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SHARP ERROR BOUNDS FOR THE TRAPEZOIDAL RULE AND SIMPSON'S RULE

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ABSTRACT. We give error bounds for the trapezoidal rule and Simpson's rule for "rough" continuous functions—for instance, functions which are Hölder continuous, of bounded variation, or which are absolutely continuous and whose derivative is in L^p. These differ considerably from the classical results, which require the functions to have continuous higher derivatives. Further, we show that our results are sharp, and in many cases precisely characterize the functions for which equality holds. One consequence of these results is that for rough functions, the error estimates for the trapezoidal rule are better (that is, have smaller constants) than those for Simpson's rule.

Key words and phrases: Numerical integration, Trapezoidal rule, Simpson's rule.

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1. INTRODUCTION

1.1. Overview of the Problem. Given a finite interval $I = [a, b]$ and a continuous function $f : I \to \mathbb{R}$, there are two elementary methods for approximating the integral

$$\int_I f(x)\,dx,$$

the trapezoidal rule and Simpson's rule. Partition the interval I into n intervals of equal length with endpoints $x_i = a + i|I|/n$, $0 \le i \le n$. Then the trapezoidal rule approximates the integral

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with the sum

$$T_n(f) = \frac{|I|}{2n}\big(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\big). \tag{1.1}$$

Similarly, if we partition I into $2n$ intervals, Simpson's rule approximates the integral with the sum

$$S_{2n}(f) = \frac{|I|}{6n}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \cdots + 4f(x_{2n-1}) + f(x_{2n})\big). \tag{1.2}$$

Both approximation methods have well-known error bounds in terms of higher derivatives:

$$E_n^T(f) = \left|T_n(f) - \int_I f(x)\,dx\right| \le \frac{|I|^3\|f''\|_\infty}{12n^2},$$
$$E_{2n}^S(f) = \left|S_{2n}(f) - \int_I f(x)\,dx\right| \le \frac{|I|^5\|f^{(4)}\|_\infty}{180n^4}.$$

(See, for example, Ralston [13].)

Typically, these estimates are derived using polynomial approximation, which leads naturally to the higher derivatives on the righthand sides. However, the assumption that f is not only continuous but has continuous higher order derivatives means that we cannot use them to estimate directly the error when approximating the integral of such a well-behaved function as $f(x) = \sqrt{x}$ on $[0,1]$. (It is possible to use them indirectly by approximating f with a smooth function; see, for example, Davis and Rabinowitz [3].)

In this paper we consider the problem of approximating the error $E_n^T(f)$ and $E_{2n}^S(f)$ for continuous functions which are much rougher. We prove estimates of the form

$$E_n^T(f), E_{2n}^S(f) \le c_n\|f\|; \tag{1.3}$$

where the constants c_n are independent of f, $c_n \to 0$ as $n \to \infty$, and $\|\cdot\|$ denotes the norm in one of several Banach function spaces which are embedded in $C(I)$. In particular, in order (roughly) of increasing smoothness, we consider functions in the following spaces:

- $\Lambda_\alpha(I)$, $0 < \alpha \le 1$: Hölder continuous functions with norm
$$\|f\|_{\Lambda_\alpha} = \sup_{x,y\in I}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$
- $CBV(I)$: continuous functions of bounded variation, with norm
$$\|f\|_{BV,I} = \sup_\Gamma \sum_{i=1}^n |f(x_i) - f(x_{i-1})|,$$
where $a = x_0 < x_1 < \cdots < x_n = b$, and the supremum is taken over all such partitions $\Gamma = \{x_i\}$ of I.
- $W_1^p(I)$, $1 \le p \le \infty$: absolutely continuous functions such that $f' \in L^p(I)$, with norm $\|f'\|_{p,I}$.
- $W_1^{pq}(I)$, $1 \le p \le \infty$: absolutely continuous functions such that f' is in the Lorentz space $L^{pq}(I)$, with norm
$$||f'||_{pq,I} = \left(\int_0^\infty t^{q/p-1}(f')^{*q}(t)dt\right)^{1/q} = \left(\frac{p}{q}\int_0^\infty \lambda_{f'}(y)^{q/p}d(y^q)\right)^{1/q}.$$
(For precise definitions, see the proof of Theorem 1.15 in Section 4 below, or see Stein and Weiss [17].)
- $W_2^p(I)$, $1 \le p \le \infty$: differentiable functions such that f' is absolutely continuous and $f'' \in L^p(I)$, with norm $\|f''\|_{p,I}$.

(Properly speaking, some of these norms are in fact semi-norms. For our purposes we will ignore this distinction.)

In order to prove inequalities like (1.3), it is necessary to make some kind of smoothness assumption, since the supremum norm on $C(I)$ is not adequate to produce this kind of estimate. For example, consider the family of functions $\{f_n\}$ defined on $[0,1]$ as follows: on $[0,1/n]$ let the graph of f_n be the trapezoid with vertices $(0,1),(1/n^2,0),(1/n-1/n^2,0),(1/n,1)$, and extend periodically with period $1/n$. Then $E_n(f_n)=1-1/n$ but $||f_n||_{\infty,I}=1$.

Our proofs generally rely on two simple techniques, albeit applied in a sometimes clever fashion: integration by parts and elementary inequalities. The idea of applying integration by parts to this problem is not new, and seems to date back to von Mises [19] and before him to Peano [11]. (This is described in the introduction to Ghizzetti and Ossicini [7].) But our results themselves are either new or long-forgotten. After searching the literature, we found the following papers which contain related results, though often with more difficult proofs and weaker bounds: Pólya and Szegö [12], Stroud [18], Rozema [15], Rahman and Schmeisser [14], Büttgenbach *et al.* [2], and Dragomir [5]. Also, as the final draft of this paper was being prepared we learned that Dragomir, *et al.* [4] had independently discovered some of the same results with similar proofs. (We would like to thank A. Fiorenza for calling our attention to this paper.)

1.2. **Statement of Results.** Here we state our main results and make some comments on their relationship to known results and on their proofs. Hereafter, given a function f, define $f_r(x)=f(x)-rx$, $r\in\mathbb{R}$, and $f_s(x)=f(x)-s(x)$, where s is any polynomial of degree at most three such that $s(0)=0$. Also, in the statements of the results, the intervals J_i and the points c_i, $1\le i\le n$, are defined in terms of the partition for the trapezoidal rule, and the intervals I_i and points a_i and b_i are defined in terms of the partition for Simpson's rule. Precise definitions are given in Section 2 below.

Theorem 1.1. *Let $f\in\Lambda_\alpha(I)$, $0<\alpha\le 1$. Then for $n\ge 1$,*

$$E_n^T(f)\le \frac{|I|^{1+\alpha}}{(1+\alpha)2^\alpha n^\alpha}\inf_r \|f_r\|_{\Lambda_\alpha}, \tag{1.4}$$

and

$$E_{2n}^S(f)\le \frac{2(1+2^{\alpha+1})|I|^{1+\alpha}}{(1+\alpha)6^{1+\alpha}n^\alpha}\inf_s \|f_s\|_{\Lambda_\alpha}, \tag{1.5}$$

Further, inequality (1.4) *is sharp, in the sense that for each n there exists a function f such that equality holds.*

Remark 1.2. We conjecture that inequality (1.5) is sharp, but we have been unable to construct an example which shows this.

Remark 1.3. Inequality (1.4) should be compared to the examples of increasing functions in Λ_α, $0<\alpha<1$, constructed by Dubuc and Topor [6], for which $E_n^T(f)=O(1/n)$.

In the special case of Lipschitz functions (i.e., functions in Λ_1) Theorem 1.1 can be improved.

Corollary 1.4. *Let $f\in\Lambda_1$. Then for $n\ge 1$,*

$$|E_n^T(f)|\le \frac{|I|^2}{8n}(M-m), \tag{1.6}$$

$$|E_{2n}^S(f)|\le \frac{5|I|^2}{72n}(M-m), \tag{1.7}$$

where $M = \sup_I f'$, $m = \inf_I f'$. Furthermore, equality holds in (1.6) *if and only if f is such that its derivative is given by*

$$f'(t) = \pm \left(\sum_{i=1}^{n} \left(M\chi_{J_i^+}(t) + m\chi_{J_i^-}(t) \right) \right). \tag{1.8}$$

Similarly, equality holds in (1.7) *if and only if*

$$f'(t) = \pm \left(\sum_{i=1}^{n} \left(m\chi_{I_i^1}(t) + M\chi_{I_i^2}(t) + m\chi_{I_i^3}(t) + M\chi_{I_i^4}(t) \right) \right). \tag{1.9}$$

Remark 1.5. Inequality (1.6) was first proved by Kim and Neugebauer [9] as a corollary to a theorem on integral means.

