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LITTLEWOOD'S INEQUALITY FOR $p-$BIMEASURES

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ABSTRACT. In this paper we extend an inequality of Littlewood concerning the higher variations of functions of bounded Fréchet variations of two variables (bimeasures) to a class of functions that are p-bimeasures, by using the machinery of vector measures. Using random estimates of Kahane-Salem-Zygmund, we show that the inequality is sharp.

Key words and phrases: Inequalities, Bimeasures, Fréchet variation, p-variations, Bounded variations.

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1. INTRODUCTION

Let μ be a set function defined on the product $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$ of 2 σ-fields, such that it is a finite complex measure in each coordinate. More precisely, for each fixed $A \in \sigma(\mathcal{B}_1)$ the set function $\mu(A, \cdot)$ is a complex measure defined on $\sigma(\mathcal{B}_2)$. Similarly for each $B \in \sigma(\mathcal{B}_2)$, the set function μ gives rise to a measure in the first coordinate. Such set functions dubbed *bimeasures* by Morse and Transue were studied extensively by these and other authors (see [1, 2, 3, 5, 6, 7, 10, 11, 12]). It is well known that such set functions need not be extendible to a measure on the $\sigma-$Algebra generated by $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$. Now suppose that μ is a set function defined on $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$, such that it has finite *semi-variation*; that is,

$$\|\mu\|_F = \sup \left\{ \left\| \sum_{j,k} \mu(A_j \times B_k) r_j \otimes r_k \right\|_\infty \right\} < \infty, \tag{1.1}$$

where $\sup$ is taken over all measurable partitions $\{A_j\}$, $\{B_k\}$ of Ω_1 and Ω_2, respectively. Here $\{r_j\}$ is the usual system of Rademachers, realized as functions on the interval $[0, 1]$. By a partition of Ω, we mean a finite collection of mutually disjoint measurable sets whose union is Ω. F in $|| \cdot ||_F$ is for Fréchet. It is clear that a set function μ with finite semi-variation is also a bimeasure. It is interesting that the converse also holds. That is, a bimeasure has finite semi-variation. This follows easily from the machinery of vector measure theory. On the other hand,

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it is well known that a set function which has finite semi-variation need not have finite total variation (in the sense of Vitali), hence it may not be extendible to a measure [2, 9]. However, all is not lost, in his 1930 paper, Littlewood showed that a bimeasure has finite 4/3-variation. To make this precise we first introduce the notion of mixed variation of μ. Let $p, q > 0$, and define the mixed (p, q)-variation of μ to be

$$\|\mu\|_{p,q} = \sup \left\{ \left(\sum_k \left(\sum_j |\mu(A_j \times B_k)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right\}, \tag{1.2}$$

where the $\sup$ is taken over all finite measurable partitions $\{A_j\}$ and $\{B_k\}$ of Ω_1 and Ω_2 respectively. In the case that $p = q$, we simply write $\|\mu\|_p$, that is $\|\mu\|_p = \|\mu\|_{p,p}$. We now state Littlewood's $4/3$ inequalities.

1.1. Littlewood's Inequalities.

$$\|\mu\|_{2,1} + \|\mu\|_{1,2} + \|\mu\|_{4/3} \leq c \|\mu\|_F , \tag{1.3}$$

where c is a fixed universal constant. The result is sharp in the sense that, there exists $\mu \in$ such that $\|\mu\|_p$ and $\|\mu\|_{q,1/q}$ are infinite for all $p < 4/3$ and for all $q < 2$. Extension of Littlewood's inequality to a larger class of functions of two variables is the main result of this paper.

Definition 1.1. A set function μ defined on product of two algebras $\mathcal{B}_1 \times \mathcal{B}_2$ is called a pre-p-bimeasure, if it is finitely additive in each coordinate, and for each fixed $A \in \mathcal{B}_1$, the quantity

$$BV_p(\mu(A, \cdot)) := \sup \left\{ \sum_k |\mu(A \times B_k|^p \right\}$$

is finite, and for each fixed $B \in \mathcal{B}_2$, $BV_p(\mu(\cdot, B))$ is finite. Here $\sup$ is taken over all finite measurable partitions of Ω_2.

If the set function is defined on the product of two σ-algebras with above properties, then it is called a p-bimeasure.

Definition 1.2. A pre-p-bimeasure μ defined on product of two algebras $\mathcal{B}_1 \times \mathcal{B}_2$, is said to be bounded, if there exists a positive constant M such that $BV_p(\mu(A, \cdot)) + BV_p(\mu(\cdot, B)) \leq M$, for all $A \in \mathcal{B}_1$ and for all $B \in \mathcal{B}_2$.

We prove the following result.

