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FINDING DISCONTINUITIES OF PIECEWISE-SMOOTH FUNCTIONS

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ABSTRACT. Formulas for stable differentiation of piecewise-smooth functions are given. The data are noisy values of these functions. The locations of discontinuity points and the sizes of the jumps across these points are not assumed known, but found stably from the noisy data.

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1. INTRODUCTION

Let f be a piecewise-$C^2([0,1])$ function, $0 < x_1 < x_2 < \cdots < x_J$, $1 \le j \le J$, are discontinuity points of f. We do not assume their locations x_j and their number J known a priori. We assume that the limits $f(x_j \pm 0)$ exist, and

$$\sup_{x \neq x_j, 1\le j \le J} |f^{(m)}(x)| \le M_m, \quad m = 0,1,2. \tag{1.1}$$

Assume that f_δ is given, $\|f - f_\delta\| := \sup_{x\neq x_j, 1\le j\le J} |f - f_\delta| \le \delta$, where $f_\delta \in L^\infty(0,1)$ are the noisy data.

The problem is: given $\{f_\delta, \delta\}$, where $\delta \in (0, \delta_0)$ and $\delta_0 > 0$ is a small number, estimate stably f', find the locations of discontinuity points x_j of f and their number J, and estimate the jumps $p_j := f(x_j + 0) - f(x_j - 0)$ of f across x_j, $1 \le j \le J$.

A stable estimate $R_\delta f_\delta$ of f' is an estimate satisfying the relation $\lim_{\delta\to 0} ||R_\delta f_\delta - f'|| = 0$.

There is a large literature on stable differentiation of noisy smooth functions (e.g., see references in [3]), but the problem stated above was not solved for piecewise-smooth functions by the method given below. A statistical estimation of the location of discontinuity points from noisy discrete data is given in [1]. In [5], [7], [2], various approaches to finding discontinuities of functions from the measured values of these functions are developed.

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The following formula was proposed originally (in 1968, see [4], and [3]) for stable estimation of $f'(x)$, assuming $f \in C^2([0,1])$, $M_2 \neq 0$, and given noisy data f_δ:

$$(1.2) \quad R_\delta f_\delta := \frac{f_\delta(x+h(\delta)) - f_\delta(x-h(\delta))}{2h(\delta)}, \quad h(\delta) := \left(\frac{2\delta}{M_2}\right)^{\frac{1}{2}}, \quad h(\delta) \leq x \leq 1 - h(\delta),$$

and

$$(1.3) \qquad \|R_\delta f_\delta - f'\| \leq \sqrt{2M_2\delta} := \varepsilon(\delta),$$

where the norm in (1.3) is the $L^\infty(0,1)-$norm. The numerical efficiency and stability of the stable differentiation method proposed in [4] has been demonstrated in [6]. Moreover, (cf [3]),

$$(1.4) \qquad \inf_T \sup_{f \in K(M_2,\delta)} \|Tf_\delta - f'\| \geq \varepsilon(\delta),$$

where $T : L^\infty(0,1) \to L^\infty(0,1)$ runs through the set of all bounded operators, $K(M_2,\delta) := \{f : \|f''\| \leq M_2,\ \|f - f_\delta\| \leq \delta\}$. Therefore (1.2) is the best possible estimate of f', given noisy data f_δ, and assuming $f \in K(M_2,\delta)$.

In [3] this result was generalized to the case $f \in K(M_a,\delta)$, $\|f^{(a)}\| \leq M_a$, $1 < a \leq 2$, where $\|f^{(a)}\| := \|f\| + \|f'\| + \sup_{x,x'} \frac{|f'(x)-f'(x')|}{|x-x'|^{a-1}}$, $1 < a \leq 2$, and $f^{(a)}$ is the fractional-order derivative of f.

The aim of this paper is to extend the above results to the case of piecewise-smooth functions. In Section 2 the results are formulated, and proofs are given. In Section 3 the case of continuous piecewise-smooth functions is treated.

