorem 2.1. *If $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ ($j = 1, \dots, m$), then*

$$(f_1 * \cdots * f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where

$$(2.1) \quad \beta = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)}{\prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

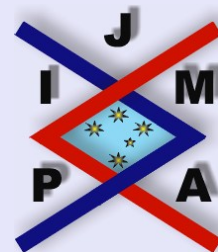
$$(2.2) \quad f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j} \right) z^n \quad (j = 1, \dots, m).$$

Proof. Following the work of Owa [2], we use the principle of mathematical induction in our proof of Theorem 2.1. Let $f_1(z) \in \mathcal{M}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{M}_n^*(\alpha_2)$. Then the inequality:

$$\sum_{k=n}^{\infty} (k - \alpha_j) |a_{k,j}| \leq \alpha_j - 1 \quad (j = 1, 2)$$

implies that

$$(2.3) \quad \sum_{k=n}^{\infty} \sqrt{\frac{k - \alpha_j}{\alpha_j - 1}} |a_{k,j}| \leq 1 \quad (j = 1, 2).$$



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Thus, by applying the Cauchy-Schwarz inequality, we have

$$\left| \sum_{k=n}^{\infty} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} |a_{k,1}| |a_{k,2}| \right|^2 \leq \left(\sum_{k=n}^{\infty} \left(\frac{k-\alpha_1}{\alpha_1-1} \right) |a_{k,1}| \right) \left(\sum_{k=n}^{\infty} \left(\frac{k-\alpha_2}{\alpha_2-1} \right) |a_{k,2}| \right) \leq 1.$$

Therefore, if

$$\sum_{k=n}^{\infty} \left(\frac{k-\delta}{\delta-1} \right) |a_{k,1}| |a_{k,2}| \leq \sum_{k=n}^{\infty} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} |a_{k,1}| |a_{k,2}|,$$

that is, if

$$\sqrt{|a_{k,1}| |a_{k,2}|} \leq \left(\frac{\delta-1}{k-\delta} \right) \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}} \quad (k = n, n+1, n+2, \dots),$$

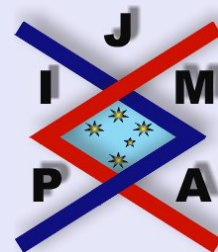
then $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$.

We also note that the inequality (2.3) yields

$$\sqrt{|a_{k,j}|} \leq \sqrt{\frac{\alpha_j-1}{k-\alpha_j}} \quad (j = 1, 2; k = n, n+1, n+2, \dots).$$

Consequently, if

$$\sqrt{\frac{(\alpha_1-1)(\alpha_2-1)}{(k-\alpha_1)(k-\alpha_2)}} \leq \frac{\delta-1}{k-\delta} \sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}},$$



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that is, if

$$(2.4) \quad \frac{k - \delta}{\delta - 1} \leq \frac{(k - \alpha_1)(k - \alpha_2)}{(\alpha_1 - 1)(\alpha_2 - 1)} \quad (k = n, n + 1, n + 2, \dots),$$

then we have $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$. It follows from (2.4) that

$$\delta \geq 1 + \frac{(k - 1)(\alpha_1 - 1)(\alpha_2 - 1)}{(k - \alpha_1)(k - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)} =: h(k) \quad (k = n, n + 1, n + 2, \dots).$$

Since $h(k)$ is decreasing for $k \geq n$, we have

$$\delta \geq 1 + \frac{(n - 1)(\alpha_1 - 1)(\alpha_2 - 1)}{(n - \alpha_1)(n - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)},$$

which shows that $(f_1 * f_2)(z) \in \mathcal{M}_n^*(\delta)$, where

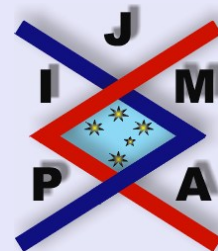
$$\delta := 1 + \frac{(n - 1)(\alpha_1 - 1)(\alpha_2 - 1)}{(n - \alpha_1)(n - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)}.$$