Theorem 1.1. *Suppose that either μ is a p-bimeasure defined on $\sigma(\mathcal{B}_1) \times \sigma(\mathcal{B}_2)$, or that it is a bounded pre-p-bimeasure defined on $\mathcal{B}_1 \times \mathcal{B}_2$. If $1 \leq p \leq 2$ then*

$$\|\mu\|_{2,p} + \|\mu\|_{p,2} + \|\mu\|_{\frac{4p}{2+p}} < \infty. \tag{1.4}$$

In the case that $p \geq 2$, then

$$\|\mu\|_p < \infty. \tag{1.5}$$

Furthermore, the result is sharp, in the sense that, there exists a p-bimeasure such that $\|\mu\|_q = \infty$, for all $q < \frac{4p}{2+p}$.

To prove Theorem 1.1 we collect some definitions and results about vector measures. Much of the following can be found in Chapter 1 of [4].

Definition 1.3. A function μ from a field $\mathcal{B}$ of a set Ω to a Banach space is called a finitely additive vector measure, or simply a vector measure, if whenever A_1 and A_2 are disjoint members of $\mathcal{B}$ then $\mu\left(A_1 \bigcup A_2\right) = \mu(A_1) + \mu(A_2)$. The *variation* of a vector measure μ is the extended nonnegative function $|\mu|$ whose value on the set E is given by

$$|\mu|(A) = \sup_{\pi} \sum_{A \in \pi} ||\mu(A)||,$$

where the sup is taken over all partitions π of A into a finite number of disjoint members of $\mathcal{B}$. If $|\mu|(\Omega)$ is finite, then μ will be called a measure of *bounded variation.*

A different type of variation related to a vector measure μ is the so called *semi-variation* of μ. More precisely, the semi-variation of μ is the extended nonnegative function $\|\mu\|_F$ whose value on a measurable set A is given by

$$\|\mu\|_F(A) = \sup\left\{|x^*(\mu)|(A) : x^* \in X^*, \|x^*\| \leq 1\right\},$$

where $|x^*(\mu)|$ is the variation of the real-valued measure (finitely additive measure) $x^*(\mu)$. If $\|\mu\|_F(\Omega)$ is finite, then μ will be called a *measure of bounded semi-variation.*

2. PROOF OF THEOREM 1.1

We now prove Theorem 1.1. Suppose that $1 \leq p < 2$. Let X_1 be the space of finitely additive set functions defined on $\sigma(\mathcal{B}_1)$, which have finite p-variations. Similarly let X_2 be the set finitely additive functions defined on $\sigma(\mathcal{B}_2)$ which have finite p-variations. It can be shown that equipped with p-variation norm, X_1 and X_2 are Banach spaces. Let L be the X_1-valued function defined on $\sigma(\mathcal{B}_2)$ as follows: $L(A) = \mu(\cdot, A)$, where $A \in \sigma(\mathcal{B}_2)$. Let R be the X_2-valued function defined on $\sigma(\mathcal{B}_1)$ as follows: $R(A) = \mu(A, \cdot)$, where $A \in \sigma(\mathcal{B}_1)$. If μ is a p-bimeasure then by the Nikodym Boundedness Theorem (see [4, Theorem 1, page 14]), L and R have finite semi-variations. If μ is a bounded pre-p-bimeasure then by general properties of vector measures (see e.g., [4, Proposition 11, page 4]), L and R have finite semi-variations. Let $\{A_n\}$ be a finite measurable partition of Ω_2 and $\{B_k\}$ be a finite measurable partition of Ω_1, then

(2.1)
$$\begin{aligned}
\infty &> ||L||_F(\Omega_2) \\
&\geq \left\| BV_p\left(\sum_n r_n \mu(A_n, \cdot)\right)\right\|_\infty \\
&\geq \left\| \left(\sum_k \left|\sum_n r_n \mu(A_n, B_k)\right|^p\right)^{\frac{1}{p}} \right\|_\infty \\
&\geq \left(\int_0^1 \sum_k \left|\sum_n r_n(x) \mu(A_n, B_k)\right|^p dx\right)^{\frac{1}{p}} \\
\text{(Khinchin's inequality)} \quad &\Rightarrow \geq c \left(\sum_k \left(\sum_n |\mu(A_n, B_k)|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}.
\end{aligned}$$

Similarly,

(2.2)
$$\infty > ||R||_F(\Omega_1) \geq c \left(\sum_n \left(\sum_k |\mu(A_n, B_k)|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

