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ON WEIGHTED PARTIAL ORDERINGS ON THE SET OF RECTANGULAR COMPLEX MATRICES

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ABSTRACT. In this paper, the relations between the weighted partial orderings on the set of rectangular complex matrices are first studied. Then, using the matrix function defined by Yang and Li [H. Yang and H.Y. LI, Weighted UDV^*-decomposition and weighted spectral decomposition for rectangular matrices and their applications, Appl. Math. Comput. 198 (2008), pp. 150–162], some weighted partial orderings of matrices are compared with the orderings of their functions.

Key words and phrases: Weighted partial ordering, Matrix function, Singular value decomposition.

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1. INTRODUCTION

Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices, $\mathbb{C}_r^{m\times n}$ denote a subset of $\mathbb{C}^{m\times n}$ comprising matrices with rank r, $\mathbb{C}_{\geq}^{m}$ denote a set of Hermitian positive semidefinite matrices of order m, and $\mathbb{C}_{>}^{m}$ denote a subset of $\mathbb{C}_{\geq}^{m}$ consisting of positive definite matrices. Let I_r be the identity matrix of order r. Given $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $A^{\#}_{MN}$, $R(A)$, and $r(A)$ stand for the conjugate transpose, weighted conjugate transpose, range, and rank, respectively, of A. Details for the concept of $A^{\#}_{MN}$ can be found in [11, 13]. Moreover, unless otherwise specified, in this paper we always assume that the given weight matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$.

In the following, we give some definitions of matrix partial orderings.

Definition 1.1. For $A, B \in \mathbb{C}^{m\times m}$, we say that A is below B with respect to:

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(1) the *Löwner partial ordering* and write $A \leq_L B$, whenever $B - A \in \mathbb{C}^m_{\geq}$.
(2) the *weighted Löwner partial ordering* and write $A \leq_{WL} B$, whenever $\tilde{M}(B-A) \in \mathbb{C}^m_{\geq}$.

Definition 1.2. For $A, B \in \mathbb{C}^{m\times n}$, we say that A is below B with respect to:

(1) the *star partial ordering* and write $A \overset{*}{\leq} B$, whenever $A^*A = A^*B$ and $AA^* = BA^*$.
(2) the *weighted star partial ordering* and write $A \overset{\#}{\leq} B$, whenever $A^{\#}_{MN}A = A^{\#}_{MN}B$ and $AA^{\#}_{MN} = BA^{\#}_{MN}$.
(3) the WG*-weighted star partial ordering* and write $A \overset{\#}{\leq}_{WG} B$, whenever $MAB^{\#}_{MN} \in \mathbb{C}^m_{\geq}$, $NA^{\#}_{MN}B \in \mathbb{C}^n_{\geq}$, and $AA^{\#}_{MN} \leq_{WL} AB^{\#}_{MN}$.
(4) the WGL *partial ordering* and write $A \leq_{WGL} B$, whenever $(AA^{\#}_{MN})^{1/2} \leq_{WL} (BB^{\#}_{MN})^{1/2}$ and $AB^{\#}_{MN} = (AA^{\#}_{MN})^{1/2}(BB^{\#}_{MN})^{1/2}$.
(5) the $WGL2$ *partial ordering* and write $A \leq_{WGL2} B$, whenever $AA^{\#}_{MN} \leq_{WL} BB^{\#}_{MN}$ and $AB^{\#}_{MN} = (AA^{\#}_{MN})^{1/2}(BB^{\#}_{MN})^{1/2}$.
(6) the *minus partial ordering* and write $A \overset{-}{\leq} B$, whenever $A^-A = A^-B$ and $AA^= = BA^=$ for some (possibly distinct) generalized inverses $A^-, A^=$ of A (satisfying $AA^-A = A = AA^=A$).

The weighted Löwner and weighted star partial orderings can be found in [6, 15] and [9], respectively. The WGL partial ordering was defined by Yang and Li in [15] and the $WGL2$ partial ordering can be defined similarly. The minus partial ordering was introduced by Hartwig [2], who also showed that the minus partial ordering is equivalent to rank subtractivity, namely $A \overset{-}{\leq} B$ if and only if $r(B - A) = r(B) - r(A)$. For the relation $\overset{\#}{\leq}_{WG}$, we can use Lemma 2.5 introduced below to verify that it is indeed a matrix partial ordering according to the three laws of matrix partial orderings.