Next, we suppose that

$$(f_1 * \dots * f_m)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma := 1 + \frac{(n - 1) \prod_{j=1}^m (\alpha_j - 1)}{\prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}.$$



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Then, by means of the above technique, we can show that

$$(f_1 * \cdots * f_{m+1})(z) \in \mathcal{M}_n^*(\beta),$$

where

$$(2.5) \quad \beta := 1 + \frac{(n-1)(\gamma-1)(\alpha_{m+1}-1)}{(n-\gamma)(n-\alpha_{m+1}) + (\gamma-1)(\alpha_{m+1}-1)}.$$

Since

$$(\gamma-1)(\alpha_{m+1}-1) = \frac{(n-1) \prod_{j=1}^{m+1} (\alpha_j - 1)}{\prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}$$

and

$$(n-\gamma)(n-\alpha_{m+1}) = \frac{(n-1) \prod_{j=1}^{m+1} (n - \alpha_j)}{\prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)},$$

Equation (2.5) shows that

$$\beta = 1 + \frac{(n-1) \prod_{j=1}^{m+1} (\alpha_j - 1)}{\prod_{j=1}^{m+1} (n - \alpha_j) + \prod_{j=1}^{m+1} (\alpha_j - 1)}.$$

Finally, for the functions $f_j(z)$ ($j = 1, \dots, m$) given by (2.2), we have

$$(f_1 * \cdots * f_m)(z) = z + \left(\prod_{j=1}^m \left(\frac{\alpha_j - 1}{n - \alpha_j} \right) \right) z^n = z + A_n z^n,$$

where

$$A_n := \prod_{j=1}^m \left(\frac{\alpha_j - 1}{n - \alpha_j} \right).$$



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It follows that

$$\sum_{k=n}^{\infty} \left(\frac{k - \beta}{\beta - 1} \right) |A_k| = 1.$$

This evidently completes the proof of Theorem 2.1. □

By setting $\alpha_j = \alpha$ ($j = 1, \dots, m$) in Theorem 2.1, we get

Corollary 2.2. *If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ ($j = 1, \dots, m$), then*

$$(f_1 * \dots * f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)(\alpha-1)^m}{(n-\alpha)^m + (\alpha-1)^m}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$f_j(z) = z + \left(\frac{\alpha-1}{n-\alpha} \right) z^n \quad (j = 1, \dots, m).$$

Next, for the Hadamard product (or convolution) of functions in the class $\mathcal{N}_n^*(z)$, we derive

Theorem 2.3. *If $f_j(z) \in \mathcal{N}_n^*(\alpha_j)$ ($j = 1, \dots, m$), then*

$$(f_1 * \dots * f_m)(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)}{n^{m-1} \prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)}.$$



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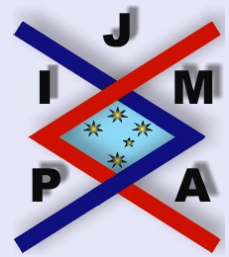


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The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$(2.6) \quad f_j(z) = z + \left(\frac{\alpha_j - 1}{n(n - \alpha_j)} \right) z^n \quad (j = 1, \dots, m).$$

Proof. As in the proof of Theorem 2.1, for $f_1(z) \in \mathcal{N}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{N}_n^*(\alpha_2)$, the following inequality:

$$\sum_{k=n}^{\infty} \left(\frac{k(k - \delta)}{\delta - 1} \right) |a_{k,1}| |a_{k,2}| \leq 1$$

implies that $(f_1 * f_2)(z) \in \mathcal{N}_n^*(\delta)$. Also, in the same manner as in the proof of Theorem 2.1, we obtain

$$(2.7) \quad \delta \geq 1 + \frac{(k - 1)(\alpha_1 - 1)(\alpha_2 - 1)}{k(k - \alpha_1)(k - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)} \\ (k = n, n + 1, n + 2, \dots).$$