(2.2) and (2.3) imply that, $\|\mu\|_{2,p}$ is finite. Applying Minkowski's inequality we obtain $\|\mu\|_{p,2} \leq \|\mu\|_{2,p} < \infty$. We now show that $\|\mu\|_{\frac{4p}{2+p}}$ is finite. Let $a_{n,k} = \mu(A_n, B_k)$. Applying Hölder's inequality with exponents $\frac{2+p}{p}$ and $\frac{2+p}{2}$, we obtain

$$
\begin{aligned}
(2.3)\qquad \sum_{n,k} |a_{n,k}|^{\frac{4p}{2+p}} &= \sum_{n,k} |a_{n,k}|^{\frac{2p}{2+p}} |a_{n,k}|^{\frac{2p}{2+p}} \\
&\leq \sum_{n} \left[\sum_{k} |a_{n,k}|^2\right]^{\frac{p}{2+p}} \left[\sum_{k} |a_{n,k}|^p\right]^{\frac{2}{p+2}} \\
&\leq \left[\sum_{n} (\sum_{k} |a_{n,k}|^2)^{\frac{p}{2}}\right]^{\frac{2}{2+p}} \left[\sum_{n} \left(\sum_{k} |a_{n,k}|^p\right)^{\frac{2}{p}}\right]^{\frac{p}{2+p}} \\
&\leq \left(\|\mu\|_{2,p} \|\mu\|_{p,2}\right)^{\frac{2p}{p+2}} < \infty.
\end{aligned}
$$

This proves inequality (1.5). If $p \geq 2$ then $p/2 \geq 1$, consequently

$$
\begin{aligned}
(2.4)\qquad ||R||_F(\Omega_1) &\geq c \left(\sum_{n} \left(\sum_{k} |\mu(A_n, B_k)|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\geq c \left(\sum_{k} \left(\sum_{n} |\mu(A_n, B_k)|^p\right)\right)^{\frac{1}{p}}.
\end{aligned}
$$

Similarly

$$
||L||_F(\Omega_2) \geq c \left(\sum_{k} \left(\sum_{n} |\mu(A_n, B_k)|^p\right)\right)^{\frac{1}{p}}.
$$

This proves inequality (2.1).

We now show that the exponent $\frac{4p}{p+2}$ is sharp. We only consider the case $1 < p < 2$. Sharpness of Theorem 1.1 for the case $p = 1$ is known [9]. Sharpness of Theorem 1.1 for $p \geq 2$ is trivial.

We need the following result, which is a consequence of Kahane-Salem-Zygmund estimates (see [8, Theorem 3, p. 70]).

Lemma 2.1. *Let $X_{n_1,n_2,\ldots n_s}$ be a subnormal collection of independent random variables. Given complex numbers $c_{n_1,n_2,\ldots,n_s}$, where the multi-index $(n_1, n_2, \ldots, n_s)$ satisfies $|n_1| + |n_2| + \cdots + |n_s| \leq N$, then*

$$
(2.5)\qquad \Pr\left\{\sup_{t_1,\ldots,t_s} \left|\sum X_{n_1,n_2,\ldots n_s} c_{n_1,n_2,\ldots,n_s} e^{i(n_1t_1+\cdots n_st_s)}\right| \geq C\left[s\sum |c_{n_1,n_2,\ldots,n_s}|^2 \log N\right]^{\frac{1}{2}}\right\} \leq N^{-2}e^{-s},
$$

where C is an independent constant.

To apply Lemma 2.1, we will need to construct an appropriate sequence of independent subnormal random variables. We will construct a Radamacher type of system, which we will call the 4-level Radamacher system.

2.1. **4-level Radamacher System.** 4-level Radamacher system is the sequence of independent random variables, $\{w_j(x)\}_{j=1}^{\infty}$, defined on the unit interval $[0,1]$, such that each w_j takes on 4 discrete values $\{2,-2,1,-1\}$, each with probability $\frac{1}{4}$. Such a system can be constructed similar to the usual Radamacher system. Observe that, M 4-level Radamacher system generate 4^M distinct vectors of length M. On the other hand the set $\{1,2,...,M\}$ has 2^M distinct subsets.

By Lemma 2.1, for $j,k=1,...,M$, there exists a vector $\vec{t}=(t_1,t_2)$ and choice of scalers $\{b_{jk}\}_{j,k=1}^{M}$ (approximately as many as $\left(1-\frac{1}{M^2}\right)4^{M^2}-2^{M^2}$), such that $b_{jk}\in\{2,-2,1,-1\}$, and for any subset A of $\{1,2,...,M\}$,

$$\left|\sum_{j\in A} b_{jk}e^{i(kt_1+jt_2)}\right| \le C[4M\log(2M)]^{\frac{1}{2}}, \tag{2.6}$$

and

$$\left|\sum_{k\in A} b_{jk}e^{i(kt_1+jt_2)}\right| \le C[4M\log(2M)]^{\frac{1}{2}}. \tag{2.7}$$