Theorem 1.6. *Let $f \in CBV(I)$. Then for $n \geq 1$,*

$$E_n^T(f) \leq \frac{|I|}{2n} \inf_r \|f_r\|_{BV,I}, \tag{1.10}$$

and

$$E_{2n}^S(f) \leq \frac{|I|}{3n} \inf_r \|f_r\|_{BV,I}. \tag{1.11}$$

Both inequalities are sharp, in the sense that for each n there exists a sequence of functions which show that the given constant is the best possible. Further, in each equality holds if and only if both sides are equal to zero.

Remark 1.7. Pölya and Szegö [12] proved an inequality analogous to (1.10) for rectangular approximations. However, they do not show that their result is sharp.

Theorem 1.8. *Let $f \in W_1^p(I)$, $1 \leq p \leq \infty$. Then for all $n \geq 1$,*

$$E_n^T(f) \leq \frac{|I|^{1+1/p'}}{2n(p'+1)^{1/p'}} \inf_r \|f_r'\|_{p,I}, \tag{1.12}$$

and

$$E_n^S(f) \leq \frac{2^{1/p'}(1+2^{p'+1})^{1/p'}|I|^{1+1/p'}}{(p'+1)^{1/p'}6^{1+1/p'}n} \inf_s \|f_s'\|_{p,I}. \tag{1.13}$$

Inequality (1.12) *is sharp, and when $1 < p < \infty$, equality holds if and only if*

$$f'(t) = d_1 \sum_{i=1}^{n} \left((t-c_i)^{p'-1}\chi_{J_i^+}(t) - (c_i - t)^{p'-1}\chi_{J_i^-}(t) \right) + d_2, \tag{1.14}$$

where $d_1, d_2 \in \mathbb{R}$. Similarly, inequality (1.13) *is sharp, and when $1 < p < \infty$, equality holds if and only if*

$$\begin{aligned} f'(t) = d_1 \sum_{i=1}^{n} \Big(&(t-a_i)^{p'-1}\chi_{I_i^2}(t) + (t-b_i)^{p'-1}\chi_{I_i^4}(t) \\ &- (a_i - t)^{p'-1}\chi_{I_i^1}(t) - (b_i - t)^{p'-1}\chi_{I_i^3}(t) \Big) + d_2 t^2 + d_3 t + d_4, \end{aligned} \tag{1.15}$$

where $d_i \in \mathbb{R}$, $1 \leq i \leq 4$.

Remark 1.9. When $p = 1$, $p' = \infty$, and we interpret $(1+p')^{1/p'}$ and $(1+2^{p'+1})^{1/p'}$ in the limiting sense as equaling 1 and 2 respectively. In this case Theorem 1.8 is a special case of Theorem 1.6 since if f is absolutely continuous it is of bounded variation, and $\|f'\|_{1,I} = \|f\|_{BV,I}$.

Remark 1.10. When $1 < p < \infty$ we can restate Theorem 1.8 in a form analogous to Theorem 1.6. We define the space BV_p of functions of bounded p-variation by

$$\|f\|_{BV_p,I} = \sup_{\Gamma} \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{|x_i - x_{i-1}|^{p-1}} < \infty,$$

where the supremum is taken over all partitions $\Gamma = \{x_i\}$ of I. Then $f \in BV_p$ if and only if it is absolutely continuous and $f' \in L^p(I)$, and $\|f\|_{BV_p,I} = ||f'||_{p,I}$. This characterization is due to F. Riesz; see, for example, Natanson [10].

Remark 1.11. When $p = \infty$, Theorem 1.8 is equivalent to Theorem 1.1 with $\alpha = 1$, since $f \in W_1^\infty(I)$ if and only if $f \in \Lambda_1(I)$, and $\|f'\|_{\infty,I} = \|f\|_{\Lambda_1,I}$. (See, for example, Natanson [10].)

Remark 1.12. Inequality (1.12), with $r = 0$ and $p > 1$ was independently proved by Dragomir [5] as a corollary to a rather lengthy general theorem. Very recently, we learned that Dragomir *et al.* [4] gave a direct proof similar to ours for (1.12) for all p but still with $r = 0$. Neither paper considers the question of sharpness.

While inequalities (1.12) and (1.13) are sharp in the sense that for a given n equality holds for a given function, $E_n^T(f)$ and $E_{2n}^S(f)$ go to zero more quickly than $1/n$.

Theorem 1.13. *Let* $f \in W_1^p(I)$, $1 \le p \le \infty$. *Then*

$$\lim_{n\to\infty} n \cdot E_n^T(f) = 0 \tag{1.16}$$

$$\lim_{n\to\infty} n \cdot E_{2n}^S(f) = 0. \tag{1.17}$$

Further, these limits are sharp in the sense that the factor of n *cannot be replaced by* n^a *for any* $a > 1$.

Remark 1.14. Unlike most of our proofs, the proof of Theorem 1.13 requires that we approximate f by smooth functions. It would be of interest to find a proof of this result which avoided this.

Theorem 1.15. *Let* $f \in W_1^{pq}(I)$, $1 \le p, q \le \infty$. *Then for* $n \ge 1$,

$$E_n^T(f) \le B(q'/p', q'+1)^{1/q'} \frac{|I|^{1+1/p'}}{2n} \inf_r \|f_r'\|_{pq,I}, \tag{1.18}$$

where B *is the Beta function,*

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1}\,dx, \quad u,v > 0.$$

Similarly,

$$E_{2n}^S(f) \le C(q'/p', q'+1)^{1/q'} \frac{|I|^{1+1/p'}}{n} \inf_s \|f_s'\|_{pq,I}, \tag{1.19}$$

where

$$C(u,v) = \int_0^{1/3} t^{u-1}\left(\frac{1}{3} - \frac{t}{2}\right)^{v-1} dt + \int_{1/3}^1 t^{u-1}\left(\frac{1}{4} - \frac{t}{4}\right)^{v-1} dt.$$

Remark 1.16. When $p = q$ then Theorem 1.15 reduces to Theorem 1.8.

Remark 1.17. Theorem (1.15) is sharp; when $1 \le q < p$ the condition for equality to hold is straightforward (f is constant), but when $q \ge p$ it is more technical, and so we defer the statement until after the proof, when we have made the requisite definitions.

Remark 1.18. The constant in (1.19) is considerably more complicated than that in (1.18); the function $C(u,v)$ can be rewritten in terms of the Beta function and the hypergeometric function ${}_2F_1$, but the resulting expression is no simpler. (Details are left to the reader.) However it is easy to show that $C(q'/p', p'+1) \le B(q'/p', p'+1)/3^{q'}$, so that we have the weaker but somewhat more tractable estimate

$$E_{2n}^S(f) \le B(q'/p', q'+1)^{1/q'} \frac{|I|^{1+1/p'}}{3n} \inf_s \|f_s'\|_{pq,I}.$$