The right-hand side of (2.7) takes its maximum value for $k = n$, because it is a decreasing function of $k \geq n$. This shows that $(f_1 * f_2)(z) \in \mathcal{N}_n^*(\delta)$, where

$$\delta = 1 + \frac{(n - 1)(\alpha_1 - 1)(\alpha_2 - 1)}{n(n - \alpha_1)(n - \alpha_2) + (\alpha_1 - 1)(\alpha_2 - 1)}.$$

Now, assuming that

$$(f_1 * \dots * f_m)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma := 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)}{n^{m-1} \prod_{j=1}^m (n - \alpha_j) + \prod_{j=1}^m (\alpha_j - 1)},$$

we have

$$(f_1 * \cdots * f_{m+1})(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\begin{aligned} \beta &= 1 + \frac{(n-1)(\gamma-1)(\alpha_{m+1}-1)}{n(n-\gamma)(n-\alpha_{m+1}) + (\gamma-1)(\alpha_{m+1}-1)} \\ &= 1 + \frac{(n-1) \prod_{j=1}^{m+1} (\alpha_j - 1)}{n^m \prod_{j=1}^{m+1} (n - \alpha_j) + \prod_{j=1}^{m+1} (\alpha_j - 1)}. \end{aligned}$$

Moreover, by taking the functions $f_j(z)$ given by (2.6), we can easily verify that the result of Theorem 2.3 is sharp. \square

By letting $\alpha_j = \alpha$ ($j = 1, \dots, m$) in Theorem 2.3, we obtain

Corollary 2.4. *If $f_j(z) \in \mathcal{N}_n^*(\alpha)$ ($j = 1, \dots, m$), then*

$$(f_1 * \cdots * f_m)(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1)(\alpha-1)^m}{n^{m-1}(n-\alpha)^m + (\alpha-1)^m}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$f_j(z) = z + \left(\frac{\alpha-1}{n(n-\alpha)} \right) z^n \quad (j = 1, \dots, m).$$



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Now we turn to the derivation of the following lemma which will be used in our investigation.

Lemma 2.5. *If $f(z) \in \mathcal{M}_n^*(\alpha)$ and $g(z) \in \mathcal{N}_n^*(\beta)$, then $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$, where*

$$\gamma := 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{n(n-\alpha)(n-\beta) + (\alpha-1)(\beta-1)}.$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \left(\frac{\alpha-1}{n-\alpha}\right) z^n$$

and

$$g(z) = z + \left(\frac{\beta-1}{n(n-\beta)}\right) z^n.$$

Proof. Let

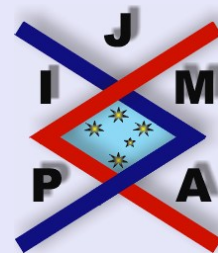
$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{M}_n^*(\alpha)$$

and

$$g(z) = z + \sum_{k=n}^{\infty} b_k z^k \in \mathcal{N}_n^*(\beta).$$

Then, by virtue of Lemma 1.1, it is sufficient to show that

$$\sum_{k=n}^{\infty} \left(\frac{k-\gamma}{\gamma-1}\right) |a_k| |b_k| \leq 1$$



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for $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$. Indeed, since

$$\sum_{k=n}^{\infty} \left(\frac{k - \alpha}{\alpha - 1} \right) |a_k| \leq 1$$

and

$$\sum_{k=n}^{\infty} \left(\frac{k(k - \beta)}{\beta - 1} \right) |b_k| \leq 1,$$

if we assume that

$$\sum_{k=n}^{\infty} \left(\frac{k - \gamma}{\gamma - 1} \right) |a_k| |b_k| \leq \sum_{k=n}^{\infty} \sqrt{\frac{k(k - \alpha)(k - \beta)}{(\alpha - 1)(\beta - 1)}} |a_k| |b_k|,$$