Baksalary and Pukelsheim showed how the partial orderings of two Hermitian positive semidefinite matrices A and B relate to the orderings of their squares A^2 and B^2 in the sense of the Löwner partial ordering, minus partial ordering, and star partial ordering in [1]. In terms of these steps, Hauke and Markiewicz [3] discussed how the partial orderings of two rectangular matrices A and B relate to the orderings of their generalized square $A^{(2)}$ and $B^{(2)}$, $A^{(2)} = A(A^*A)^{1/2}$, in the sense of the GL partial ordering, minus partial ordering, G-star partial ordering, and star partial ordering. The definitions of the GL and G-star partial orderings can be found in [3, 4].

In addition, Hauke and Markiewicz [5] also compared the star partial ordering $A \overset{*}{\leq} B$, G-star partial ordering $A \overset{*}{\leq}_G B$, and GL partial ordering $A \leq_{GL} B$ with the orderings $f(A) \overset{*}{\leq} f(B)$, $f(A) \overset{*}{\leq}_G f(B)$, and $f(A) \leq_{GL} f(B)$, respectively. Here, $f(A)$ is a matrix function defined in A [7]. Legiša [8] also discussed the star partial ordering and surjective mappings on $\mathbb{C}^{n\times n}$. These results extended the work of Mathias [10] to some extent, who studied the relations between the Löwner partial ordering $A \leq_L B$ and the ordering $f(A) \leq_L f(B)$.

In the present paper, based on the definition $A^{(2)} = A(A^{\#}_{MN}A)^{1/2}$ (also called the generalized square of A), we study how the partial orderings of two rectangular matrices A and B relate to the orderings of their generalized squares $A^{(2)}$ and $B^{(2)}$ in the sense of the WGL partial ordering, WG-weighted star partial ordering, weighted star partial ordering, and minus partial ordering. Further, adopting the matrix functions presented in [14], we also compare the weighted partial orderings $A \overset{\#}{\leq} B$, $A \overset{\#}{\leq}_{WG} B$, and $A \leq_{WGL} B$ with the orderings $f(A) \overset{\#}{\leq} f(B)$, $f(A) \overset{\#}{\leq}_{WG} f(B)$, and $f(A) \leq_{WGL} f(B)$, respectively. These works generalize the results of Hauke and Markiewicz [3, 5].

Now we introduce the (M, N) weighted singular value decomposition [11, 12] (MN-SVD) and the matrix functions based on the MN-SVD, which are useful in this paper,

Lemma 1.1. *Let $A \in \mathbb{C}_r^{m\times n}$. Then there exist $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ satisfying $U^*MU = I_m$ and $V^*N^{-1}V = I_n$ such that*

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{1.1}$$

where $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i = \sqrt{\lambda_i} > 0$, and $\lambda_1 \geq \cdots \geq \lambda_r > 0$ are the nonzero eigenvalues of $A^{\#}_{MN}A = (N^{-1}A^*M)A$. Here, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are called the nonzero (M, N) weighted singular values of A. If, in addition, we let $U = (U_1, U_2)$ and $V = (V_1, V_2)$, where $U_1 \in \mathbb{C}^{m\times r}$ and $V_1 \in \mathbb{C}^{n\times r}$, then

$$U_1^*MU_1 = V_1^*N^{-1}V_1 = I_r, \quad A = U_1DV_1^*. \tag{1.2}$$

Considering the MN-SVD, from [14], we can rewrite the matrix function $f(A) : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ by way of $f(A) = U_1f(D)V_1^*$ using the real function f, where $f(D)$ is the diagonal matrix with diagonal elements $f(\sigma_1), \ldots, f(\sigma_r)$. More information on the matrix function can be found in [14].