Theorem 1.19. *Let $f \in W_2^p(I)$, $1 \le p \le \infty$. Then for $n \ge 1$,*

$$E_n^T(f) \le B(p'+1, p'+1)^{1/p'} \frac{|I|^{2+1/p'}}{2n^2} \|f''\|_{p,I} \tag{1.20}$$

and

$$E_{2n}^S(f) \le D(p'+1, p'+1)^{1/p'} \frac{|I|^{2+1/p'}}{2^{1/p}3^{2+1/p'}n^2} \inf_s \|f_s''\|_{p,I}, \tag{1.21}$$

where

$$D(u,v) = \int_0^{3/2} t^{u-1}|1-t|^{v-1}\,dt.$$

Inequality (1.20) *is sharp, and when $1 < p < \infty$ equality holds if and only if*

$$f''(t) = d\sum_{i=1}^n \left(\frac{|J_i|^2}{4} - (t-c_i)^2\right)^{p'-1} \chi_{J_i}(t), \tag{1.22}$$

where $d \in \mathbb{R}$. Similarly, inequality (1.21) *is sharp, and when $1 < p < \infty$ equality holds if and only if*

$$\begin{aligned} f''(t) = d_1 \sum_{i=1}^n \Bigg(\left(\frac{|I_i|^2}{36} - (t-a_i)^2\right)^{p'-1} \chi_{\tilde I_i^1}(t) - \left((t-a_i)^2 - \frac{|I_i|^2}{36}\right)^{p'-1} \chi_{\tilde I_i^2}(t) \\ + \left(\frac{|I_i|^2}{36} - (t-b_i)^2\right)^{p'-1} \chi_{\tilde I_i^3}(t) \\ - \left((t-b_i)^2 - \frac{|I_i|^2}{36}\right)^{p'-1} \chi_{\tilde I_i^4}(t) \Bigg) + d_2 t + d_3, \end{aligned} \tag{1.23}$$

where $d_i \in \mathbb{R}$, $1 \le i \le 3$, and the intervals $\tilde I_i^j$, $1 \le j \le 4$, defined in (5.2) *below, are such that the corresponding functions are positive.*

Remark 1.20. When $p = 1$, $p' = \infty$, and we interpret $B(p'+1, p'+1)^{1/p'}$ as the limiting value $1/4$. This follows immediately from the identity $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$ and from Stirling's formula. (See, for instance, Whittaker and Watson [20].)

Remark 1.21. When $p = \infty$, (1.20) reduces to the classical estimate given above.

Remark 1.22. Like the function $C(u,v)$ in Theorem 1.15, the function $D(u,v)$ can be rewritten in terms of the Beta function and the hypergeometric function ${}_2F_1$. However, the resulting expression does not seem significantly better, and details are left to the reader.

Prior to Theorem 1.19, each of our results shows that for rough functions, the trapezoidal rule is better than Simpson's rule. More precisely, the constants in the sharp error bounds for $E_{2n}^T(f)$ are less than or equal to the constants in the sharp error bounds for $E_{2n}^S(f)$. (We use $E_{2n}^T(f)$ instead of $E_n^T(f)$ since we want to compare numerical approximations with the same number of data points.)

This is no longer the case for twice differentiable functions. Numerical calculations show that, for instance, when $p = 10/9$, the constant in (1.20) is smaller, but when $p = 10$, (1.21) has the smaller constant. Furthermore, the following analogue of Theorem 1.13 shows that though the constants in Theorem 1.19 are sharp, Simpson's rule is asymptotically better than the trapezoidal rule.

Theorem 1.23. *Given* $f \in W_2^p(I)$, $1 \le p \le \infty$,

$$\lim_{n\to\infty} n^2 E_n^T(f) = \left| \frac{|I|^2}{12} \int_I f''(t)\,dt \right|, \tag{1.24}$$

but

$$\lim_{n\to\infty} n^2 E_{2n}^S(f) = 0. \tag{1.25}$$

Remark 1.24. (*Added in proof.*) Given Theorems 1.13 and 1.23, it would be interesting to compare the asymptotic behavior of $E_n^T(f)$ and $E_{2n}^S(f)$ for extremely rough functions, say those in $\Lambda_\alpha(I)$ and $CBV(I)$. We suspect that in these cases their behavior is the same, but we have no evidence for this. (We want to thank the referee for raising this question with us.)

1.3. **Organization of the Paper.** The remainder of this paper is organized as follows. In Section 2 we make some preliminary observations and define notation that will be used in all of our proofs. In Section 3 we prove Theorems 1.1 and 1.6 and Corollary 1.4. In Section 4 we prove Theorems 1.8, 1.13 and 1.15. In Section 5 we prove Theorems 1.19 and 1.23.

Throughout this paper all notation is standard or will be defined when needed. Given an interval I, $|I|$ will denote its length. Given p, $1 \le p \le \infty$, p' will denote the conjugate exponent: $1/p + 1/p' = 1$.

2. PRELIMINARY REMARKS

In this section we establish notation and make some observations which will be used in the subsequent proofs.

2.1. **Estimating the Error.** Given an interval $I = [a, b]$, for the trapezoidal rule we will always have an equally spaced partition of $n + 1$ points, $x_i = a + i|I|/n$. Define the intervals $J_i = [x_{i-1}, x_i]$, $1 \le i \le n$; then $|J_i| = |I|/n$.

For each i, $1 \le i \le n$, define

$$L_i = \frac{|J_i|}{2}\big(f(x_{i-1}) + f(x_i)\big) - \int_{J_i} f(t)\,dt. \tag{2.1}$$

If we divide each J_i into two intervals J_i^- and J_i^+ of equal length, then (2.1) can be rewritten as

$$L_i = \int_{J_i^-} \big(f(x_{i-1}) - f(t)\big)\,dt + \int_{J_i^+} \big(f(x_i) - f(t)\big)\,dt. \tag{2.2}$$

Alternatively, if f is absolutely continuous, then we can apply integration by parts to (2.1) to get that

$$L_i = \int_{J_i} (t - c_i) f'(t)\,dt, \tag{2.3}$$

where $c_i = (x_{i-1} + x_i)/2$ is the midpoint of J_i. If f' is absolutely continuous, then we can apply integration by parts again to get

$$L_i = \frac{1}{2} \int_{J_i} \left(\frac{|J_i|^2}{4} - (t - c_i)^2 \right) f''(t)\,dt. \tag{2.4}$$

From the definition of the trapezoidal rule (1.1) it follows immediately that

$$E_n^T(f) = \left|\sum_{i=1}^n L_i\right| \le \sum_{i=1}^n |L_i|, \tag{2.5}$$

and our principal problem will be to estimate $|L_i|$.

We make similar definitions for Simpson's rule. Given I, we form a partition with $2n+1$ points, $x_j = a + j|I|/2n$, $0 \le j \le 2n$, and form the intervals $I_i = [x_{2i-2}, x_{2i}]$, $1 \le i \le n$. Then $|I_i| = |I|/n$.

For each i, $1 \le i \le n$, define

$$K_i = \frac{|I|}{6n}\big(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\big) - \int_{I_i} f(t)\,dt. \tag{2.6}$$

To get an identity analogous to (2.2), we need to partition I_i into four intervals of different lengths. Define

$$a_i = \frac{2x_{2i-2} + x_{2i-1}}{3} \qquad b_i = \frac{2x_{2i} + x_{2i-1}}{3},$$

and let

$$I_i^1 = [x_{2i-2}, a_i], \quad I_i^2 = [a_i, x_{2i-1}], \quad I_i^3 = [x_{2i-1}, b_i], \quad I_i^4 = [b_i, x_{2i}].$$

Then $|I_i^1| = |I_i^4| = |I|/6n$ and $|I_i^2| = |I_i^3| = |I|/3n$, and we can rewrite (2.6) as

$$\begin{aligned} K_i = &\int_{I_i^1} \big(f(x_{2i-2}) - f(t)\big)\,dt + \int_{I_i^2} \big(f(x_{2i-1}) - f(t)\big)\,dt \\ &+ \int_{I_i^3} \big(f(x_{2i-1}) - f(t)\big)\,dt + \int_{I_i^4} \big(f(x_{2i}) - f(t)\big)\,dt. \end{aligned} \tag{2.7}$$

If f is absolutely continuous we can apply integration by parts to (2.6) to get

$$K_i = \int_{I_i^-} (t-a_i)f'(t)\,dt + \int_{I_i^+} (t-b_i)f'(t)\,dt. \tag{2.8}$$

If f' is absolutely continuous we can integrate by parts again to get

$$K_i = \frac{1}{2}\int_{I_i^-} \left(\frac{|I_i|^2}{36} - (t-a_i)^2\right) f''(t)\,dt + \frac{1}{2}\int_{I_i^+} \left(\frac{|I_i|^2}{36} - (t-b_i)^2\right) f''(t)\,dt. \tag{2.9}$$

Whichever expression we use, it follows from the definition of Simpson's rule (1.2) that

$$E_{2n}^S(f) = \left|\sum_{i=1}^n K_i\right| \le \sum_{i=1}^n |K_i|. \tag{2.10}$$

Finally, we want to note that there is a connection between Simpson's rule and the trapezoidal rule: it follows from the definitions (1.1) and (1.2) that

$$S_{2n}(f) = \frac{4}{3}T_{2n}(f) - \frac{1}{3}T_n(f). \tag{2.11}$$

2.2. **Modifying the Norm.** In all of our results, we estimate the error in the trapezoidal rule with an expression of the form

$$\inf_r \|f_r\|,$$

where the infimum is taken over all $r \in \mathbb{R}$. It will be enough to prove the various inequalities with $\|f\|$ on the righthand side: since the trapezoidal rule is exact on linear functions, $E_n^T(f_r) = E_n^T(f)$ for all f and r. Further, we note that for each f, there exists $r_0 \in \mathbb{R}$ such that

$$\|f_{r_0}\| = \inf_r \|f_r\|.$$

This follows since the norm is continuous in r and tends to infinity as $|r| \to \infty$.