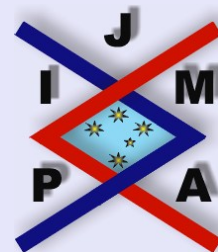
so that

$$\sqrt{|a_k| |b_k|} \leq \left(\frac{\gamma - 1}{k - \gamma} \right) \sqrt{\frac{k(k - \alpha)(k - \beta)}{(\alpha - 1)(\beta - 1)}} \quad (k = n, n + 1, n + 2, \dots)$$

then we prove that $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$. Consequently, if γ satisfies the inequality:

$$\gamma \geq 1 + \frac{(k - 1)(\alpha - 1)(\beta - 1)}{k(k - \alpha)(k - \beta) + (\alpha - 1)(\beta - 1)} \quad (k = n, n + 1, n + 2, \dots),$$

then $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$. Thus it is easy to see that $(f * g)(z) \in \mathcal{M}_n^*(\gamma)$ with γ given already in Lemma 2.5. \square



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By combining Theorem 2.1 and Theorem 2.3 with Lemma 2.5, we arrive at

Theorem 2.6. *If $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ ($j = 1, \dots, p$) and $g_j(z) \in \mathcal{N}_n^*(\beta_j)$ ($j = 1, \dots, q$), then*

$$(f_1 * \dots * f_p * g_1 * \dots * g_q)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{n(n-\alpha)(n-\beta) + (\alpha-1)(\beta-1)},$$

$$(2.8) \quad \alpha = 1 + \frac{(n-1) \prod_{j=1}^p (\alpha_j - 1)}{\prod_{j=1}^p (n - \alpha_j) + \prod_{j=1}^p (\alpha_j - 1)},$$

and

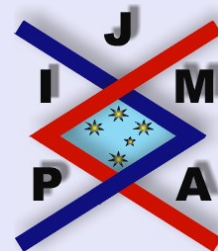
$$(2.9) \quad \beta = 1 + \frac{(n-1) \prod_{j=1}^q (\beta_j - 1)}{n^{q-1} \prod_{j=1}^q (n - \beta_j) + \prod_{j=1}^q (\beta_j - 1)}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, p$) and $g_j(z)$ ($j = 1, \dots, q$) given by

$$(2.10) \quad f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j} \right) z^n \quad (j = 1, \dots, p)$$

and

$$(2.11) \quad g_j(z) = z + \left(\frac{\beta_j - 1}{n(n - \beta_j)} \right) z^n \quad (j = 1, \dots, q).$$



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For $\alpha_j = \alpha$ ($j = 1, \dots, p$) and $\beta_j = \beta$ ($j = 1, \dots, q$), Theorem 2.6 immediately yields

Corollary 2.7. *If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ ($j = 1, \dots, p$) and $g_j(z) \in \mathcal{N}_n^*(\beta)$ ($j = 1, \dots, q$), then*

$$(f_1 * \dots * f_p * g_1 * \dots * g_q)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1)(\alpha-1)^p(\beta-1)^q}{n^q(n-\alpha)^p(n-\beta)^q + (\alpha-1)^p(\beta-1)^q}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, p$) and $g_j(z)$ ($j = 1, \dots, q$) given by

$$(2.12) \quad f_j(z) = z + \left(\frac{\alpha-1}{n-\alpha}\right) z^n \quad (j = 1, \dots, p)$$

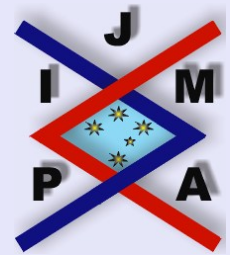
and

$$(2.13) \quad g_j(z) = z + \left(\frac{\beta-1}{n(n-\beta)}\right) z^n \quad (j = 1, \dots, q).$$

We also have the following results analogous to Theorem 2.6 and Corollary 2.7:

Theorem 2.8. *If $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ ($j = 1, \dots, p$) and $g_j(z) \in \mathcal{N}_n^*(\beta_j)$ ($j = 1, \dots, q$), then*

$$(f_1 * \dots * f_p * g_1 * \dots * g_q)(z) \in \mathcal{N}_n^*(\gamma),$$



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where

$$(2.14) \quad \gamma = 1 + \frac{(n-1)(\alpha-1)(\beta-1)}{(n-\alpha)(n-\beta) + (\alpha-1)(\beta-1)},$$

α and β are given by (2.8) and (2.9), respectively. The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, p$) and $g_j(z)$ ($j = 1, \dots, q$) given by (2.10) and (2.11), respectively.

