

## A COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Abstract

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## Abstract

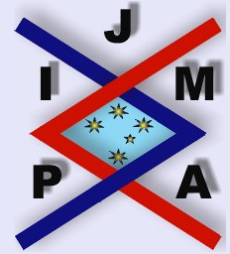
In the present investigation, we obtain the Fekete-Szegő inequality for a certain normalized analytic function  $f(z)$  defined on the open unit disk for which  $1 + \frac{1}{b} \left[ \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right]$  ( $\alpha \geq 0$  and  $b \neq 0$ , a complex number) lies in a region starlike with respect to 1 and symmetric with respect to real axis. Also certain application of the main result for a class of functions of complex order defined by convolution is given. The motivation of this paper is to give a generalization of the Fekete-Szegő inequalities for subclasses of starlike functions of complex order.

*2000 Mathematics Subject Classification:* Primary 30C45.

*Key words:* Starlike functions of complex order, Convex functions of complex order, Subordination, Fekete-Szegő inequality.

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# 1. Introduction

Let  $\mathcal{A}$  denote the class of all analytic functions  $f(z)$  of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} / |z| < 1\})$$

and  $\mathcal{S}$  be the subclass of  $\mathcal{A}$  consisting of univalent functions. Let  $\phi(z)$  be an analytic function with positive real part on  $\Delta$  with  $\phi(0) = 1$ ,  $\phi'(0) > 0$  which maps the unit disk  $\Delta$  onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let  $S^*(\phi)$  be the class of functions in  $f \in \mathcal{S}$  for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta)$$

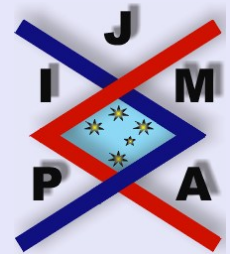
and  $C(\phi)$  be the class of functions  $f \in \mathcal{S}$  for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where  $\prec$  denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [4]. They have obtained the Fekete-Szegő inequality for functions in the class  $C(\phi)$ . Since  $f \in C(\phi)$  iff  $zf'(z) \in S^*(\phi)$ , we get the Fekete-Szegő inequality for functions in the class  $S^*(\phi)$ .

The class  $S_b^*(\phi)$  consists of all analytic functions  $f \in \mathcal{A}$  satisfying

$$1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z)$$



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and the class  $C_b(\phi)$  consists of functions  $f \in \mathcal{A}$  satisfying

$$1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \prec \phi(z).$$

These classes were defined and studied by Ravichandran et al. [7]. They have obtained the Fekete-Szegő inequalities for functions in these classes.

For a brief history of the Fekete-Szegő problem for the class of starlike, convex and close to convex functions, see the recent paper by Srivastava et al. [10].

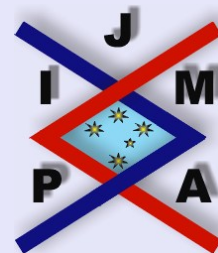
In the present paper, we obtain the Fekete-Szegő inequality for functions in a more general class  $M_{\alpha,b}(\phi)$  of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class  $M_{\alpha,b}^\lambda(\phi)$  of functions defined by fractional derivatives.

**Definition 1.1.** Let  $b \neq 0$  be a complex number. Let  $\phi(z)$  be an analytic function with positive real part on  $\Delta$  with  $\phi(0) = 1$ ,  $\phi'(0) > 0$  which maps the unit disk  $\Delta$  onto a region starlike with respect to 1 which is symmetric with respect to the real axis. A function  $f \in \mathcal{A}$  is in the class  $M_{\alpha,b}(\phi)$  if

$$1 + \frac{1}{b} \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right) \prec \phi(z) \quad (\alpha \geq 0).$$

For fixed  $g \in \mathcal{A}$ , we define the class  $M_{\alpha,b}^g(\phi)$  to be the class of functions  $f \in \mathcal{A}$  for which  $(f * g) \in M_{\alpha,b}(\phi)$ .

To prove our result, we need the following:




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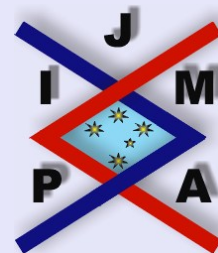
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**Lemma 1.1 ([7]).** *If  $p(z) = 1 + c_1z + c_2z^2 + \dots$  is a function with positive real part, then for any complex number  $\mu$ ,*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

*and the result is sharp for the functions given by*

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$



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## 2. The Fekete-Szegö Problem

Our main result is the following:

**Theorem 2.1.** Let  $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots$ . If  $f(z)$  given by (1.1) belongs to  $M_{\alpha,b}(\phi)$ , then

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+3\alpha)} \max \left\{ 1, \left| \frac{B_2}{B_1} + \left[ \frac{(1+2\alpha) - 2\mu(1+3\alpha)}{(1+2\alpha)^2} \right] bB_1 \right| \right\}.$$

The result is sharp.