Let

$$(a)=\{a_{jk}\}_{j,k}=\left\{b_{jk}e^{i(jt_1+kt_2)}\right\}_{j,k=1}^{M}. \tag{2.8}$$

Let $A,B\subset\{1,2,...,M\}$ and define

$$a(A,B)=\sum_{j\in A}\sum_{k\in B} a_{jk}, \tag{2.9}$$

then by virtue of inequalities (2.7) and (2.8),

$$||a||_F \le C_p M^{\frac{1}{2}+\frac{1}{p}}\sqrt{\log(2M)}. \tag{2.10}$$

On the other hand for any $r>0$,

$$||a||_r=\left[\sum_{j=1}^{M}\sum_{k=1}^{M}|a_{jk}|^r\right]^{\frac{1}{r}} \ge M^{\frac{2}{r}}. \tag{2.11}$$

We see that if $r<\frac{4p}{p+2}$,

$$\lim_{M\to\infty}\frac{||a||_r}{||a||_F}=\infty. \tag{2.12}$$

This shows that $\frac{4p}{p+2}$ is sharp. The proof for the case that μ is a bounded-pre-bimeasure is similar.

3. FUNCTIONS OF BOUNDED p-VARIATIONS AND RELATED FUNCTION SPACES

Let $p\ge 1$ and f be a function defined on $[0,1]^2$. Let

$$V_p^{(2)}(f,[0,1]^2)=\left(\sup_{\pi_1,\pi_2}\sum_{i,j}|\Delta_{i,j}^{\pi_1,\pi_2}f|^p\right)^{1/p}.$$

Here $\pi_1=\{0=x_0<x_1,<\cdots<x_m=1\}$, and $\pi_2=\{0=y_0<y_1,<\cdots<y_n=1\}$, are partitions of $[0,1]$ and

$$\Delta_{i,j}^{\pi_1,\pi_2}(f)=f(x_i,y_j)-f(x_i,y_{j-1})-f(x_{i-1},y_j)+f(x_{i-1},y_{j-1}).$$

Let $W_p^{(2)}([0,1]^2)=W_p^{(2)}$ denote the class of functions f on $[0,1]^2$ such that,

$$||f||_{W_p^2}=V_p^{(2)}(f,[0,1]^2)+V_p^{(2)}(f(0,\cdot),[0,1])+V_p^{(1)}(f(\cdot,0),[0,1])+|f(0,0,0)|<\infty.$$

Let $\vec{x} = (x_1, x_2),\ \ \vec{y} = (y_1, y_2) \in [0,1]^2$, and f be a function defined on $[0,1]^2$. Let

$$f_{\vec{y}}(\vec{x}) = f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2).$$

We say that f is a *Lipschitz function of order α of first type,* if there exists a constant C such that for all $\vec{x}$ and $\vec{y}$ in $[0,1]^2$,

$$|f(\vec{x}) - f(\vec{y})| \leq C||\vec{x} - \vec{y}||_2^\alpha. \tag{3.1}$$

Here $||\cdot||_2$ refers to the usual l_2-norm. The class of Lipschitz functions of order α of first type is denoted by $\Lambda^1_\alpha(2)$. We say that f is a *Lipschitz function of order α of second type,* if there exists a constant C such that for all $\vec{x}$ and $\vec{y}$ in $[0,1]^2$,

$$|f_{\vec{y}}(\vec{x})| \leq C||\vec{x} - \vec{y}||_2^\alpha. \tag{3.2}$$

The class of Lipschitz functions of order α of second type is denoted by $\Lambda^2_\alpha(2)$. If $f \in \Lambda^1_\alpha(2)$ then

$$|f_{\vec{y}}(\vec{x})| \leq 4C \min\{|x_j - y_j|^\alpha : 1 \leq j \leq 2\} \leq C_2||\vec{x} - \vec{y}||_2^\alpha\}.$$

Therefore, $\Lambda^1_\alpha(2) \subset \Lambda^2_\alpha(2)$. Using Theorem 1.1 we obtain

Corollary 3.1. *Let f be a function defined on $[0,1]^2$. Suppose that for any $1 \leq j \leq n$ and for any fixed partitions π_1 and π_2 of the interval $[0,1]$, we have*

$$\sup_\pi \left[\sum_{i,j} |\Delta^{\pi_1,\pi}_{i,j} f|^p\right]^{1/p} + \sup_\pi \left[\sum_{i,j} |\Delta^{\pi,\pi_2}_{i,j} f|^p\right]^{1/p} \leq M < \infty, \tag{3.3}$$

then $f \in W^{(2)}_{\frac{4p}{2+p}}$.

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