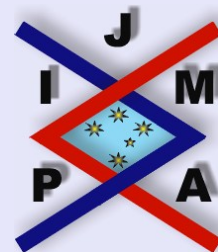
Corollary 2.9. If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ ($j = 1, \dots, p$) and $g_j(z) \in \mathcal{N}_n^*(\beta)$ ($j = 1, \dots, q$), then

$$(f_1 * \dots * f_p * g_1 * \dots * g_q)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$(2.15) \quad \gamma = 1 + \frac{(n-1)(\alpha-1)^p(\beta-1)^q}{n^{q-1}(n-\alpha)^p(n-\beta)^q + (\alpha-1)^p(\beta-1)^q}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, p$) and $g_j(z)$ ($j = 1, \dots, q$) given by (2.12) and (2.13), respectively.



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3. Generalizations of Convolution Properties

For functions $f_j(z)$ ($j = 1, \dots, m$) given by (1.11), the *generalized convolution* (or the *generalized Hadamard product*) is defined here by

$$(3.1) \quad (f_1 \bullet \cdots \bullet f_m)(z) := z + \sum_{k=n}^{\infty} \left(\prod_{j=1}^m (a_{k,j})^{\frac{1}{p_j}} \right) z^k$$

$$\left(\sum_{j=1}^m \frac{1}{p_j} = 1; p_j > 1; j = 1, \dots, m \right).$$

Our first result for the generalized convolution defined by (3.1) is contained in

Theorem 3.1. *If $f_j(z) \in \mathcal{M}_n^*(\alpha_j)$ ($j = 1, \dots, m$), then*

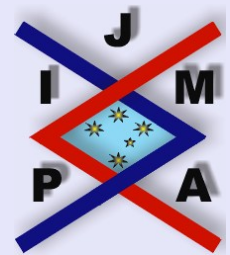
$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where

$$(3.2) \quad \beta = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^m (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}$$

and

$$\frac{n-1}{\prod_{j=1}^m (\alpha_j - 1)} \left(\sum_{j=1}^m \left(\frac{\alpha_j}{p_j} \right) - 1 \right) \geq 2.$$



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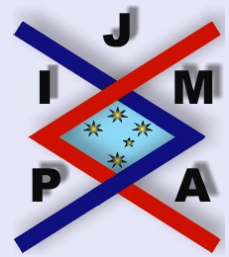


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The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$(3.3) \quad f_j(z) = z + \left(\frac{\alpha_j - 1}{n - \alpha_j} \right) z^n \quad (j = 1, \dots, m).$$

Proof. We use the principle of mathematical induction once again for the proof of Theorem 3.1. Since, for $f_1(z) \in \mathcal{M}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{M}_n^*(\alpha_2)$,

$$\sum_{k=n}^{\infty} \left(\frac{k - \alpha_j}{\alpha_j - 1} \right) |a_{k,j}| \leq 1 \quad (j = 1, 2),$$

we have

$$(3.4) \quad \prod_{j=1}^2 \left(\sum_{k=n}^{\infty} \left\{ \left(\frac{k - \alpha_j}{\alpha_j - 1} \right)^{\frac{1}{p_j}} |a_{k,j}|^{\frac{1}{p_j}} \right\}^{p_j} \right)^{\frac{1}{p_j}} \leq 1.$$

Therefore, by appealing to the Hölder inequality, we find from (3.4) that

$$\sum_{k=n}^{\infty} \left\{ \prod_{j=1}^2 \left(\frac{k - \alpha_j}{\alpha_j - 1} \right)^{\frac{1}{p_j}} |a_{k,j}|^{\frac{1}{p_j}} \right\} \leq 1,$$

which implies that

$$(3.5) \quad \prod_{j=1}^2 |a_{k,j}|^{\frac{1}{p_j}} \leq \prod_{j=1}^2 \left(\frac{\alpha_j - 1}{k - \alpha_j} \right)^{\frac{1}{p_j}} \quad (k = n, n + 1, n + 2, \dots).$$