2. FORMULATION OF THE RESULT

Theorem 2.1. *Formula* (1.2) *gives stable estimate of f' on the set*

$$S_\delta := [h(\delta), 1 - h(\delta)] \Big\backslash \bigcup_{j=1}^{J} (x_j - h(\delta), x_j + h(\delta)) \,,$$

and (1.3) *holds with the norm $\|\cdot\|$ taken on the set S_δ. Assuming $M_2 > 0$ and computing the quantities $f_j := \frac{f_\delta(jh+h)-f_\delta(jh-h)}{2h}$, where $h := h(\delta) := \left(\frac{2\delta}{M_2}\right)^{\frac{1}{2}}$, $1 \leq j < [\frac{1}{h}]$, for sufficiently small δ, one finds the location of discontinuity points of f with accuracy $2h$, and their number J. Here $[\frac{1}{h}]$ is the integer smaller than $\frac{1}{h}$ and closest to $\frac{1}{h}$. The discontinuity points of f are located on the intervals $(jh-h, jh+h)$ such that $|f_j| \gg 1$ for sufficiently small δ, where $\varepsilon(\delta)$ is defined in (1.3). The size p_j of the jump of f across the discontinuity point x_j is estimated by the formula $p_j \approx f_\delta(jh+h) - f_\delta(jh-h)$, and the error of this estimate is $O(\sqrt{\delta})$.*

Let us assume that $\min_j |p_j| := p \gg h(\delta)$, where $\gg$ means "much greater than". Then x_j is located on the j^{th} interval $[jh-h, jh+h]$, $h := h(\delta)$, such that

$$(2.1) \qquad |f_j| := \left|\frac{f_\delta(jh+h) - f_\delta(jh-h)}{2h}\right| \gg 1,$$

so that x_j is localized with the accuracy $2h(\delta)$. More precisely, $|f_j| \geq \frac{|f(jh+h)-f(jh-h)|}{2h} - \frac{\delta}{h}$, and $\frac{\delta}{h} = 0.5\varepsilon(\delta)$, where $\varepsilon(\delta)$ is defined in (1.3). One has

$$\begin{aligned} |f(jh+h) - f(jh-h)| &\geq |p_j| - |f(jh+h) - f(x_j+0)| - |f(jh-h) - f(x_j-0)| \\ &\geq |p_j| - 2M_1 h. \end{aligned}$$

Thus,

$$|f_j| \geq \frac{|p_j|}{2h} - M_1 - 0.5\varepsilon(\delta) = c_1 \frac{|p_j|}{\sqrt{\delta}} - c_2 \gg 1,$$

where $c_1 := \frac{\sqrt{M_2}}{2\sqrt{2}}$, and $c_2 := M_1 + 0.5\varepsilon(\delta)$.

The jump p_j is estimated by the formula:

$$p_j \approx [f_\delta(jh+h) - f_\delta(jh-h)], \tag{2.2}$$

and the error estimate of this formula can be given:

$$|p_j - [f_\delta(jh+h) - f_\delta(jh-h)]| \leq 2\delta + 2M_1 h = 2\delta + 2M_1\sqrt{\frac{2\delta}{M_2}} = O(\sqrt{\delta}). \tag{2.3}$$

Thus, the error of the calculation of p_j by the formula $p_j \approx f_\delta(jh+h) - f_\delta(jh-h)$ is $O(\delta^{\frac{1}{2}})$ as $\delta \to 0$.

Proof of Theorem 2.1. If $x \in S_\delta$, then using Taylor's formula one gets:

$$|(R_\delta f_\delta)(x) - f'(x)| \leq \frac{\delta}{h} + \frac{M_2 h}{2}. \tag{2.4}$$

Here we assume that $M_2 > 0$ and the interval $(x - h(\delta), x + h(\delta)) \subset S_\delta$, i.e., this interval does not contain discontinuity points of f. If, for all sufficiently small h, not necessarily for $h = h(\delta)$, inequality (2.4) fails, i.e., if $|(R_\delta f_\delta)(x) - f'(x)| > \frac{\delta}{h} + \frac{M_2 h}{2}$ for all sufficiently small $h > 0$, then the interval $(x-h, x+h)$ contains a point $x_j \notin S_\delta$, i.e., a point of discontinuity of f or f'. This observation can be used for locating the position of an isolated discontinuity point x_j of f with any desired accuracy provided that the size $|p_j|$ of the jump of f across x_j is greater than 4δ, $|p_j| > 4\delta$, and that h can be taken as small as desirable. Indeed, if $x_j \in (x-h, x+h)$, then we have

$$|p_j| - 2hM_1 - 2\delta \leq |f_\delta(x+h) - f_\delta(x-h)| \leq |p_j| + 2hM_1 + 2\delta.$$