Similarly, in our estimates for $E_{2n}^S(f)$, it will suffice to prove the inequalities with $\|f\|$ on the righthand side instead of $\inf_s \|f_s\|$: because Simpson's rule is exact for polynomials of degree 3 or less, $E_n^S(f) = E_n^T(f_s)$, for $s(x) = ax^3 + bx^2 + cx$. Again the infimum is attained, since the norm is continuous in the coefficients of s and tends to infinity as $|a| + |b| + |c| \to \infty$.

(The exactness of the Trapezoidal rule and Simpson's rule is well-known; see, for example, Ralston [13].)

3. FUNCTIONS IN Λ_α(I), $0 < \alpha \leq 1$, AND $CBV(I)$

Proof of Theorem 1.1. We first prove inequality (1.4). By (2.2), for each i, $1 \leq i \leq n$,

$$\begin{aligned}|L_i| &\leq \int_{J_i^-} |f(x_{i-1}) - f(t)|dt + \int_{J_i^+} |f(x_i) - f(t)|dt \\ &\leq \|f\|_{\Lambda_\alpha} \int_{J_i^-} |x_{i-1} - t|^\alpha \, dt + \|f\|_{\Lambda_\alpha} \int_{J_i^-} |x_i - t|^\alpha \, dt;\end{aligned}$$

by translation and reflection,

$$\begin{aligned}&= 2\|f\|_{\Lambda_\alpha} \int_{J_i^-} (t - x_{i-1})^\alpha \, dt \\ &= 2\|f\|_{\Lambda_\alpha} \frac{|J_i^-|^{1+\alpha}}{1+\alpha} \\ &= \frac{|I|^{1+\alpha}\|f\|_{\Lambda_\alpha}}{(1+\alpha)2^\alpha n^{1+\alpha}}.\end{aligned}$$

Therefore, by (2.5)

$$E_n^T(f) \leq \frac{|I|^{1+\alpha}\|f\|_{\Lambda_\alpha}}{(1+\alpha)2^\alpha n^\alpha}, \tag{3.1}$$

and by the observation in Section 2.2 we get (1.4).

The proof of inequality (1.5) is almost identical to the proof of (1.4): we begin with inequality (2.7) and argue as before to get

$$|K_i| \leq \frac{2(1 + 2^{\alpha+1})|I|^{1+\alpha}\|f\|_{\Lambda_\alpha}}{(1+\alpha)6^{1+\alpha}n^{1+\alpha}},$$

which in turn implies (1.5).

To see that inequality (1.4) is sharp, fix $n \geq 1$ and define the function f as follows: on $[0, 1/n]$ let

$$f(x) = \begin{cases} x^\alpha, & 0 \leq x \leq \dfrac{1}{2n} \\ \left|x - \dfrac{1}{n}\right|^\alpha, & \dfrac{1}{2n} \leq x \leq \dfrac{1}{n}. \end{cases}$$

Now extend f to the interval $[0, 1]$ as a periodic function with period $1/n$. It is clear that $\|f\|_{\Lambda_\alpha} = 1$, and it is immediate from the definition that $T_n(f) = 0$. Therefore,

$$E_n^T(f) = \int_0^1 f(x)\, dx = 2n \int_0^{1/2n} x^\alpha \, dx = \frac{1}{(1+\alpha)2^\alpha n^\alpha},$$

which is precisely the righthand side of (3.1). □

Proof of Corollary 1.4. Inequalities (1.6) and (1.7) follow immediately from (1.4) and (1.5). Recall that if $f \in \Lambda_1(I)$, then f is differentiable almost everywhere, $f' \in L^\infty(I)$ and $\|f\|_{\Lambda_1} = \|f'\|_\infty$. (See, for example, Natanson [10].) Let $r = (M+m)/2$; then

$$\|f - rx\|_{\Lambda_1} = \|f' - r\|_\infty = \frac{M-m}{2}.$$

We now show that (1.6) is sharp and that equality holds exactly when (1.8) holds. First note that if (1.8) holds, then by (2.3),

$$L_i = \int_{J_i^+} (t - c_i) M \, dt + \int_{J_i^-} (t - c_i) m \, dt = \pm \frac{|I|^2}{8n^2}(M - m),$$

and it follows at once from (2.5) that equality holds in (1.6).

To prove that (1.8) is necessary for (1.6) to hold, we consider two cases.

Case 1. $M > 0$ and $m = -M$. In this case,

$$E_n^T(f) = \frac{|I|^2}{4n} M.$$

Again by (2.3),

$$\begin{aligned}
E_n^T(f) &\le \left| \sum_{i=1}^n \int_{J_i} (t - c_i) f'(t) \, dt \right| \\
&\le \sum_{i=1}^n \left| \int_{J_i^-} (t - c_i) f'(t) \, dt \right| + \left| \int_{J_i^+} (t - c_i) f'(t) \, dt \right| \\
&\le \frac{|I|^2}{8n^2} \sum_{i=1}^n \left(\|f'\|_{\infty, J_i^-} + \|f'\|_{\infty, J_i^+} \right) \\
&\le \frac{|I|^2}{8n} M + \frac{|I|^2}{8n} M,
\end{aligned}$$

and since the first and last terms are equal, equality must hold throughout. Therefore, we must have that

$$|L_i^-| = \|f'\|_{\infty, J_i^-} \int_{J_i^-} (c_i - t) \, dt, \quad |L_i^+| = \|f'\|_{\infty, J_i^+} \int_{J_i^+} (t - c_i) \, dt, \tag{3.2}$$

and

$$\frac{|I|^2}{8n^2} \sum_{i=1}^n \|f'\|_{\infty, J_i^+} = \frac{|I|^2}{8n^2} \sum_{i=1}^n \|f'\|_{\infty, J_i^-} = \frac{|I|^2}{8n} M. \tag{3.3}$$

Hence, by (3.2), on J_i

$$f'(t) = \alpha_i \chi_{J_i^+}(t) - \beta_i \chi_{J_i^-}(t),$$

with either $\alpha_i, \beta_i > 0$ for all i, or $\alpha_i, \beta_i < 0$ for all i. Without loss of generality we assume that $\alpha_i, \beta_i > 0$.

Further, we must have that $M = \sup\{\alpha_i : 1 \le i \le n\}$, so it follows from (3.3) that $\alpha_i = M$ for all i. Similarly, we must have that $\beta_i = M$, $1 \le i \le n$. This completes the proof of Case 1.

Case 2. The general case: $m < M$. Let $r = (M+m)/2$; then

$$E_n^T(f) = \frac{|I|^2}{8n}(M - m) = E_n^T(f_r).$$

Since Case 1 applies to f_r, we have that

$$f_r'(t) = \frac{M-m}{2} \sum_{i=1}^{n} \big(\chi_{J_i^+}(t) - \chi_{J_i^-}(t)\big).$$

This completes the proof since $f' = f_r' + r$.