2. RELATIONS BETWEEN THE WEIGHTED PARTIAL ORDERINGS

Firstly, it is easy to obtain that on the cone of generalized Hermitian positive semidefinite matrices (namely the cone comprising all matrixes which multiplied by a given Hermitian positive definite matrix become Hermitian positive semidefinite matrices) the WGL partial ordering coincides with the weighted Löwner partial ordering, i.e., for matrices $A, B \in \mathbb{C}^{m\times m}$ satisfying $MA, MB \in \mathbb{C}^m_{\geq}$,

$$A \leq_{WGL} B \text{ if and only if } A \leq_{WL} B$$

and the $WGL2$ partial ordering coincides with the WGL partial ordering of the squares of matrices, i.e., for matrices $A, B \in \mathbb{C}^{mm}$ satisfying $MA, MB \in \mathbb{C}^m_{\geq}$,

$$A \leq_{WGL2} B \text{ if and only if } A^2 \leq_{WGL} B^2.$$

On the set of rectangular matrices, for the generalized square of A, i.e., $A^{(2)} = A(A^{\#}_{MN}A)^{1/2}$, the above relation takes the form:

$$A \leq_{WGL2} B \text{ if and only if } A^{(2)} \leq_{WGL} B^{(2)}, \tag{2.1}$$

which will be proved in the following theorem.

Theorem 2.1. *Let $A, B \in \mathbb{C}^{m\times n}$, $r(A) = a$, and $r(B) = b$. Then (2.1) holds.*

Proof. It is easy to find that the first conditions in the definitions of $WGL2$ partial ordering for A and B and WGL partial ordering for $A^{(2)}$ and $B^{(2)}$ are equivalent. To prove the equivalence of the second conditions, let us use the MN-SVD introduced in Lemma 1.1.

Let $A = U_1D_aV_1^*$ and $B = U_2D_bV_2^*$ be the MN-SVDs of A and B, where $U_1 \in \mathbb{C}^{m\times a}$, $U_2 \in \mathbb{C}^{m\times b}$, $V_1 \in \mathbb{C}^{n\times a}$, and $V_2 \in \mathbb{C}^{n\times b}$ satisfying $U_1^*MU_1 = V_1^*N^{-1}V_1 = I_a$ and $U_2^*MU_2 = V_2^*N^{-1}V_2 = I_b$, and $D_a \in \mathbb{C}^a_>, D_b \in \mathbb{C}^b_>$ are diagonal matrices. Then

$$\begin{aligned} AB^{\#}_{MN} =& (AA^{\#}_{MN})^{1/2}(BB^{\#}_{MN})^{1/2} \\ &\Leftrightarrow U_1D_aV_1^*N^{-1}V_2D_bU_2^*M \\ &\qquad = (U_1D_aV_1^*N^{-1}V_1D_aU_1^*M)^{1/2}(U_2D_bV_2^*N^{-1}V_2D_bU_2^*M)^{1/2} \\ &\Leftrightarrow U_1D_aV_1^*N^{-1}V_2D_bU_2^*M = U_1D_aU_1^*MU_2D_bU_2^*M \\ &\Leftrightarrow V_1^*N^{-1}V_2 = U_1^*MU_2. \end{aligned} \tag{2.2}$$

Note that

$$A^{(2)} = A(A^{\#}_{MN}A)^{1/2} = U_1 D_a V_1^* (N^{-1} V_1 D_a U_1^* M U_1 D_a V_1^*)^{1/2} \\ = U_1 D_a V_1^* N^{-1} V_1 D_a V_1^* = U_1 D_a^2 V_1^*. \tag{2.3}$$

Similarly,

$$B^{(2)} = U_2 D_b^2 V_2^*. \tag{2.4}$$

Then

$$\begin{aligned} A^{(2)}(B^{(2)})^{\#}_{MN} =& (A^{(2)}(A^{(2)})^{\#}_{MN})^{1/2}(B^{(2)}(B^{(2)})^{\#}_{MN})^{1/2} \\ &\Leftrightarrow U_1 D_a^2 V_1^* N^{-1} V_2 D_b^2 U_2^* M \\ &\qquad = (U_1 D_a^2 V_1^* N^{-1} V_1 D_a^2 U_1^* M)^{1/2} (U_2 D_b^2 V_2^* N^{-1} V_2 D_b^2 U_2^* M)^{1/2} \\ &\Leftrightarrow U_1 D_a^2 V_1^* N^{-1} V_2 D_b^2 U_2^* M = U_1 D_a^2 U_1^* M U_2 D_b^2 U_2^* M \\ &\Leftrightarrow V_1^* N^{-1} V_2 = U_1^* M U_2, \end{aligned}$$

which together with (2.2) gives

$$\begin{aligned} AB^{\#}_{MN} = (AA^{\#}_{MN})^{1/2}(BB^{\#}_{MN})^{1/2} \\ \Leftrightarrow A^{(2)}(B^{(2)})^{\#}_{MN} = (A^{(2)}(A^{(2)})^{\#}_{MN})^{1/2}(B^{(2)}(B^{(2)})^{\#}_{MN})^{1/2}. \end{aligned}$$

Therefore, the proof is completed. □

Before studying the relation between the WGL partial orderings for A and B and that for their generalized squares, we first introduce a lemma from [1].