The above estimate follows from the relation

$$\begin{aligned}
&|f_\delta(x+h) - f_\delta(x-h)| \\
&\qquad = |f(x+h) - f(x_j+0) + p_j + f(x_j-0) - f(x-h) \pm 2\delta| \\
&\qquad = ||p_j| \pm (2hM_1 + 2\delta)|.
\end{aligned}$$

Here $|p \pm b|$, where $b > 0$, denotes a quantity such that $|p| - b \leq |p \pm b| \leq |p| + b$. Thus, if h is sufficiently small and $|p_j| > 4\delta$, then the inequality $2\delta - 2hM_1 \leq |f_\delta(x+h) - f_\delta(x-h)|$ can be checked, and therefore the inclusion $x_j \in (x-h, x+h)$ can be checked. Since $h > 0$ is arbitrarily small in this argument, it follows that the location of the discontinuity point x_j of f is established with arbitrary accuracy. Additional discussion of the case when a discontinuity point x_j belongs to the interval $(x - h(\delta), x + h(\delta))$ will be given below.

Minimizing the right-hand side of (2.4) with respect to h yields formula (1.2) for the minimizer $h = h(\delta)$ defined in (1.2), and estimate (1.3) for the minimum of the right-hand side of (2.4).

If $p \gg h(\delta)$, and (2.1) holds, then the discontinuity points are located with the accuracy $2h(\delta)$, as we prove now.

Consider the case when a discontinuity point x_j of f belongs to the interval $(jh-h, jh+h)$, where $h = h(\delta)$. Then estimate (2.2) can be obtained as follows. For $jh - h \leq x_j \leq jh + h$, one has

$$\begin{aligned}|f(x_j+0)-f(x_j-0)-f_\delta(jh+h)+f_\delta(jh-h)|\\ \leq 2\delta+|f(x_j+0)-f(jh+h)|+|f(x_j-0)-f(jh-h)|\\ \leq 2\delta+2hM_1, \qquad h=h(\delta).\end{aligned}$$

This yields formulas (2.2) and (2.3). Computing the quantities f_j for $1 \leq j < [\frac{1}{h}]$, and finding the intervals on which (2.1) holds for sufficiently small δ, one finds the location of discontinuity points of f with accuracy $2h$, and the number J of these points. For a small fixed $\delta > 0$ the above method allows one to recover the discontinuity points of f at which $|f_j| \geq \frac{|p_j|}{2h} - \frac{\delta}{h} - M_1 \gg 1$. This is the inequality (2.1). If $h = h(\delta)$, then $\frac{\delta}{h} = 0.5\varepsilon(\delta) = O(\sqrt{\delta})$, and $|2hf_j - p_j| = O(\sqrt{\delta})$ as $\delta \to 0$ provided that $M_2 > 0$. Theorem 2.1 is proved. □

Remark 2.2. Similar results can be derived if $\|f^{(a)}\|_{L^\infty(S_\delta)} := \|f^{(a)}\|_{S_\delta} \leq M_a$, $1 < a \leq 2$. In this case $h = h(\delta) = c_a\delta^{\frac{1}{a}}$, where $c_a = \left[\frac{2}{M_a(a-1)}\right]^{\frac{1}{a}}$, $R_\delta f_\delta$ is defined in (1.2), and the error of the estimate is:

$$\|R_\delta f_\delta - f'\|_{S_\delta} \leq aM_a^{\frac{1}{a}}\left(\frac{2}{a-1}\right)^{\frac{a-1}{a}}\delta^{\frac{a-1}{a}}.$$

The proof is similar to that given in Section 3. It is proved in [3] that for C^a-functions given with noise it is possible to construct stable differentiation formulas if $a > 1$ and it is impossible to construct such formulas if $a \leq 1$. The obtained formulas are useful in applications. One can also use the L^p-norm on S_δ in the estimate $\|f^{(a)}\|_{S_\delta} \leq M_a$ (cf. [3]).