Now we need to find the smallest δ ($1 < \delta < \frac{n+1}{2}$) which satisfies the inequality:

$$\sum_{k=n}^{\infty} \left(\frac{k-\delta}{\delta-1} \right) \left(\prod_{j=1}^2 |a_{k,j}|^{\frac{1}{p_j}} \right) \leq 1.$$

By virtue of the inequality (3.5), this means that we find the smallest δ ($1 < \delta < \frac{n+1}{2}$) such that

$$\sum_{k=n}^{\infty} \left(\frac{k-\delta}{\delta-1} \right) \left(\prod_{j=1}^2 |a_{k,j}|^{\frac{1}{p_j}} \right) \leq \sum_{k=n}^{\infty} \left(\frac{k-\delta}{\delta-1} \right) \left(\prod_{j=1}^2 \left(\frac{\alpha_j-1}{k-\alpha_j} \right)^{\frac{1}{p_j}} \right) \leq 1,$$

that is, that

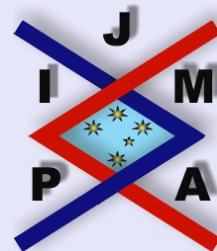
$$\frac{k-\delta}{\delta-1} \leq \prod_{j=1}^2 \left(\frac{k-\alpha_j}{\alpha_j-1} \right)^{\frac{1}{p_j}} \quad (k = n, n+1, n+2, \dots),$$

which yields

$$\delta \geq 1 + \frac{(k-1) \prod_{j=1}^2 (\alpha_j-1)^{\frac{1}{p_j}}}{\prod_{j=1}^2 (k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^2 (\alpha_j-1)^{\frac{1}{p_j}}} \quad (k = n, n+1, n+2, \dots).$$

Let us define

$$h(k) := \frac{k-1}{\prod_{j=1}^2 (k-\alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^2 (\alpha_j-1)^{\frac{1}{p_j}}} \quad (k \geq n).$$



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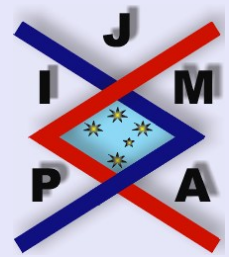


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Then, for the numerator $N(k)$ of $h'(k)$, we have

$$\begin{aligned} N(k) &= (\alpha_1 - 1)^{\frac{1}{p_1}} (\alpha_2 - 1)^{\frac{1}{p_2}} - (k - \alpha_1)^{\frac{1}{p_1} - 1} (k - \alpha_2)^{\frac{1}{p_2} - 1} \\ &\quad \cdot \left(\frac{k - 1}{p_1} (k - \alpha_2) + \frac{k - 1}{p_2} (k - \alpha_1) - (k - \alpha_1)(k - \alpha_2) \right) \\ &\leq (\alpha_1 - 1)^{\frac{1}{p_1}} (\alpha_2 - 1)^{\frac{1}{p_2}} - (k - \alpha_1)^{\frac{1}{p_1} - 1} (k - \alpha_2)^{\frac{1}{p_2} - 1} \\ &\quad \cdot \left(\frac{1}{p_1} (k - \alpha_2) (\alpha_1 - 1) + \frac{1}{p_2} (k - \alpha_1) (\alpha_2 - 1) \right). \end{aligned}$$

Since $k \geq n$ and $1 < \alpha_j < \frac{n+1}{2}$, we note that $k - \alpha_j > \alpha_j - 1$ ($j = 1, 2$). This implies that