The proof that (1.7) is sharp and equality holds if and only if (1.9) holds is essentially the same as the above argument, and we omit the details. □

Proof of Theorem 1.6. We first prove (1.10). By (2.2) and the definition of the norm in $CBV(I)$, for each i, $1 \le i \le n$,

$$\begin{aligned} |L_i| &\le \int_{J_i^-} |f(x_{i-1}) - f(t)|dt + \int_{J_i^+} |f(x_i) - f(t)|dt \\ &\le \|f\|_{BV,J_i^-}|J_i^-| + \|f\|_{BV,J_i^+}|J_i^+| \\ &= \frac{1}{2n}\|f\|_{BV,J_i}. \end{aligned}$$

If we sum over i, we get

$$E_n^T(f) \le \frac{1}{2n}\sum_{i=1}^{n}\|f\|_{BV,J_i} = \frac{1}{2n}\|f\|_{BV,I};$$

inequality (1.10) now follows from the remark in Section 2.2.

To show that inequality (1.10) is sharp, fix $n \ge 1$ and for $k \ge 1$ define $a_k = 4^{-k}/n$. We now define the function f_k on $I = [0,1]$ as follows: on $[0, 1/n]$ let

$$f_k(x) = \begin{cases} 1 - \dfrac{x}{a_n} & 0 \le x \le a_n \\[2ex] 0 & a_n \le x \le \dfrac{1}{n} - a_n \\[2ex] 1 + \dfrac{\left(x - \frac{1}{n}\right)}{a_n} & \dfrac{1}{n} - a_n \le x \le \dfrac{1}{n}. \end{cases}$$

Extend f_k to $[0,1]$ periodically with period $1/n$. It follows at once from the definition that $\|f_k\|_{BV,[0,1]} = 2n$. Furthermore,

$$E_n^T(f_k) = \left| T_n(f_k) - \int_0^1 f_k(t)\,dt \right| = 1 - a_k n = 1 - 4^{-k}.$$

Thus the constant $1/2n$ in (1.10) is the best possible.

We now consider when equality can hold in (1.10). If $f(t) = mt + b$, then we have equality since both sides are zero.

For the converse implication we first show that if $f \in CBV(I)$ is not constant on I, then

$$E_n^T(f) < \frac{|I|}{2n}\|f\|_{BV,I}. \tag{3.4}$$

By the above argument, it will suffice to show that for some i,

$$|L_i| < \frac{|I|}{2n}\|f\|_{BV,J_i}.$$

Since f is non-constant, choose i such that f is not constant on J_i. Since f is continuous, the function $|f(x_{i-1}) + f(x_i) - 2f(t)|$ achieves its maximum at some $t \in J_i$, and, again because

f is non-constant, it must be strictly smaller than its maximum on a set of positive measure. Hence on a set of positive measure, $|f(x_{i-1}) + f(x_i) - 2f(t)| < \|f\|_{BV,J_i}$, and so (3.4) follows from (2.1), since we can rewrite this as

$$L_i = \frac{1}{2}\int_{J_i} \big(f(x_{i-1}) + f(x_i) - 2f(t)\big)\,dt.$$

To finish the proof, note that as we observed in Section 2.2, there exists r_0 such that $\|f_{r_0}\|_{BV,I} = \inf_r \|f_r\|_{BV,I}$. Hence, we would have that

$$E_n^T(f) = E_n^T(f_{r_0}) \le \frac{|I|}{2n}\|f_{r_0}\|_{BV,I}.$$

If $f(t)$ were not of the form $mt+b$, so that f_{r_0} could not be a constant function, then by (3.4), the inequality would be strict. Hence equality can only hold if f is linear.

The proof that inequality (1.11) holds is almost identical to the proof of (1.10): we begin with inequality (2.7) and argue exactly as we did above.

The proof that inequality (1.11) is sharp requires a small modification to the example given above. Fix $n \ge 1$ and, as before, let $a_k = 4^{-k}/n$. Define the function f_k on $I = [0,1]$ as follows: on $[0, 1/n]$ let

$$f_k(x) = \begin{cases} 0 & 0 \le x \le \dfrac{1}{2n} - a_n \\[2ex] 1 + \dfrac{\left(x - \frac{1}{2n}\right)}{a_n} & \dfrac{1}{2n} - a_n \le x \le \dfrac{1}{2n} \\[2ex] 1 + \dfrac{\left(\frac{1}{2n} - x\right)}{a_n} & \dfrac{1}{2n} \le x \le \dfrac{1}{2n} + a_n \\[2ex] 0 & \dfrac{1}{2n} + a_n \le x \le \dfrac{1}{n}. \end{cases}$$

Extend f_k to $[0,1]$ periodically with period $1/n$. Then we again have $\|f_k\|_{BV,[0,1]} = 2n$; furthermore,

$$E_{2n}^S(f_k) = \left| S_{2n}(f_k) - \int_0^1 f_k(t)\,dt \right| = \frac{2}{3} - a_k n = \frac{2}{3} - 4^{-k}.$$

Thus the constant $1/3n$ in (1.11) is the best possible.

The proof that equality holds in (1.11) only when both sides are zero is again very similar to the above argument, replacing L_i by K_i and using (2.6) instead of (2.1). □

4. FUNCTIONS IN $W_1^p(I)$ AND $W_1^{pq}(I)$, $1 \le p,\ q \le \infty$

Proof of Theorem 1.8. As we noted in Remarks 1.9 and 1.11, it suffices to consider the case $1 < p < \infty$.

We first prove inequality (1.12). If we apply Hölder's inequality to (2.3), then for all i, $1 \le i \le n$,

$$|L_i| \le \|f'\|_{p,J_i} \left(\int_{J_i} |t - c_i|^{p'}\,dt \right)^{1/p'}.$$

An elementary calculation shows that

$$\int_{J_i} |t - c_i|^{p'}\,dt = \frac{|J_i|^{p'+1}}{(p'+1)2^{p'}}.$$

Hence, by (2.5) and by Hölder's inequality for series,

$$\begin{aligned}E_n^T(f) &\le \frac{|I|^{1+1/p'}}{2(p'+1^{1/p'})n^{1+1/p'}}\sum_{i=1}^{n}\left(\int_{J_i}|f'(t)|^p\,dt\right)^{1/p}\\ &\le \frac{|I|^{1+1/p'}}{2(p'+1)^{1/p'}n^{1+1/p'}}\left(\int_I |f'(t)|^p\,dt\right)^{1/p} n^{1/p'}\\ &= \frac{|I|^{1+1/p'}}{2(p'+1)^{1/p'}n}\|f'\|_{p,I}.\end{aligned}$$

Inequality (1.12) now follows from the observation in Section 2.2.

The proof of inequality (1.13) is essentially the same as the proof of inequality (1.12), beginning instead with (2.8) and using the fact that

$$\int_{I_i^-}|t-a_i|^{p'}\,dt+\int_{I_i^+}|t-b_i|^{p'}\,dt=\frac{2(1+2^{p'+1})|I|^{p'+1}}{(p'+1)6^{p'+1}n^{p'+1}}.$$

We will now show that inequality (1.12) is sharp. We write $L_i = L_i^+ + L_i^-$, where

$$L_i^+ = \int_{J_i^+}(t-c_i)f'(t)dt,\quad L_i^- = \int_{J_i^-}(t-c_i)f'(t)dt. \tag{4.1}$$

Also note that

$$\int_{J_i^+}(t-c_i)^{p'}dt = \int_{J_i^-}(c_i-t)^{p'}dt = \frac{|I|^{p'+1}}{(p'+1)(2n)^{p'+1}}.$$

We first assume that f' has the desired form. A pair of calculations shows that

$$E_n^T(f) = 2|d_1|\frac{|I|^{p'+1}n}{(p'+1)(2n)^{p'+1}},\quad \|f'-d_2\|_{p,I} = |d_1|\frac{|I|^{(p'+1)/p}(2n)^{1/p}}{(p'+1)^{1/p}(2n)^{(p'+1)/p}},$$

and since

$$\frac{p'+1}{p} = p' - \frac{1}{p'},\qquad \text{and}\qquad \frac{2n}{(2n)^{p'}} = \frac{1}{(2n)^{p'-1}},$$

we have the desired equality.