*Proof.* If  $f(z) \in M_{\alpha,b}(\phi)$ , then there is a Schwarz function  $w(z)$ , analytic in  $\Delta$  with  $w(0) = 0$  and  $|w(z)| < 1$  in  $\Delta$  such that

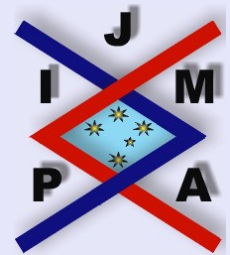
$$(2.1) \quad 1 + \frac{1}{b} \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right) = \phi(w(z)).$$

Define the function  $p_1(z)$  by

$$(2.2) \quad p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \dots$$

Since  $w(z)$  is a Schwarz function, we see that  $\Re p_1(z) > 0$  and  $p_1(0) = 1$ . Define the function  $p(z)$  by

$$(2.3) \quad p(z) := 1 + \frac{1}{b} \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right) = 1 + b_1z + b_2z^2 + \dots$$



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In view of the equations (2.1), (2.2), (2.3), we have

$$(2.4) \quad p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right)$$

and from this equation (2.4), we obtain

$$(2.5) \quad b_1 = \frac{1}{2}B_1c_1$$

and

$$(2.6) \quad b_2 = \frac{1}{2}B_1 \left( c_2 - \frac{1}{2}c_1^2 \right) + \frac{1}{4}B_2c_1^2.$$

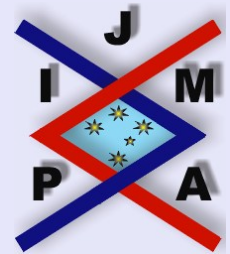
From equation (2.3), we obtain

$$\begin{aligned} (1 + 2\alpha)a_2 &= bb_1, \\ (2 + 6\alpha)a_3 &= bb_2 + (1 + 2\alpha)a_2^2 \end{aligned}$$

or equivalently we have

$$(2.7) \quad a_2 = \frac{bb_1}{1 + 2\alpha},$$

$$(2.8) \quad a_3 = \frac{1}{2 + 6\alpha} \left[ bb_2 + \frac{b^2b_1^2}{1 + 2\alpha} \right].$$




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Applying (2.5) in (2.7) and (2.5), (2.6) in (2.8), we have

$$a_2 = \frac{bB_1c_1}{2(1+2\alpha)},$$

$$a_3 = \frac{bB_1c_2}{4(1+3\alpha)} + \frac{c_1^2}{8(1+3\alpha)} \left[ \frac{b^2B_1^2}{1+2\alpha} - b(B_1 - B_2) \right].$$

Therefore we have

$$(2.9) \quad a_3 - \mu a_2^2 = \frac{bB_1}{4(1+3\alpha)} \{c_2 - vc_1^2\},$$

where

$$v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \left( \frac{2\mu(1+3\alpha) - (1+2\alpha)}{(1+2\alpha)^2} \right) bB_1 \right].$$

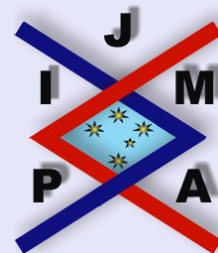
Our result now follows by an application of Lemma 1.1. The result is sharp for the function defined by

$$1 + \frac{1}{b} \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right) = \phi(z^2)$$

and

$$1 + \frac{1}{b} \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1 \right) = \phi(z).$$

□



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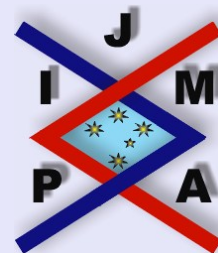
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**Example 2.1.** By taking  $b = (1 - \beta)e^{-i\lambda} \cos \lambda$ ,  $\phi(z) = \frac{1+z}{1-z}$ , we obtain the following sharp inequality

$$|a_3 - \mu a_2^2| \leq \frac{(1 - \beta) \cos \lambda}{1 + 3\alpha} \times \max \left\{ 1, \left| e^{i\lambda} - 2 \left[ \frac{2\mu(1 + 3\alpha) - (1 + 2\alpha)}{(1 + 2\alpha)^2} \right] (1 - \beta) \cos \lambda \right| \right\}.$$

**Remark 1.** When  $\alpha = 0$ , Example 2.1 reduces to a result of [7] for  $\lambda$ -spirallike function  $f(z)$  of order  $\beta$ .