$$\begin{aligned} N(k) &\leq - (k - \alpha_1)^{\frac{1}{p_1} - 1} (k - \alpha_2)^{\frac{1}{p_2} - 1} \\ &\quad \cdot \left(\frac{1}{p_1} (k - \alpha_2) (\alpha_1 - 1) + \frac{1}{p_2} (k - \alpha_1) (\alpha_2 - 1) - (\alpha_1 - 1) (\alpha_2 - 1) \right) \\ &\leq - (k - \alpha_1)^{\frac{1}{p_1} - 1} (k - \alpha_2)^{\frac{1}{p_2} - 1} \\ &\quad \cdot \left(\frac{1}{p_1} (n - \alpha_2) (\alpha_1 - 1) + \frac{1}{p_2} (n - \alpha_1) (\alpha_2 - 1) - (\alpha_1 - 1) (\alpha_2 - 1) \right) \\ &= - (k - \alpha_1)^{\frac{1}{p_1} - 1} (k - \alpha_2)^{\frac{1}{p_2} - 1} \\ &\quad \cdot \left\{ (n - 1) \left(\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} - 1 \right) - 2 (\alpha_1 - 1) (\alpha_2 - 1) \right\} \\ &\leq 0, \end{aligned}$$

by means of the condition of Theorem 3.1. This implies that $h(k)$ is decreasing

for $k \geq n$. Consequently, we have

$$\delta = 1 + \frac{(n-1) \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^2 (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Thus the assertion of Theorem 3.1 holds true when $m = 2$.

Next we suppose that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^m (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Then, clearly, the first half of the above proof implies that

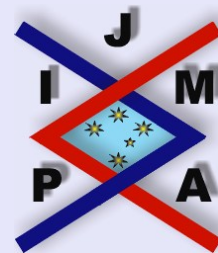
$$(f_1 \bullet \cdots \bullet f_{m+1})(z) \in \mathcal{M}_n^*(\beta)$$

with

$$\beta = 1 + \frac{(n-1)(\gamma-1)^{1-\frac{1}{p_{m+1}}} (\alpha_{m+1}-1)^{\frac{1}{p_{m+1}}}}{(n-\gamma)^{1-\frac{1}{p_{m+1}}} (n-\alpha_{m+1})^{\frac{1}{p_{m+1}}} + (\gamma-1)^{1-\frac{1}{p_{m+1}}} (\alpha_{m+1}-1)^{\frac{1}{p_{m+1}}}}.$$

It is easy to verify that

$$\beta = 1 + \frac{(n-1) \prod_{j=1}^{m+1} (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m+1} (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m+1} (\alpha_j - 1)^{\frac{1}{p_j}}}.$$



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Thus, by the principle of mathematical induction, we conclude that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\beta),$$

where β is given by (3.2).

Finally, by taking the functions $f_j(z)$ ($j = 1, \dots, m$) given by (3.3), we have

$$(f_1 \bullet \cdots \bullet f_m)(z) = z + \left(\prod_{j=1}^m \left(\frac{\alpha_j - 1}{n - \alpha_j} \right) \right) z^n,$$

which shows that

$$\left(\frac{n - \beta}{\beta - 1} \right) \left(\prod_{j=1}^m \left(\frac{\alpha_j - 1}{n - \alpha_j} \right)^{\frac{1}{p_j}} \right) = 1.$$

Therefore, Theorem 3.1 is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by (3.3). This completes the proof of Theorem 3.1. \square

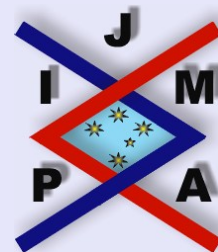
By putting $\alpha_j = \alpha$ ($j = 1, \dots, m$) in Theorem 3.1, we obtain

Corollary 3.2. *If $f_j(z) \in \mathcal{M}_n^*(\alpha)$ ($j = 1, \dots, m$), then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{M}_n^*(\alpha).$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$f_j(z) = z + \left(\frac{\alpha - 1}{n - \alpha} \right) z^n \quad (j = 1, \dots, m).$$



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Similarly, for the generalized convolution defined by (3.1) for functions in the class $\mathcal{N}_n^*(\alpha)$, we derive

Theorem 3.3. *If $f_j(z) \in \mathcal{N}_n^*(\alpha_j)$ ($j = 1, \dots, m$), then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\beta),$$

where

$$(3.6) \quad \beta = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^m (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}.$$

