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# A COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. In the present investigation, we obtain the Fekete-Szegö inequality for a certain normalized analytic function $f(z)$ defined on the open unit disk for which $1+\frac{1}{b}\left[\frac{zf'(z)+\alpha z^2 f''(z)}{f(z)}-1\right]$ ($\alpha \geq 0$ and $b \neq 0$, a complex number) lies in a region starlike with respect to 1 and symmetric with respect to real axis. Also certain application of the main result for a class of functions of complex order defined by convolution is given. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities for subclasses of starlike functions of complex order.

*Key words and phrases:* Starlike functions of complex order, Convex functions of complex order, Subordination, Fekete-Szegö inequality.

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## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C}/|z| < 1\}) \tag{1.1}$$

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and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta)$$

and $C(\phi)$ be the class of functions $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [4]. They have obtained the Fekete-Szegö inequality for functions in the class $C(\phi)$. Since $f \in C(\phi)$ iff $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$.

The class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z)$$

and the class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \prec \phi(z).$$

These classes were defined and studied by Ravichandran et al. [7]. They have obtained the Fekete-Szegö inequalities for functions in these classes.

For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close to convex functions, see the recent paper by Srivastava et al. [10].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_{\alpha,b}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M_{\alpha,b}^{\lambda}(\phi)$ of functions defined by fractional derivatives.

**Definition 1.1.** Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. A function $f \in \mathcal{A}$ is in the class $M_{\alpha,b}(\phi)$ if

$$1 + \frac{1}{b}\left(\frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1\right) \prec \phi(z) \quad (\alpha \geq 0).$$

For fixed $g \in \mathcal{A}$, we define the class $M_{\alpha,b}^{g}(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M_{\alpha,b}(\phi)$.

To prove our result, we need the following:

**Lemma 1.1** ([7]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then for any complex number $\mu$,*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

*and the result is sharp for the functions given by*

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$

## 2. THE FEKETE-SZEGÖ PROBLEM

Our main result is the following:

**Theorem 2.1.** *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to* $M_{\alpha,b}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+3\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{(1+2\alpha) - 2\mu(1+3\alpha)}{(1+2\alpha)^2}\right] bB_1\right|\right\}.$$

*The result is sharp.*

*Proof.* If $f(z) \in M_{\alpha,b}(\phi)$, then there is a Schwarz function $w(z)$, analytic in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$ in $\Delta$ such that

$$1 + \frac{1}{b}\left(\frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1\right) = \phi(w(z)). \tag{2.1}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.2}$$

Since $w(z)$ is a Schwarz function, we see that $\mathcal{R} p_1(z) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := 1 + \frac{1}{b}\left(\frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} - 1\right) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.3}$$

In view of the equations (2.1), (2.2), (2.3), we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.4}$$

and from this equation (2.4), we obtain

$$b_1 = \frac{1}{2} B_1 c_1 \tag{2.5}$$

and

$$b_2 = \frac{1}{2} B_1 \left(c_2 - \frac{1}{2} c_1^2\right) + \frac{1}{4} B_2 c_1^2. \tag{2.6}$$

From equation (2.3), we obtain

$$(1 + 2\alpha) a_2 = b b_1,$$
$$(2 + 6\alpha) a_3 = b b_2 + (1 + 2\alpha) a_2^2$$

or equivalently we have

$$a_2 = \frac{b b_1}{1 + 2\alpha}, \tag{2.7}$$

$$a_3 = \frac{1}{2 + 6\alpha}\left[b b_2 + \frac{b^2 b_1^2}{1 + 2\alpha}\right]. \tag{2.8}$$

Applying (2.5) in (2.7) and (2.5), (2.6) in (2.8), we have

$$a_2 = \frac{bB_1c_1}{2(1+2\alpha)},$$
$$a_3 = \frac{bB_1c_2}{4(1+3\alpha)} + \frac{c_1^2}{8(1+3\alpha)}\left[\frac{b^2B_1^2}{1+2\alpha} - b(B_1-B_2)\right].$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4(1+3\alpha)}\left\{c_2 - vc_1^2\right\}, \tag{2.9}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{2\mu(1+3\alpha)-(1+2\alpha)}{(1+2\alpha)^2}\right)bB_1\right].$$

Our result now follows by an application of Lemma 1.1. The result is sharp for the function defined by

$$1 + \frac{1}{b}\left(\frac{zf'(z)+\alpha z^2f''(z)}{f(z)} - 1\right) = \phi(z^2)$$

and

$$1 + \frac{1}{b}\left(\frac{zf'(z)+\alpha z^2f''(z)}{f(z)} - 1\right) = \phi(z).$$

□

**Example 2.1.** By taking $b = (1-\beta)e^{-i\lambda}\cos\lambda$, $\phi(z) = \frac{1+z}{1-z}$, we obtain the following sharp inequality

$$|a_3 - \mu a_2^2| \le \frac{(1-\beta)\cos\lambda}{1+3\alpha} \times \max\left\{1, \left|e^{i\lambda} - 2\left[\frac{2\mu(1+3\alpha)-(1+2\alpha)}{(1+2\alpha)^2}\right](1-\beta)\cos\lambda\right|\right\}.$$

**Remark 2.2.** When $\alpha = 0$, Example 2.1 reduces to a result of [7] for $\lambda$-spirallike function $f(z)$ of order $\beta$.

## 3. APPLICATION TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES

In order to introduce the class $M^\lambda_{\alpha,b}(\phi)$, we need the following:

**Definition 3.1.** (See [5, 6]; see also [11, 12]). Let the function $f(z)$ be analytic in a simply connected region of the $z$-plane containing the origin. The fractional derivative of $f$ of order $\lambda$ is defined by

$$D^\lambda_z f(z) := \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta \quad (0 \le \lambda < 1)$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda D^\lambda_z f(z), \quad (\lambda \ne 2, 3, 4, \dots).$$

The class $M^\lambda_{\alpha,b}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M_{\alpha,b}(\phi)$. Note that $M^0_{0,b}(\phi) = S^*_b(\phi)$ and $M^0_{0,1}(\phi) = S^*(\phi)$. Also $M^\lambda_{\alpha,b}(\phi)$ is the special case of the class $M^g_{\alpha,b}(\phi)$ when

$$g(z) = z + \sum_{n=2}^\infty \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n. \tag{3.1}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M^g_{\alpha,b}(\phi)$$

if and only if

$$(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha,b}(\phi),$$

we obtain the coefficient estimate for functions in the class $M^g_{\alpha,b}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,b}(\phi)$. Applying Theorem 2.1 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get the following theorem after an obvious change of the parameter $\mu$:

**Theorem 3.1.** *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^g_{\alpha,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1|b|}{2g_3(1+3\alpha)} \max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{(1+2\alpha)g_2^2 - 2\mu(1+3\alpha)g_3}{(1+2\alpha)^2 g_2^2}\right] bB_1\right|\right\}.$$

*The result is sharp.*

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 = \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 = \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For $g_2$ and $g_3$ given by (3.2) and (3.3), Theorem 3.1 reduces to the following:

**Theorem 3.2.** *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^\lambda_{\alpha,b}(\phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1|b|(2-\lambda)(3-\lambda)}{12(1+3\alpha)} \times \max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{(1+2\alpha)(3-\lambda) - 3\mu(1+3\alpha)(2-\lambda)}{(3-\lambda)(1+2\alpha)^2} bB_1\right]\right|\right\}.$$

*The result is sharp.*

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