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### 3. Application to Functions Defined by Fractional Derivatives

In order to introduce the class  $M_{\alpha,b}^\lambda(\phi)$ , we need the following:

**Definition 3.1.** (See [5, 6]; see also [11, 12]). Let the function  $f(z)$  be analytic in a simply connected region of the  $z$ -plane containing the origin. The fractional derivative of  $f$  of order  $\lambda$  is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1)$$

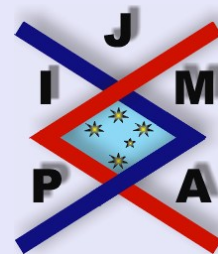
where the multiplicity of  $(z-\zeta)^\lambda$  is removed by requiring  $\log(z-\zeta)$  to be real when  $z-\zeta > 0$ .

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator  $\Omega^\lambda : \mathcal{A} \rightarrow \mathcal{A}$  defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \quad (\lambda \neq 2, 3, 4, \dots).$$

The class  $M_{\alpha,b}^\lambda(\phi)$  consists of functions  $f \in \mathcal{A}$  for which  $\Omega^\lambda f \in M_{\alpha,b}(\phi)$ . Note that  $M_{0,b}^0(\phi) = S_b^*(\phi)$  and  $M_{0,1}^0(\phi) = S^*(\phi)$ . Also  $M_{\alpha,b}^\lambda(\phi)$  is the special case of the class  $M_{\alpha,b}^g(\phi)$  when

$$(3.1) \quad g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n.$$



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Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_{\alpha,b}^g(\phi)$$

if and only if

$$(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha,b}(\phi),$$

we obtain the coefficient estimate for functions in the class  $M_{\alpha,b}^g(\phi)$ , from the corresponding estimate for functions in the class  $M_{\alpha,b}(\phi)$ . Applying Theorem 2.1 for the function  $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots$ , we get the following theorem after an obvious change of the parameter  $\mu$ :

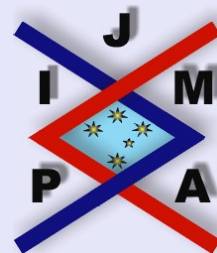
**Theorem 3.1.** *Let the function  $\phi(z)$  be given by  $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ . If  $f(z)$  given by (1.1) belongs to  $M_{\alpha,b}^g(\phi)$ , then*

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \frac{B_1 |b|}{2g_3(1+3\alpha)} \max \left\{ 1, \left| \frac{B_2}{B_1} + \left[ \frac{(1+2\alpha)g_2^2 - 2\mu(1+3\alpha)g_3}{(1+2\alpha)^2 g_2^2} \right] b B_1 \right| \right\}. \end{aligned}$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$




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we have

$$(3.2) \quad g_2 = \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}$$

and

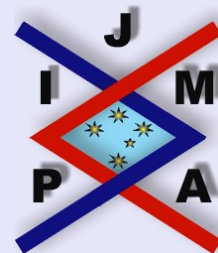
$$(3.3) \quad g_3 = \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}.$$

For  $g_2$  and  $g_3$  given by (3.2) and (3.3), Theorem 3.1 reduces to the following:

**Theorem 3.2.** *Let the function  $\phi(z)$  be given by  $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots$ . If  $f(z)$  given by (1.1) belongs to  $M_{\alpha,b}^\lambda(\phi)$ , then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|(2-\lambda)(3-\lambda)}{12(1+3\alpha)} \\ \times \max \left\{ 1, \left| \frac{B_2}{B_1} + \left[ \frac{(1+2\alpha)(3-\lambda) - 3\mu(1+3\alpha)(2-\lambda)}{(3-\lambda)(1+2\alpha)^2} b B_1 \right] \right| \right\}.$$

*The result is sharp.*



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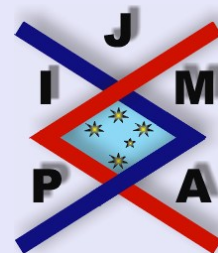
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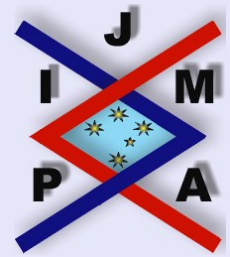
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