Lemma 2.2. *Let* $A, B \in \mathbb{C}^m_{\geq}$. *Then*

(a) *If* $A^2 \leq_L B^2,$ *then* $A \leq_L B.$

(b) *If* $AB = BA$ *and* $A \leq_L B,$ *then* $A^2 \leq_L B^2.$

Theorem 2.3. *Let* $A, B \in \mathbb{C}^{m\times n}$, $r(A) = a$, $r(B) = b$, *and*

(a) $A \leq_{WGL} B,$

(b) $A^{(2)} \leq_{WGL} B^{(2)},$

(c) $(AB^{\#}_{MN})^{\#}_{MM} = AB^{\#}_{MN}.$

Then (b) *implies* (a), *and* (a) *and* (c) *imply* (b).

Proof. (i). $(b) \Rightarrow (a)$.

Together with Theorem 2.1 and the definitions of $WGL2$ and WGL partial orderings, it suffices to show that

$$(A^{(2)}(A^{(2)})^{\#}_{MN})^{1/2} \leq_{WL} (B^{(2)}(B^{(2)})^{\#}_{MN})^{1/2} \Rightarrow (AA^{\#}_{MN})^{1/2} \leq_{WL} (BB^{\#}_{MN})^{1/2}. \tag{2.5}$$

From the proof of Theorem 2.1 and the definition of weighted Löwner partial ordering, we have

$$\begin{aligned} &(A^{(2)}(A^{(2)})^{\#}_{MN})^{1/2} \leq_{WL} (B^{(2)}(B^{(2)})^{\#}_{MN})^{1/2} \\ &\Leftrightarrow U_1 D_a^2 U_1^* M \leq_{WL} U_2 D_b^2 U_2^* M \\ &\Leftrightarrow M U_1 D_a^2 U_1^* M \leq_L M U_2 D_b^2 U_2^* M \\ &\Leftrightarrow M^{1/2} U_1 D_a^2 U_1^* M^{1/2} \leq_L M^{1/2} U_2 D_b^2 U_2^* M^{1/2} \\ &\Leftrightarrow M^{1/2} U_1 D_a U_1^* M^{1/2} M^{1/2} U_1 D_a U_1^* M^{1/2} \\ &\qquad \leq_L M^{1/2} U_2 D_b U_2^* M^{1/2} M^{1/2} U_2 D_b U_2^* M^{1/2}. \end{aligned} \tag{2.6}$$

Applying Lemma 2.2 (a) to (2.6) leads to

$$(2.7)\qquad \begin{aligned} M^{1/2}U_1D_aU_1^*M^{1/2} &\leq_L M^{1/2}U_2D_bU_2^*M^{1/2} \\ &\Leftrightarrow MU_1D_aU_1^*M \leq_L MU_2D_bU_2^*M \\ &\Leftrightarrow M(AA^{\#}_{MN})^{1/2} \leq_L M(BB^{\#}_{MN})^{1/2} \\ &\Leftrightarrow (AA^{\#}_{MN})^{1/2} \leq_{WL} (BB^{\#}_{MN})^{1/2}. \end{aligned}$$

Then, by (2.6) and (2.7), we show that (2.5) holds.

(ii). (a) and $(c) \Rightarrow (b)$.