To show the converse, we first consider when equality holds with $r=0$. Observe that by the above argument,

$$\begin{aligned}E_n^T(f) &= \left|\sum_{i=1}^{n} L_i\right|\\ &= \left|\sum_{i=1}^{n}(L_i^+ + L_i^-)\right|\\ &\le \sum_{i=1}^{n}|L_i^+| + \sum_{i=1}^{n}|L_i^-|\\ &\le \frac{|I|^{1+1/p'}}{(p'+1)^{1/p'}(2n)^{1+1/p'}}\sum_{i=1}^{n}\left(\|f'\|_{p,J_i^+}+\|f'\|_{p,J_i^-}\right)\\ &\le \frac{|I|^{1+p'}}{(p'+1)^{1/p'}(2n)^{1+1/p'}}\|f'\|_{p,I}(2n)^{1/p'}\\ &= \frac{|I|^{1+p'}}{(p'+1)^{1/p'}2n}\|f'\|_{p,I}.\end{aligned}$$

Since the first and last terms are equal, each inequality must be an equality. Hence all L_i^+ and L_i^- have the same sign; without loss of generality we may assume they are all positive. By the criterion for equality in Hölder's inequality on J_i (see, for example, Rudin [16, p. 63]),

$$f'(t) = \alpha_i(t-c_i)^{p'-1}\chi_{J_i^+}(t) - \beta_i(c_i-t)^{p'-1}\chi_{J_i^-}(t),$$

where $\alpha_i, \beta_i > 0$. (Here we used the assumption that $L_i^+, L_i^- > 0$.)

Next we claim that $\alpha_1 = \beta_1 = \cdots = \alpha_n = \beta_n$. To see this, first note that

$$\begin{aligned} E_n^T(f) &= \sum_{i=1}^n \left(\int_{J_i^+} (t-c_i)f'(t)dt + \int_{J_i^-} (c_i-t)f'(t)dt \right) \\ &= \sum_{i=1}^n (\alpha_i+\beta_i)\frac{|I|^{p'+1}}{(p'+1)(2n)^{p'+1}}, \end{aligned}$$

and this equals

$$\begin{aligned} \frac{|I|^{1+1/p'}}{(p'+1)^{1/p'}}\|f'\|_{p,I} &= \frac{|I|^{1+1/p'}}{(p'+1)^{1/p'}2n}\left(\sum_{i=1}^n(\alpha_i^p+\beta_i^p)\frac{|I|^{p'+1}}{(p'+1)(2n)^{p'+1}}\right)^{1/p} \\ &= \left(\sum_{i=1}^n(\alpha_i^p+\beta_i^p)\right)^{1/p}\frac{|I|^{1+1/p'+(p'+1)/p}}{(p'+1)(2n)^{1+(p'+1)/p}}. \end{aligned}$$

Since $1+1/p'+(p'+1)/p = p'+1$ and $2n(2n)^{(p'+1)/p} = (2n)^{p'+1-1/p'}$, it follows that

$$\sum_{i=1}^n(\alpha_i+\beta_i) = \left(\sum_{i=1}^n(\alpha_i^p+\beta_i^p)\right)^{1/p}(2n)^{1/p'}.$$

This is equality in Hölder's inequality for series, which occurs precisely when all the α_i's and β_i's are equal. (See, for example, Hardy, Littlewood and Pólya [8, p. 22].) This establishes when equality holds when $r=0$.

Finally, as we observed in Section 2.2, $\inf_r \|f_r'\|_{p,I} = \|f_{r_0}'\|_{p,I}$ for some $r_0 \in \mathbb{R}$. Since $E_n(f) = E_n(f_{r_0})$ we conclude that (1.12) holds if and only if (1.14) holds.

The proof that (1.13) is sharp and that equality holds if and only if (1.15) holds is essentially the same as the above argument and we omit the details. □

Proof of Theorem 1.13. We first prove the limit (1.16) for $f \in C^1(I)$. Define L_i^- and L_i^+ as in (4.1), and define the four values $M_i^\pm = \max\{f'(t) : t \in J_i^\pm\}$, $m_i^\pm = \min\{f'(t) : t \in J_i^\pm\}$. Then, since

$$\int_{J_i^+}(t-c_i)dt = \frac{1}{8}\frac{|I|^2}{n^2} = \int_{J_i^-}(c_i-t)dt,$$

we have that

$$m_i^+\frac{|I|^2}{8n^2} \le L_i^+ \le M_i^+\frac{|I|^2}{8n^2}, \quad -M_i^-\frac{|I|^2}{8n^2} \le L_i^- \le -m_i^-\frac{|I|^2}{8n^2}.$$

Hence,

$$\frac{|I|^2}{8n}(m_i^+-M_i^-) \le nL_i \le \frac{|I|^2}{8n}(M_i^+-m_i^-);$$

this in turn implies that

$$\frac{|I|}{8}\sum_{i=1}^n\frac{|I|}{n}(m_i^+-M_i^-) \le n\sum_{i=1}^n L_i \le \frac{|I|}{8}\sum_{i=1}^n\frac{|I|}{n}(M_i^+-m_i^-). \tag{4.2}$$

Since $f' \in C(I)$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{|I|}{n}(M_i^+ - m_i^-) = \int_I f'(t)\,dt - \int_I f'(t)\,dt = 0.$$

Similarly, the left side of (4.2) converges to 0 as $n \to \infty$. This yields (1.16) if $f \in C^1(I)$.

We will now show that (1.16) holds in general. If $1 < p \le \infty$, $W_1^p(I) \subset W_1^1(I)$, so we may assume without loss of generality that $p = 1$. Fix $\varepsilon > 0$ and choose $g \in C^1(I)$ such that $\|f' - g'\|_{1,I} \le 2\varepsilon/|I|$. Then

$$E_n^T(f) \le E_n^T(g) + E_n^T(f - g). \tag{4.3}$$

If we let

$$\phi_n(t) = \sum_{i=1}^{n} (t - c_i)\chi_{J_i}(t), \tag{4.4}$$

then

$$E_n^T(f-g) = \left| \int_I \phi_n(t)(f' - g')(t)dt \right|.$$

Hence,

$$|nE_n^T(f-g)| \le n\|f' - g'\|_{1,I}\|\phi_n\|_{\infty,I} \le \varepsilon.$$

Therefore, by (4.3) and the special case above,

$$0 \le \limsup_{n\to\infty} nE_n^T(f) \le \varepsilon;$$

since $\varepsilon > 0$ is arbitrary, we get that (1.16) holds.

We can prove (1.17) in essentially the same way, beginning by rewriting (2.8) as

$$K_i = \int_{I_i^1} (a_i - t)f'(t)\,dt + \int_{I_i^2} (t - a_i)f'(t)\,dt + \int_{I_i^3} (b_i - t)f'(t)\,dt + \int_{I_i^4} (t - b_i)f'(t)\,dt,$$

where the intervals I_j, $1 \le j \le 4$, are defined as in (2.7). Alternatively, it follows from the identity (2.11), the triangle inequality, and (1.16):

$$\begin{aligned} E_{2n}^S(f) &= \left| S_{2n}(f) - \int_I f(t)\,dt \right| \\ &\le \frac{4}{3}\left| T_{2n}(f) - \int_I f(t)\,dt \right| + \frac{1}{3}\left| T_n(f) - \int_I f(t)\,dt \right| \\ &= \frac{4}{3}E_{2n}^T(f) + \frac{1}{3}E_n^T(f). \end{aligned}$$

To see that (1.16) is sharp, fix $a > 1$; without loss of generality we may assume $a = 1 + r$, $0 < r < 1$.

We define a function f on $[0,1]$ as follows: for $j \ge 1$ define the intervals $I_j = (2^{-j}, 2^{-j+1}]$. Define the function

$$g(t) = \sum_{j=1}^{\infty} 2^{(1-r)j}\chi_{I_j}.$$

It follows immediately that $g \in L^p[0,1]$ if $1 \le p < 1/(1-r)$. Now define f by

$$f(t) = \int_0^t g(s)\,ds;$$

then $f \in W_1^p[0,1]$ for p in the same range.