Remark 2.3. The case when $M_2 = 0$ requires a special discussion. In this case the last term on the right-hand side of formula (2.4) vanishes and the minimization with respect to h becomes void: it requires that h be as large as possible, but one cannot take h arbitrarily large because estimate (2.4) is valid only on the interval $(x - h, x + h)$ which does not contain discontinuity points of f, and these points are unknown. If $M_2 = 0$, then f is a piecewise-linear function. The discontinuity points of a piecewise-linear function can be found if the sizes $|p_j|$ of the jumps of f across these points satisfy the inequality $|p_j| \gg 2\delta + 2M_1h$ for some choice of h. For instance, if $h = \frac{\delta}{M_1}$, then $2\delta + 2M_1h = 4\delta$. So, if $|p_j| \gg 4\delta$, then the location of discontinuity points of f can be found in the case when $M_2 = 0$. These points are located on the intervals for which $|f_\delta(jh + h) - f_\delta(jh - h)| \gg 4\delta$, where $h = \frac{\delta}{M_1}$.

The size $|p_j|$ of the jump of f across a discontinuity point x_j can be estimated by formula (2.2) with $h = \frac{\delta}{M_1}$, and one assumes that $x_j \in (jh - h, jh + h)$ is the only discontinuity point on this interval. The error of the formula (2.2) is estimated as in the proof of Theorem 2.1. This error is not more than $2\delta + 2M_1h = 4\delta$ for the above choice of $h = \frac{\delta}{M_1}$.

One can estimate the derivative of f at the point of smoothness of f assuming $M_2 = 0$ provided that this derivative is not too small. If $M_2 = 0$, then $f = a_jx + b_j$ on every interval Δ_j between the discontinuity points x_j, where a_j and b_j are some constants. If $(jh - h, jh + h) \subset \Delta_j$, and $f_j := \frac{f_\delta(jh+h)-f_\delta(jh-h)}{2h}$, then $|f_j - a_j| \leq \frac{\delta}{h}$. Choose $h = \frac{t\delta}{M_1}$, where $t > 0$ is a parameter, and $M_1 = \max_j |a_j|$. Then the relative error of the approximate formula $a_j \approx f_j$ for the derivative $f' = a_j$ on Δ_j equals to $\frac{|f_j-a_j|}{|a_j|} \leq \frac{M_1}{t|a_j|}$. Thus, if, e.g., $|a_j| \geq \frac{M_1}{2}$ and $t = 20$, then the relative error of the above approximate formula is not more than 0.1.

3. CONTINUOUS PIECEWISE-SMOOTH FUNCTIONS

Suppose now that $\xi \in (mh-h, mh+h)$, where $m>0$ is an integer, and ξ *is a point at which* f *is continuous but* $f'(\xi)$ *does not exist*. Thus, the jump of f across ξ is zero, but ξ is not a point of smoothness of f. *How does one locate the point* ξ*?*

The algorithm we propose consists of the following. We assume that $M_2>0$ on S_δ. Calculate the numbers $f_j := \frac{f_\delta(jh+h)-f_\delta(jh-h)}{2h}$ and $|f_{j+1}-f_j|$, $j=1,2,\dots$, $h=h(\delta)=\sqrt{\frac{2\delta}{M_2}}$. Inequality (1.3) implies $f_j-\varepsilon(\delta) \le f'(jh) \le f_j+\varepsilon(\delta)$, where $\varepsilon(\delta)$ is defined in (1.3).

Therefore, if $|f_j|>\varepsilon(\delta)$, then $\operatorname{sgn} f_j = \operatorname{sgn} f'(jh)$.

One has:

$$J-\frac{2\delta}{h} \le |f_{j+1}-f_j| \le J+\frac{2\delta}{h},$$

where $\frac{\delta}{h}=0.5\varepsilon(\delta)$ and $J := |\frac{f(jh+2h)-f(jh)-f(jh+h)+f(jh-h)}{2h}|$. Using Taylor's formula, one derives the estimate:

$$0.5[J_1-\varepsilon(\delta)] \le J \le 0.5[J_1+\varepsilon(\delta)], \tag{3.1}$$

where $J_1 := |f'(jh+h)-f'(jh)|$.