Similarly, combining with Theorem 2.1 and the definitions of $WGL2$ and WGL partial orderings, we only need to prove that

$$(2.8)\qquad (AA^{\#}_{MN})^{1/2} \leq_{WL} (BB^{\#}_{MN})^{1/2} \Rightarrow (A^{(2)}(A^{(2)})^{\#}_{MN})^{1/2} \leq_{WL} (B^{(2)}(B^{(2)})^{\#}_{MN})^{1/2}.$$

From the proof of Theorem 2.1 and the definition of weighted Löwner partial orderings, we have

$$(2.9)\qquad \begin{aligned} (AA^{\#}_{MN})^{1/2} &\leq_{WL} (BB^{\#}_{MN})^{1/2} \\ &\Leftrightarrow U_1D_aU_1^*M \leq_{WL} U_2D_bU_2^*M \\ &\Leftrightarrow MU_1D_aU_1^*M \leq_L MU_2D_bU_2^*M \\ &\Leftrightarrow M^{1/2}U_1D_aU_1^*M^{1/2} \leq_L M^{1/2}U_2D_bU_2^*M^{1/2}. \end{aligned}$$

According to (c), we have

$$(2.10)\qquad U_2D_bV_2^*N^{-1}V_1D_aU_1^*M = U_1D_aV_1^*N^{-1}V_2D_bU_2^*M.$$

Thus, together with (2.10) and (2.2), we can obtain

$$(2.11)\qquad \begin{aligned} U_2D_bU_2^*MU_1D_aU_1^*M &= U_1D_aU_1^*MU_2D_bU_2^*M \\ &\Leftrightarrow M^{1/2}U_1D_aU_1^*M^{1/2}M^{1/2}U_2D_bU_2^*M^{1/2} \\ &\qquad = M^{1/2}U_2D_bU_2^*M^{1/2}M^{1/2}U_1D_aU_1^*M^{1/2}. \end{aligned}$$

Applying Lemma 2.2 (b) to (2.11) and (2.9), we have

$$(2.12)\qquad M^{1/2}U_1D_aU_1^*M^{1/2}M^{1/2}U_1D_aU_1^*M^{1/2} \leq_L M^{1/2}U_2D_bU_2^*M^{1/2}M^{1/2}U_2D_bU_2^*M^{1/2}.$$

Then, combining with (2.12) and (2.6), we can show that (2.8) holds. □

The weighted star partial ordering was characterized by Liu in [9], using the simultaneous weighted singular value decomposition of matrices [9]. He obtained the following result.

Lemma 2.4. *Let $A, B \in \mathbb{C}^{m\times n}$ and $r(B) = b > r(A) = a \geq 1$. Then $A \overset{\#}{\leq} B$ if and only if there exist matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ satisfying $U^*MU = I_m$ and $V^*N^{-1}V = I_n$ such that*

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^* = U_1D_aV_1^*,$$

$$B = U \begin{pmatrix} D_a & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{pmatrix} V^* = U_2 \begin{pmatrix} D_a & 0 \\ 0 & D \end{pmatrix} V_2^*,$$

where $U_1 \in \mathbb{C}^{m\times a}$, $V_1 \in \mathbb{C}^{n\times a}$ and $U_2 \in \mathbb{C}^{m\times b}$, $V_2 \in \mathbb{C}^{n\times b}$ denote the first a and b columns of U,V, respectively, and satisfy $U_1^*MU_1 = V_1^*N^{-1}V_1 = I_a$ and $U_2^*MU_2 = V_2^*N^{-1}V_2 = I_b$, and $D_a \in \mathbb{C}^a_>$ and $D \in \mathbb{C}^{b-a}_>$ are diagonal matrices.

Similarly to Lemma 2.4, we can take the following form to characterize the WG-weighted star partial ordering. A detailed proof is omitted.

Lemma 2.5. *Let $A, B \in \mathbb{C}^{m\times n}$ and $r(B) = b > r(A) = a \geq 1$. Then $A \overset{\#}{\leq}_{WG} B$ if and only if there exist matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ satisfying $U^*MU = I_m$ and $V^*N^{-1}V = I_n$ such that*

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^* = U_1 D_a V_1^*,$$

$$B = U \begin{pmatrix} D_{a'} & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{pmatrix} V^* = U_2 \begin{pmatrix} D_{a'} & 0 \\ 0 & D \end{pmatrix} V_2^*,$$

where $U_1 \in \mathbb{C}^{m\times a}$, $V_1 \in \mathbb{C}^{n\times a}$ and $U_2 \in \mathbb{C}^{m\times b}$, $V_2 \in \mathbb{C}^{n\times b}$ denote the first a and b columns of U,V, respectively, and satisfy $U_1^*MU_1 = V_1^*N^{-1}V_1 = I_a$ and $U_2^*MU_2 = V_2^*N^{-1}V_2 = I_b$, and $D_a, D_{a'} \in \mathbb{C}^a_>$ and $D \in \mathbb{C}^{b-a}_>$ are diagonal matrices, and $D_{a'} - D_a \in \mathbb{C}^a_\geq$.