Fix $k > 1$ and let $n = 2^k$. Then, since f is linear on each interval I_j, $f(0) = 0$, and since the trapezoidal rule is exact on linear functions,

$$E_n^T(f) = \left| 2^{-k-1} f(2^{-k}) - \int_0^{2^{-k}} f(t)\,dt \right|. \tag{4.5}$$

Again since f is linear on each I_j,

$$\int_0^{2^{-k}} f(t)\,dt = \sum_{j=k+1}^{\infty} \int_{I_j} f(t)\,dt = \sum_{j=k+1}^{\infty} 2^{-j-1}\big(f(2^{-j}) + f(2^{-j+1})\big)$$

Furthermore, for all j,

$$f(2^{-j}) = \int_0^{2^{-j}} g(t)\,dt = \sum_{i=j+1}^{\infty} 2^{-ri} = \frac{2^r}{2^r - 1} \cdot 2^{-r(j+1)} = \frac{2^{-rj}}{2^r - 1}.$$

Hence,

$$2^{-k-1} f(2^{-k}) = \frac{n^{-a}}{2(2^r - 1)},$$

$$\sum_{j=k+1}^{\infty} 2^{-j-1} f(2^{-j}) = \frac{n^{-a}}{2(2^r - 1)(2^a - 1)},$$

$$\sum_{j=k+1}^{\infty} 2^{-j-1} f(2^{-j+1}) = \frac{2^r n^{-a}}{2(2^r - 1)(2^a - 1)}.$$

If we combine these three identities with (4.5) we get that

$$E_n^T(f) = \frac{n^{-a}}{2(2^r - 1)} \left| 1 - \frac{1 + 2^r}{2^a - 1} \right|.$$

The quantity in absolute values is positive if $0 < r < 1$; hence $n^a E_n^T(f)$ cannot converge to zero as $n \to \infty$.

This example also shows that (1.17) is sharp. Fix $n = 2^k$ and fix $a > 1$ as before. Then

$$E_{2n}^S(f) = \left| \frac{2^{-k+2} f(2^{-k-1}) + 2^{-k} f(2^{-k})}{3} - \int_0^{2^{-k}} f(t)\,dt \right|,$$

and the computation proceeds exactly as it did after (4.5). Alternatively, we can again argue using (2.11):

$$n^a E_{2n}^S(f) \geq \frac{4n^a}{3} E_{2n}^T(f) - \frac{n^a}{3} E_n^T(f),$$

and if $1 < a < 2$ the limit of the righthand side as $n \to \infty$ is positive. □

Proof of Theorem 1.15. We begin by recalling two definitions. For more information, see Stein and Weiss [17]. Given a function f on an interval I, define λ_f, the distribution function of f, by

$$\lambda_f(y) = |\{x \in I : |f(x)| > y\}|,$$

and define f^*, the non-increasing rearrangement of f, on $[0, |I|]$ by

$$f^*(t) = \inf\{y : \lambda_f(y) \leq t\}.$$

We can now prove that (1.18) holds. Fix n and define ϕ_n as in (4.4). It follows at once that

$$\lambda_{\phi_n}(y) = \begin{cases} 0, & y \geq \frac{|I|}{2n} \\ |I| - 2ny, & 0 < y < \frac{|I|}{2n}, \end{cases}$$

which in turn implies that $\phi_n^*(t) = (|I| - t)/(2n)$. Hence, by an inequality of Hardy and Littlewood (see, for example Bennett and Sharpley [1, p. 44]) and Hölder's inequality,

$$\begin{aligned} E_n^T(f) &= \left| \int_I \phi_n(t) f'(t)\, dt \right| \\ &\leq \int_0^{|I|} \phi_n^*(t)(f')^*(t)\, dt \\ &= \int_0^{|I|} t^{1/p-1/q}(f')^*(t) t^{1/q-1/p} \phi_n^*(t)\, dt \\ &\leq ||f'||_{pq,I} \left(\int_0^{|I|} t^{(1/q-1/p)q'} \phi_n^*(t)^{q'} dt \right)^{1/q'} \\ &= ||f'||_{pq,I} \left(\frac{1}{(2n)^{q'}} \int_0^{|I|} t^{q'/p'-1}(|I| - t)^{q'}\, dt \right)^{1/q'} \\ &= ||f'||_{pq,I} \frac{|I|^{(1+1/p')}}{2n} B\left(\frac{q'}{p'}, q' + 1 \right)^{1/q'}. \end{aligned}$$

By the observation in Section 2.2, (1.18) now follows at once.

The proof of (1.19) is similar and we sketch the details. Define

$$\psi_n(t) = \sum_{i=1}^{n} \big((t - a_i)\chi_{I_i^-}(t) + (t - b_i)\chi_{I_i^+}(t) \big).$$

Then

$$\lambda_{\psi_n}(y) = \begin{cases} |I| - 4ny & 0 \leq y \leq \frac{|I|}{6n} \\ \frac{2|I|}{3} - 2ny & \frac{|I|}{6n} \leq y \leq \frac{|I|}{3n}, \end{cases}$$

and

$$\psi_n^*(t) = \begin{cases} \frac{|I|}{3n} - \frac{t}{2n} & 0 \leq t \leq \frac{|I|}{3} \\ \frac{|I|}{4n} - \frac{t}{4n} & \frac{|I|}{3} \leq t \leq |I|. \end{cases}$$

We now argue as above:

$$\begin{aligned} E_{2n}^S(f) &\leq \int_0^{|I|} \psi_n^*(t)(f')^*(t)\, dt \\ &\leq ||f'||_{pq,I} \left(\int_0^{|I|} t^{q'/p'-1} \psi_n^*(t)^{q'}\, dt \right)^{1/q'}. \end{aligned}$$

The last integral naturally divides into two integrals on $[0, |I|/3]$ and $[|I|/3, |I|]$, and by a change of variables we get that it equals

$$\frac{|I|^{q'/p'+q'}}{n^{q'}} C\left(\frac{q'}{p'}, q'+1\right).$$

Inequality (1.19) then follows at once.

We now consider the question of when equality holds in (1.18). Examining the proof above, we see that if the first and last terms are equal, then equality must hold in Hölder's inequality and in the inequality of Hardy and Littlewood. In particular, we must have that for some $c \in \mathbb{R}$,

$$t^{1/p-1/q}(f')^*(t) = c\big(t^{1/q-1/p}(|I|-t)\big)^{q'-1},$$

or equivalently,

$$(f')^*(t) = ct^{q'/p'-1}(|I|-t)^{q'-1}, \tag{4.6}$$

and

$$\left|\int_I \phi_n(t) f'(t)\,dt\right| = \int_0^{|I|} \phi_n^*(t)(f')^*(t)\,dt. \tag{4.7}$$

Note that when $1 \le q < p$ then $q' > p'$, so (4.6) implies that $(f')^*(0) = 0$. Hence f' is identically zero so f must be constant. For a discussion of when equality (4.7) holds, see Bennett and Sharpley [1].

When $q \ge p$, these two conditions are sufficient for equality to hold in (1.18). Given a function f with these properties, we have that

$$E_n^T(f) = \frac{c}{2n}\int_0^{|I|} t^{q'/p'-1}(|I|-t)^{q'}\,dt = \frac{c}{2n}|I|^{(1+1/p')q'} B\left(\frac{q'}{p'}, q'+1\right).$$

Similarly, we have that

$$\|f'\|_{pq,I}^q = c^q \int_0^{|I|} t^{q/p-1} t^{(q'/p'-1)q}(|I|-t)^{q'}\,dt.$$

Since

$$\frac{q}{p} - 1 + \left(\frac{q'}{p'} - 1\right) q = q\left(\frac{1}{p} - 1\right) - 1 + \frac{q'q}{p'} = \frac{(q'-1)q}{p'} - 1 = \frac{q'}{p'} - 1,$$

we get

$$\|f\|_{pq,I}^q = c^q |I|^{(1+1/p')q'} B\left(\frac{q'}{p'}, q'+1\right).$$

Hence, since $q'/q + 1 = q'$,

$$B\left(\frac{q'}{p'}, q'+1\right)^{1/q'} \frac{|I|^{1+1/p'}}{2n}\|f'\|_{pq,I} = \frac{c}{2n} B\left(\frac{q'}{p'}, q'+1\right)|I|^{(1+1/p')(q'/q+1)} = E_n^T(f).$$