The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

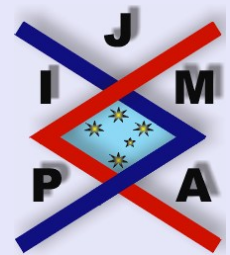
$$(3.7) \quad f_j(z) = z + \left(\frac{\alpha_j - 1}{n(n - \alpha_j)} \right) z^n \quad (j = 1, \dots, m).$$

Proof. By applying the same technique as in the proof of Theorem 3.1, we find that $(f_1 \bullet f_2)(z) \in \mathcal{N}_n^*(\delta)$, where

$$\delta \geq 1 + \frac{(k-1) \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^2 (k - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}} \quad (k = n, n+1, n+2, \dots),$$

for $f_1(z) \in \mathcal{N}_n^*(\alpha_1)$ and $f_2(z) \in \mathcal{N}_n^*(\alpha_2)$. Therefore, we have

$$(3.8) \quad \delta = 1 + \frac{(n-1) \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^2 (n - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^2 (\alpha_j - 1)^{\frac{1}{p_j}}}.$$



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Furthermore, by assuming that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\gamma),$$

where

$$\gamma = 1 + \frac{(n-1) \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^m (k - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^m (\alpha_j - 1)^{\frac{1}{p_j}}},$$

we can show that

$$(f_1 \bullet \cdots \bullet f_{m+1})(z) \in \mathcal{N}_n^*(\beta),$$

where

$$\beta = 1 + \frac{(n-1) \prod_{j=1}^{m+1} (\alpha_j - 1)^{\frac{1}{p_j}}}{\prod_{j=1}^{m+1} (k - \alpha_j)^{\frac{1}{p_j}} + \prod_{j=1}^{m+1} (\alpha_j - 1)^{\frac{1}{p_j}}}.$$

Therefore, using the principle of mathematical induction once again, we conclude that

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\beta)$$

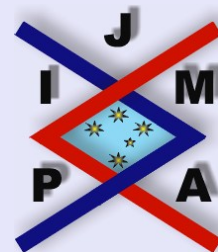
with β given by (3.6).

It is clear that the result of Theorem 3.3 is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by (3.7). \square

Finally, by letting $\alpha_j = \alpha$ ($j = 1, \dots, m$) in Theorem 3.3, we deduce

Corollary 3.4. *If $f_j(z) \in \mathcal{N}_n^*(\alpha)$ ($j = 1, \dots, m$), then*

$$(f_1 \bullet \cdots \bullet f_m)(z) \in \mathcal{N}_n^*(\alpha).$$



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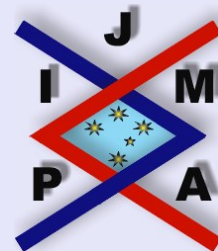
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The result is sharp for the functions $f_j(z)$ ($j = 1, \dots, m$) given by

$$(3.9) \quad f_j(z) = z + \left(\frac{\alpha - 1}{n(n - \alpha)} \right) z^n \quad (j = 1, \dots, m).$$



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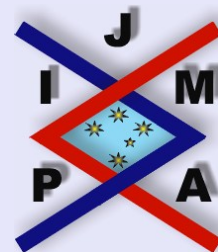
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References

- [1] J. NISHIWAKI AND S. OWA, Coefficient inequalities for certain analytic functions, *Internat. J. Math. and Math. Sci.*, **29** (2002), 285–290.
- [2] S. OWA, The quasi-Hadamard products of certain analytic functions, in *Current Topics in Analytic Function Theory* (H.M. Srivastava and S. Owa, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992, pp. 234–251.
- [3] S. SAITA AND S. OWA, Convolutions of certain analytic functions, *Algebras Groups Geom.*, **18** (2001), 375–384.
- [4] B.A. URALEGADDI AND A.R. DESAI, Convolutions of univalent functions with positive coefficients, *Tamkang J. Math.*, **29** (1998), 279–285.
- [5] B.A. URALEGADDI, M.D. GANIGI AND S.M. SARANGI, Univalent functions with positive coefficients, *Tamkang J. Math.*, **25** (1994), 225–230.
- [6] H.M. SRIVASTAVA AND S. OWA (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.



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