From the simultaneous weighted singular value decomposition of matrices [9], Lemma 2.4, and Lemma 2.5, we can derive the following theorem.

Theorem 2.6. *Let $A, B \in \mathbb{C}^{m\times n}$. Then*

(a) $A \overset{\#}{\leq} B \Leftrightarrow MAB^\#_{MN} \in C^m_\geq, NA^\#_{MN}B \in C^n_\geq$, *and* $AA^\#_{MN} = (AA^\#_{MN})^{1/2}(BB^\#_{MN})^{1/2}$.

(b) $A \overset{\#}{\leq}_{WG} B \Leftrightarrow MAB^\#_{MN} \in C^m_\geq, NA^\#_{MN}B \in C^n_\geq$, *and* $(AA^\#_{MN})^{1/2} \leq_{WL} (BB^\#_{MN})^{1/2}$.

Considering Definition 1.2(4) and Theorem 2.6, we can present the following relations between three weighted partial orderings by the sequence of implications:

$$A \overset{\#}{\leq} B \Rightarrow A \overset{\#}{\leq}_{WG} B \Rightarrow A \leq_{WGL} B.$$

As in Theorem 2.3, we now discuss the corresponding result for WG-weighted star partial ordering using Lemma 2.5.

Theorem 2.7. *Let $A, B \in \mathbb{C}^{m\times n}$, $r(A) = a$, and $r(B) = b$. Then*

$$A^{(2)} \overset{\#}{\leq}_{WG} B^{(2)} \text{ if and only if } A \overset{\#}{\leq}_{WG} B.$$

Proof. Let the MN-SVDs of A and B be as in the proof of Theorem 2.1. Considering Lemma 1.1, from (2.3), (2.4), and Lemma 2.5, we have

$$A^{(2)} = U_1 D_a^2 V_1^* = U \begin{pmatrix} D_a^2 & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

$$B^{(2)} = U_2 D_b^2 V_2^* = U \begin{pmatrix} D_b^2 & 0 \\ 0 & 0 \end{pmatrix} V^*.$$

In this case, the MN-SVDs of A and B can be rewritten as

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^*, \qquad B = U \begin{pmatrix} D_b & 0 \\ 0 & 0 \end{pmatrix} V^*.$$

Thus, from Lemma 2.5, we have

$$A^{(2)} \overset{\#}{\leq}_{WG} B^{(2)} \Rightarrow A \overset{\#}{\leq}_{WG} B.$$

Conversely, from Lemma 2.5, $A \overset{\#}{\leq}_{WG} B$ is equivalent to

$$A = U \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} V^*, \qquad B = U \begin{pmatrix} D_{a'} & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*.$$

Then

$$A^{(2)} = U \begin{pmatrix} D_a^2 & 0 \\ 0 & 0 \end{pmatrix} V^*, \qquad B^{(2)} = U \begin{pmatrix} D_b^2 & 0 & 0 \\ 0 & D^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*.$$

Therefore, from Lemma 2.5 again, the proof is completed. □

The characterization of the weighted star partial ordering can be obtained similarly using Lemma 2.4, and is given in the following theorem.

Theorem 2.8. *Let $A, B \in \mathbb{C}^{m\times n}$, $r(A) = a$, and $r(B) = b$. Then*

$$A^{(2)} \overset{\#}{\leq} B^{(2)} \text{ if and only if } A \overset{\#}{\leq} B.$$

The following result was presented by Liu [9]. It is useful for studying the relation between the minus ordering for A and B and that for $A^{(2)}$ and $B^{(2)}$.