If the interval $(jh-h, jh+2h)$ belongs to S_δ, then $J_1 = |f'(jh+h)-f'(jh)| \le M_2 h = \varepsilon(\delta)$. In this case $J \le \varepsilon(\delta)$, so

$$|f_{j+1}-f_j| \le 2\varepsilon(\delta) \quad \text{if} \quad (jh-h, jh+2h) \subset S_\delta. \tag{3.2}$$

Conclusion: *If* $|f_{j+1}-f_j|>2\varepsilon(\delta)$*, then the interval* $(jh-h, jh+2h)$ *does not belong to* S_δ*, that is, there is a point* $\xi \in (jh-h, jh+2h)$ *at which the function* f *is not twice continuously differentiable with* $|f''| \le M_2$*. Since we assume that either at a point* ξ *the function is twice differentiable, or at this point* f' *does not exist, it follows that if* $|f_{j+1}-f_j|>2\varepsilon(\delta)$*, then there is a point* $\xi \in (jh-h, jh+2h)$ *at which* f' *does not exist.*

If

$$f_j f_{j+1} < 0, \tag{3.3}$$

and

$$\min(|f_{j+1}|, |f_j|) > \varepsilon(\delta), \tag{3.4}$$

then (3.3) implies $f'(jh)f'(jh+h)<0$, so the interval $(jh, jh+h)$ contains a critical point ξ of f, or a point ξ at which f' does not exist. To determine which one of these two cases holds, let us use the right inequality (3.1). If ξ is a critical point of f and $\xi \in (jh, jh+h) \subset S_\delta$, then $J_1 \le \varepsilon(\delta)$, and in this case the right inequality (3.1) yields

$$|f_{j+1}-f_j| \le 2\varepsilon(\delta). \tag{3.5}$$

Conclusion: *If (3.3) – (3.5) hold, then* ξ *is a critical point. If (3.3) and (3.4) hold and* $|f_{j+1}-f_j|>\varepsilon(\delta)$ *then* ξ *is a point of discontinuity of* f'*.*

If ξ is a point of discontinuity of f', we would like to estimate the jump

$$P := |f'(\xi+0)-f'(\xi-0)|.$$

Using Taylor's formula one gets

$$f_{j+1}-f_j = \frac{P}{2} \pm 2.5\varepsilon(\delta). \tag{3.6}$$

The expression $A=B\pm b,\ b>0$, means that $B-b \le A \le B+b$. Therefore,

$$P = 2(f_{j+1}-f_j) \pm 5\varepsilon(\delta). \tag{3.7}$$

We have proved the following theorem:

Theorem 3.1. *If $\xi \in (jh-h, jh+2h)$ is a point of continuity of f and $|f_{j+1}-f_j| > 2\varepsilon(\delta)$, then ξ is a point of discontinuity of f'. If* (3.3) *and* (3.4) *hold, and $|f_{j+1}-f_j| \le 2\varepsilon(\delta)$, then ξ is a critical point of f. If* (3.3) *and* (3.4) *hold and $|f_{j+1}-f_j| > 2\varepsilon(\delta)$, then $\xi \in (jh, jh+h)$ is a point of discontinuity of f'. The jump P of f' across ξ is estimated by formula (3.7).*

4. FINDING NONSMOOTHNESS POINTS OF PIECEWISE-LINEAR FUNCTIONS

Assume that f is a piecewise-linear function on the interval $[0,1]$ and $0 < x_1 < \cdots < x_J < 1$ are its nonsmoothness points, i.e, the discontinuity points of f or those of f'. Assume that f_δ is known at a grid mh, $m = 0,1,2,\dots,M$, $h = \frac{1}{M}$, $f_{\delta,m} = f_\delta(mh)$, $|f(mh) - f_{\delta,m}| \le \delta \ \forall m$, $f_m = f(mh)$. If mh is a discontinuity point, $mh = x_j$, then we define its value as $f(x_j - 0)$ or $f(x_j + 0)$, depending on which of these two numbers satisfy the inequalty $|f(mh) - f_{\delta,m}| \le \delta$.