A similar argument shows that equality holds in (1.19) if and only if

$$\left|\int_I \psi_n(t) f'(t)\,dt\right| = \int_0^{|I|} \phi_n^*(t)(f')^*(t)\,dt,$$

and for some $c \in \mathbb{R}$,

$$t^{1/p-1/q}(f')^*(t) = c\big(t^{1/q-1/p}\psi_n^*(t)\big)^{q'-1}.$$

Again, when $1 \le q < p$ this implies that f' is identically zero, so equality holds only when f is constant. □

5. FUNCTIONS IN $W_2^p(I)$, $1 \le p \le \infty$

Proof of Theorem 1.19. We first prove inequality (1.20). When $1 < p \le \infty$ we apply Hölder's inequality to (2.4) to get

$$|L_i| \le \frac{1}{2}\|f''\|_{p,J_i}\left(\int_{J_i}\left(\frac{|J_i|^2}{4}-(t-c_i)^2\right)^{p'}dt\right)^{1/p'}. \tag{5.1}$$

We evaluate the integral on the righthand side. By translation and a change of variables, if $x = |I|/n$, we have that

$$\begin{aligned}\int_{J_i}\left(\frac{|J_i|^2}{4}-(t-c_i)^2\right)^{p'}dt &= \int_0^x\left(\frac{x^2}{4}-\left(t-\frac{x}{2}\right)^2\right)^{p'}dt\\ &= \int_0^x (xt-t^2)^{p'}\,dt\\ &= x^{2p'+1}\int_0^1 s^{p'}(1-s)^{p'}\,ds\\ &= B(p'+1,p'+1)\frac{|I|^{2p'+1}}{n^{2p'+1}}.\end{aligned}$$

If we combine this with (5.1) and apply Hölder's inequality for series we get

$$\begin{aligned}E_n^T(f) &\le \sum_{i=1}^n |L_i|\\ &\le \sum_{i=1}^n B(p'+1,p'+1)^{1/p'}\frac{|I|^{2+1/p'}}{2n^{2+1/p'}}\|f''\|_{p,J_i}\\ &\le B(p'+1,p'+1)^{1/p'}\frac{|I|^{2+1/p'}}{2n^2}\|f''\|_{p,I},\end{aligned}$$

and this is inequality (1.20).

When $p = 1$ and $p' = \infty$, a nearly identical argument again yields (1.20).

The proof of (1.21) is very similar to that of (1.20). We begin by applying Hölder's inequality to (2.9):

$$\begin{aligned}|K_i| \le \frac{1}{2}\|f''\|_{p,I_i^-}&\left(\int_{I_i^-}\left|\frac{|I_i|^2}{36}-(t-a_i)^2\right|^{p'}dt\right)^{1/p'}\\ &+\frac{1}{2}\|f''\|_{p,I_i^+}\left(\int_{I_i^+}\left|\frac{|I_i|^2}{36}-(t-b_i)^2\right|^{p'}dt\right)^{1/p'},\end{aligned}$$

we estimate each integral in turn. If we let $x = |I|/n$, then by translation and a change of variables we have that

$$\begin{aligned}\int_{I_i^-}\left|\frac{|I_i|^2}{36}-(t-a_i)^2\right|^{p'}dt &= \int_0^{x/2}\left|\frac{x^2}{36}-\left(t-\frac{x}{6}\right)^2\right|^{p'}dt\\ &= \int_0^{x/2} t^{p'}\left|\frac{x}{3}-t\right|^{p'}dt\end{aligned}$$

$$= \left(\frac{x}{3}\right)^{2p'+1} \int_0^{3/2} s^{p'} |1-s|^{p'} ds$$
$$= \left(\frac{|I|}{3n}\right)^{2p'+1} D(p'+1, p'+1).$$

A similar argument shows that

$$\int_{I_i^+} \left| \frac{|I_i|^2}{36} - (t-b_i)^2 \right|^{p'} dt = \left(\frac{|I|}{3n}\right)^{2p'+1} D(p'+1, p'+1).$$

Therefore, by Hölder's inequality for series, we have that

$$E_{2n}^S(f) \le \sum_{i=1}^n |K_i|$$
$$\le D(p'+1, p'+1)^{1/p'} \frac{|I|^{2+1/p'}}{2(3n)^{2+1/p'}} \sum_{i=1}^n \left(\|f''\|_{p,I_i^-} + \|f''\|_{p,I_i^+} \right)$$
$$\le D(p'+1, p'+1)^{1/p'} \frac{|I|^{2+1/p'}}{2^{1/p} 3^{2+1/p'} n^2} \|f''\|_{p,I}.$$

If we now apply the observation in Section 2.2 we get (1.21).

The proofs that (1.20) holds if and only if (1.22) holds, and that (1.22) holds if and only if (1.23) holds, are essentially the same as the proof of sharpness in Theorem 1.8 and we omit the details, except to note that in (1.23) we define the intervals $\tilde{I}_i^k$, $1 \le k \le 4$, as follows. Let

$$\tilde{a}_i = \frac{a_i + x_{2i-1}}{2}, \quad \tilde{b}_i = \frac{b_i + x_{2i-1}}{2},$$

and define

(5.2) $$\tilde{I}_i^1 = [x_{2i-2}, \tilde{a}_i], \quad \tilde{I}_i^2 = [\tilde{a}_i, x_{2i-1}], \quad \tilde{I}_i^3 = [x_{2i-1}, \tilde{b}_i], \quad \tilde{I}_i^4 = [\tilde{b}_i, x_{2i}].$$

□

Proof of Theorem 1.23. We first prove that (1.24) holds if $f \in C^2(I)$. Define $M_i = \max\{f''(i) : t \in J_i\}$ and $m_i = \min\{f''(t) : t \in J_i\}$. Since

$$\int_{J_i} \left(\frac{|J_i|^2}{4} - (t-c_i)^2 \right) dt = \frac{|I|^3}{6n^3},$$

it follows from (2.4) that

$$\frac{|I|^3}{12n^3} m_i \le L_i \le \frac{|I|^3}{12n^3} M_i.$$

If we sum over i we get that

$$\frac{|I|^2}{12} \sum_i \frac{|I|}{n} m_i \le n^2 \sum_{i=1}^n L_i \le \frac{|I|^2}{12} \sum_i \frac{|I|}{n} M_i.$$

Since f'' is continuous, the left and righthand sides converge to

$$\frac{|I|^2}{12} \int_I f''(t)\, dt,$$

and (1.24) follows at once.

We will now show that (1.24) holds in general. Since $W_2^p(I) \subset W_2^1(I)$ if $p > 1$, we may assume without loss of generality that $p = 1$. Fix $\varepsilon > 0$ and choose $g \in C^2(I)$ such that

$$\|f'' - g''\|_{1,I} < \frac{4\varepsilon}{3|I|^2}.$$

In particular, this implies that

$$\left| \frac{|I|^2}{12} \int_I f''(t)\,dt - \frac{|I|^2}{12} \int_I g''(t)\,dt \right| < \frac{\varepsilon}{3},$$

By inequality (1.20) this also implies that

$$n^2 E_n^T(f-g) < \frac{\varepsilon}{3}.$$

Further, by the special case above, if we choose n sufficiently large,

$$\left| n^2 E_n^T(g) - \left| \frac{|I|^2}{12} \int_I g''(t)\,dt \right| \right| < \frac{\varepsilon}{3}.$$

Therefore, since

$$n^2 E_n^T(g) - n^2 E_n^T(f-g) \le n^2 E_n^T(f) \le n^2 E_n^T(g) + n^2 E_n^T(f-g),$$

it follows that

$$\left| n^2 E_n T(f) - \left| \frac{|I|^2}{12} \int_I f''(t)\,dt \right| \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown (1.24) holds in general.

Finally, to show (1.25) we first note that the above argument proves the slightly stronger result that

$$\lim_{n\to\infty} n^2 \left(T_n(f) - \int_I f(t)\,dt \right) = \frac{|I|^2}{12} \int_I f''(t)\,dt.$$

Then by the identity (2.11),

$$n^2 E_{2n}^S(f) = \left| (2n)^2 \left(\frac{1}{3} T_{2n}(f) - \frac{1}{3} \int_I f(t)\,dt \right) - n^2 \left(\frac{1}{3} T_n(f) - \frac{1}{3} \int_I f(t)\,dt \right) \right|,$$

and (1.25) follows immediately. □

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