Lemma 2.9. *Let $A, B \in \mathbb{C}^{m\times n}$. Then*

$$A \overset{\#}{\leq} B \text{ if and only if } A \overset{-}{\leq} B,$$
$$(AB^{\#}_{MN})^{\#}_{MM} = AB^{\#}_{MN}, \text{ and } (A^{\#}_{MN}B)^{\#}_{NN} = A^{\#}_{MN}B.$$

Theorem 2.10. *Let $A, B \in \mathbb{C}^{m\times n}$, $r(A) = a$, $r(B) = b$, $(AB^{\#}_{MN})^{\#}_{MM} = AB^{\#}_{MN}$, and $(A^{\#}_{MN}B)^{\#}_{NN} = A^{\#}_{MN}B$. Then*

$$A^{(2)} \overset{-}{\leq} B^{(2)} \text{ if and only if } A \overset{-}{\leq} B.$$

Proof. According to $(AB^{\#}_{MN})^{\#}_{MM} = AB^{\#}_{MN}$, $(A^{\#}_{MN}B)^{\#}_{NN} = A^{\#}_{MN}B$, the proof of Theorem 5.3.2 of [9], and the simultaneous unitary equivalence theorem [7], we have

$$A = U \begin{pmatrix} E_c & 0 \\ 0 & 0 \end{pmatrix} V^*, \qquad B = U \begin{pmatrix} F_c & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

where $U \in \mathbb{C}^{m\times m}$and $V \in \mathbb{C}^{n\times n}$ satisfy $U^*MU = I_m$ and $V^*N^{-1}V = I_n$, and $E_c \in \mathbb{C}^{c\times c}_{\geq}$ and F_c are real diagonal matrices, $c = \max\{a, b\}$.

As in (2.3) and (2.4), we can obtain

$$A^{(2)} = U \begin{pmatrix} E_c^2 & 0 \\ 0 & 0 \end{pmatrix} V^*, \qquad B^{(2)} = U \begin{pmatrix} F_c\,|F_c| & 0 \\ 0 & 0 \end{pmatrix} V^*.$$

Thus, it is easy to verify that

$$(A^{(2)}(B^{(2)})^{\#}_{MN})^{\#}_{MM} = A^{(2)}(B^{(2)})^{\#}_{MN} \quad \text{and}$$
$$((A^{(2)})^{\#}_{MN}B^{(2)})^{\#}_{NN} = (A^{(2)})^{\#}_{MN}B^{(2)}.$$

As a result,

$$A^{(2)} \overset{-}{\leq} B^{(2)} \Leftrightarrow A^{(2)} \overset{\#}{\leq} B^{(2)}.$$

By Theorem 2.8 and Lemma 2.9, the proof is completed. □

3. WEIGHTED MATRIX PARTIAL ORDERINGS AND MATRIX FUNCTIONS

In this section, we study the relations between some weighted partial orderings of matrices and the orderings of their functions. Here, we are interested in such matrix functions for which $r[f(A)] = r(A)$, i.e., functions for which $f(x) = 0$ only for $x = 0$. These functions are said to be nondegenerating.

The following properties of f gathered in Lemma 3.1 will be used in subsequent parts of this section.

Lemma 3.1. *Let* $A, B \in \mathbb{C}^{m\times n}$ *and let* f *be a nondegenerating matrix function. Then*

(a) $R(A) = R(f(A))$.

(b) $AB^{\#}_{MN} = (AA^{\#}_{MN})^{1/2}(BB^{\#}_{MN})^{1/2} \Leftrightarrow f(A)f(B^{\#}_{MN}) = f((AA^{\#}_{MN})^{1/2})f((BB^{\#}_{MN})^{1/2})$.

Proof. (a). From the MN-SVD of A, i.e., (1.2), and the property of f, we have

$$R(A) = R(U_1DV_1^*) = R(U_1) = R(U_1f(D)V_1^*) = R(f(A)).$$

(b). Similar to the proof of Theorem 2.1, let $A = U_1D_aV_1^*$ and $B = U_2D_bV_2^*$ be the MN-SVDs of A and B respectively. Considering the definition of matrix functions, we obtain

$$\begin{aligned} f(A)f(B^{\#}_{MN}) =& f((AA^{\#}_{MN})^{1/2})f((BB^{\#}_{MN})^{1/2}) \\ &\Leftrightarrow U_1f(D_a)V_1^*N^{-1}V_2f(D_b)U_2^*M = U_1f(D_a)U_1^*MU_2f(D_b)U_2^*M \\ &\Leftrightarrow V_1^*N^{-1}V_2 = U_1^*MU_2, \end{aligned}$$

which together with (2.2) implies the proof. □

In the following theorems, we compare some weighted partial orderings of matrices with orderings of their functions.