The problem is: given $f_{\delta,m}$ $\forall m$, estimate the location of the discontinuity points x_j, their number J, find out which of these points are points of discontinuity of f and which are points of discontinuity of f' but points of continuity of f, and estimate the sizes of the jumps $p_j = |f(x_j+0) - f(x_j-0)|$ and the sizes of the jumps $q_j = |f'(x_j+0) - f'(x_j-0)|$ at the continuity points of f which are discontinuity points of f'.

Let us solve this problem. Consider the quantities

$$G_m := \frac{f_{\delta,m+1} - 2f_{\delta,m} + f_{\delta,m-1}}{2h^2} = g_m + w_m,$$

where

$$g_m := \frac{f_{m+1} - 2f_m + f_{m-1}}{2h^2}, \qquad w_m := \frac{f_{\delta,m+1} - f_{m+1} - 2(f_{\delta,m} - f_m) + f_{\delta,m-1} - f_m}{2h^2}.$$

We have

$$|w_m| \le \frac{4\delta}{2h^2} = \frac{2\delta}{h^2},$$

and

$$g_m = 0 \text{ if } x_j \notin (mh-h, mh+h) \quad \forall j.$$

Therefore, *if* $\min_j |x_{j+1} - x_j| > 2h$ *and*

$$|G_m| > \frac{2\delta}{h^2}, \tag{4.1}$$

then the interval $(mh-h, mh+h)$ must contain a discontinuity point of f. This condition is sufficient for the interval $(mh-h, mh+h)$ to contain a discontinuity point of f, but not a necessary one: it may happen that the interval $(mh-h, mh+h)$ contains more than one discontinuity point without changing g_m or G_m, so that one cannot detect these points by the above method. We have proved the following result.

Theorem 4.1. *Condition (4.1) is a sufficient condition for the interval $(mh-h, mh+h)$ to contain a nonsmoothness point of f. If one knows a priori that $x_{j+1} - x_j > 2h$ then condition (4.1) is a necessary and sufficient condition for the interval $(mh-h, mh+h)$ to contain exactly one point of nonsmoothness of f.*

Let us estimate the size of the jump p_j. Let us assume that (4.1) holds, $x_{j+1} - x_j > 2h$ and $x_j \in (mh-h, mh)$. The case when $x_j \in (mh, mh+h)$ is treated similarly. Let $f(x) = a_j x + b_j$ when $mh < x < x_j$, and $f(x) = a_{j+1}x + b_{j+1}$ when $x_j < x < (m+1)h$, where a_j, b_j are constants. One has

$$g_m = \frac{-(a_{j+1} - a_j)(mh-h) - (b_{j+1} - b_j)}{2h^2},$$

and

$$p_j = |(a_{j+1} - a_j)x_j + b_{j+1} - b_j|.$$

Thus

$$\begin{aligned} |g_m| &= \left| \frac{-(a_{j+1} - a_j)x_j - (b_{j+1} - b_j) - (a_{j+1} - a_j)(mh - h - x_j)}{2h^2} \right| \\ &= \frac{p_j}{2h^2} \pm \frac{|a_{j+1} - a_j||x_j - (mh - h)|}{2h^2}, \end{aligned}$$

where the symbol $a \pm b$ means $a - b \le a \pm b \le a + b$. The quantity $|a_{j+1} - a_j| = q_j$, and $|x_j - (mh - h)| \le h$ if $mh - h < x_j < mh$.

Thus,

$$|G_m| = \frac{p_j}{2h^2} \pm \left(\frac{q_j h}{2h^2} + \frac{2\delta}{h^2} \right),$$

and

$$|G_m| = \frac{p_j}{2h^2} \left(1 \pm \frac{q_j h + 4\delta}{p_j} \right),$$

provided that $p_j > 0$.

If

$$\frac{q_j h + 4\delta}{p_j} \ll 1 \text{ and } p_j > 0,$$

then

$$p_j \approx 2h^2 |G_m|.$$

If $p_j = 0$ then

$$|G_m| = \frac{q_j}{2h} \pm \frac{2\delta}{h^2}.$$

Thus,

$$q_j \approx 2h|G_m|.$$

Finally, the number of the nonsmoothness points of f can be determined as the number of intervals on which (4.1) holds.

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