Theorem 3.2. *Let* $A, B \in \mathbb{C}^{m\times n}$ *and let* f *be a positive one-to-one function. Then*

$$A \overset{\#}{\leq} B \text{ if and only if } f(A) \overset{\#}{\leq} f(B).$$

Proof. From Definition 1.2(2) and Lemma 2.4, we have that $A \overset{\#}{\leq} B$ is equivalent to

$$AB^{\#}_{MN} = U_1D_a^2U_1^*M = AA^{\#}_{MN} \quad \text{and} \quad A^{\#}_{MN}B = N^{-1}V_1D_a^2V_1^* = A^{\#}_{MN}A,$$

and $f(A) \overset{\#}{\leq} f(B)$ is equivalent to

$$\begin{aligned} &f(A)f(B)^{\#}_{MN} = U_1f(D_a)^2U_1^*M = f(A)f(A^{\#}_{MN}) \quad \text{and} \\ &f(A)^{\#}_{MN}f(B) = N^{-1}V_1f(D_a)^2V_1^* = f(A^{\#}_{MN})f(A). \end{aligned}$$

Then, using the properties of f, the proof is completed. □

Theorem 3.3. *Let* $A, B \in \mathbb{C}^{m\times n}$ *and let* f *be a positive strictly increasing function. Then*

$$A \overset{\#}{\leq}_{WG} B \text{ if and only if } f(A) \overset{\#}{\leq}_{WG} f(B).$$

Proof. From Definition 1.1(2), Definition 1.2(3), and Lemma 2.5, we obtain that $A \overset{\#}{\leq}_{WG} B$ is equivalent to

$$MAA^{\#}_{MN} = MU_1D_a^2U_1^*M \leq_L MU_1D_aD_{a'}U_1^*M = MAB^{\#}_{MN},$$
$$MAB^{\#}_{MN} = MU_1D_aD_{a'}U_1^*M \in \mathbb{C}^m_{\geq},$$

and

$$NA^{\#}_{MN}B = V_1D_aD_{a'}V_1^* \in \mathbb{C}^n_{\geq};$$

and $f(A) \overset{\#}{\leq}_{WG} f(B)$ is equivalent to

$$\begin{aligned} Mf(A)f(A)^{\#}_{MN} &= MU_1f(D_a)^2U_1^*M \leq_L MU_1f(D_a)f(D_{a'})U_1^*M \\ &= Mf(A)f(B)^{\#}_{MN}, \end{aligned}$$

$$Mf(A)f(B)^{\#}_{MN} = MU_1f(D_a)f(D_{a'})U_1^*M \in \mathbb{C}^m_{\geq}$$

and

$$Nf(A)^{\#}_{MN}B = V_1f(D_a)f(D_{a'})V_1^* \in \mathbb{C}^n_{\geq}.$$

Therefore, the proof follows from the property of f. □

We need to point out that the above results are not valid for the WGL partial ordering or for the weighted Löwner partial ordering. However, it is possible to reduce the problem of comparing the WGL partial ordering of matrices and the WGL partial ordering of their functions to a suitable problem involving the weighted Löwner partial ordering. Thus, from Definition 1.1(2), Definition 1.2(4), and Lemma 3.1, we can deduce the following theorem.

Theorem 3.4. *Let $A, B \in \mathbb{C}^{m\times n}$ and let f be a positive strictly increasing function. The following statements are equivalent:*

(a) $A \leq_{WGL} B$ *if and only if* $f(A) \leq_{WGL} f(B)$.
(b) $(AA^{\#}_{MN})^{1/2} \leq_{WL} (BB^{\#}_{MN})^{1/2}$ *if and only if* $f((AA^{\#}_{MN})^{1/2}) \leq_{WL} f((AA^{\#}_{MN})^{1/2})$.

Remark 1. It is worthwhile to note that some of the results of Section 3 can be regarded as generalizations of those in Section 2. For example, if $f(t) = t^2$, then $f(A) = U_1D^2V_1^* = A^{(2)}$, hence, in this case, Theorem 3.2 and Theorem 3.3 will reduce to Theorem 2.8 and Theorem 2.7, respectively.

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