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GENERALIZED AUXILIARY PROBLEM PRINCIPLE AND SOLVABILITY OF A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES INVOLVING COCOERCIVE AND CO-LIPSCHITZIAN MAPPINGS

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ABSTRACT. The approximation-solvability of the following class of nonlinear variational inequality (NVI) problems, based on a new generalized auxiliary problem principle, is discussed.

Find an element $x^* \in K$ such that

$$\langle (S-T)(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0 \text{ for all } x \in K,$$

where $S, T : K \to H$ are mappings from a nonempty closed convex subset K of a real Hilbert space H into H, and $f : K \to \mathbb{R}$ is a continuous convex functional on K. The generalized auxiliary problem principle is described as follows: for given iterate $x^k \in K$ and, for constants $\rho > 0$ and $\sigma > 0$), find x^{k+1} such that

$$\left\langle \rho(S-T)\left(y^k\right) + h'\left(x^{k+1}\right) - h'\left(y^k\right), x - x^{k+1}\right\rangle + \rho(f(x) - f(x^{k+1})) \geq 0 \text{ for all } x \in K,$$

where

$$\left\langle \sigma(S-T)\left(x^k\right) + h'\left(y^k\right) - h'\left(x^k\right), x - y^k\right\rangle + \sigma(f(x) - f(y^k)) \geq 0 \text{ for all } x \in K,$$

where h is a functional on K and h' the derivative of h.

Key words and phrases: Generalized auxiliary variational inequality problem, Cocoercive mappings, Approximation-solvability, Approximate solutions, Partially relaxed monotone mappings.

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1. INTRODUCTION

Recently, Zhu and Marcotte [23], based on the auxiliary problem principle introduced by Cohen [3], investigated the approximation-solvability of a class of variational inequalities involving the cocoercive and partially cocoercive mappings in the $\mathbb{R}^n$ space. The auxiliary problem technique introduced by Cohen [3], is quite similar to that of the iterative algorithm characterized as the auxiliary variational inequality studied by Marcotte and Wu [12], but the estimates

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for the approximate solutions seem to be significantly different, which makes a difference establishing the convergence of the sequence of approximate solutions to a given solution of the original variational inequality under consideration. On the top of that, using the auxiliary problem principle, one does not require any projection formula leading to a fixed point and eventually the solution of the variational inequality, which has been the case following the variational inequality type algorithm adopted by Marcotte and Wu [12]. Recently Verma [21] introduced an iterative scheme characterized as an auxiliary variational inequality type of algorithm and applied to the approximation-solvability of a class of nonlinear variational inequalities involving cocoercive as well as partially relaxed monotone mappings [18] in a Hilbert space setting. The partially relaxed monotone mappings seem to be weaker than cocoercive and strongly monotone mappings. In this paper, we first intend to introduce the generalized auxiliary problem principle, and then apply the generalized auxiliary problem principle, which includes the auxiliary problem principle of Cohen [3] as a special case, to approximation-solvability of a class of nonlinear variational inequalities involving cocoercive mappings. The obtained results do complement the earlier works of Cohen [3], Zhu and Marcotte [23] and Verma [18] on the approximation- solvability of nonlinear variational inequalities in different space settings.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let S,$T : K \to H$ be any mappings and K a closed convex subset of H. Let $f : K \to \mathbb{R}$ be a continuous convex function. We consider a class of nonlinear variational inequality (abbreviated as NVI) problems: find an element $x^* \in K$ such that

$$\langle (S-T)(x^*), x - x^*\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K. \tag{1.1}$$

Now we need to recall the following auxiliary result, most commonly used in the context of the approximation-solvability of the nonlinear variational inequality problems based on the iterative procedures.

Lemma 1.1. *An element $u \in K$ is a solution of the NVI problem (1.1) if*

$$\langle (S-T)(u), x-u\rangle + f(x) - f(u) \geq 0 \quad \text{for all } x \in K.$$

A mapping $S : H \to H$ is said to be α-*cocoercive* [19] if for all $x, y \in H$, we have

$$\|x-y\|^2 \geq \alpha^2 \|S(x)-S(y)\|^2 + \|\alpha(S(x)-S(y)) - (x-y)\|^2,$$

where $\alpha > 0$ is a constant.

A mapping $S : H \to H$ is called α-*cocoercive* [12] if there exists a constant $\alpha > 0$ such that

$$\langle S(x)-S(y), x-y\rangle \geq \alpha \|S(x)-S(y)\|^2 \quad \text{for all } x, y \in H.$$

S is called r-*strongly monotone* if for each $x, y \in H$, we have

$$\langle S(x)-S(y), x-y\rangle \geq r\|x-y\|^2 \text{ for a constant } r > 0.$$

This implies that

$$\|S(x)-S(y)\| \geq r\|x-y\|,$$

that is, S is r-expanding, and when $r = 1$, it is expanding. The mapping S is called $\beta-$*Lipschitz continuous* (or $\beta-$Lipschitzian) if there exists a constant $\beta \geq 0$ such that

$$\|S(x)-S(y)\| \leq \beta\|x-y\| \quad \text{for all } x, y \in H.$$

We note that if S is α-cocoercive and expanding, then S is α-strongly monotone. On the top of that, if S is α-strongly monotone and $\beta-$Lipschitz continuous, then S is $\left(\frac{\alpha}{\beta^2}\right)$ cocoercive for $\beta > 0$. Clearly every α-cocoercive mapping S is $\left(\frac{1}{\alpha}\right)$ $-$Lipschitz continuous.

Proposition 1.2. [21]. *Let $S : H \to H$ be a mapping from a Hilbert space H into itself. Then the following statements are equivalent:*

(i) *For each* $x, y \in H$ *and for a constant* $\alpha > 0$, *we have*

$$\|x-y\|^2 \geq \alpha^2 \|S(x)-S(y)\|^2 + \|\alpha(S(x)-S(y))-(x-y)\|^2 .$$

(ii) *For each* $x, y \in H$, *we have*

$$\langle S(x)-S(y), x-y\rangle \geq \alpha \|S(x)-S(y)\|^2 ,$$

where $\alpha > 0$ *is a constant.*

Lemma 1.3. *For all elements* $v, w \in H$, *we have*

$$\|v\|^2 + \langle v, w\rangle \geq -\frac{1}{4}\|w\|^2 .$$

A mapping $S : H \to H$ is said to be $\gamma-$*partially relaxed monotone* [18] if for all $x, y, z \in H$, we have

$$\langle S(x)-S(y), z-y\rangle \geq -\gamma \|z-x\|^2 \quad \text{for} \quad \gamma > 0.$$

Proposition 1.4. [18]. *Let* $S : H \to H$ *be an* α*-cocoercive mapping on* H. *Then* S *is* $\left(\frac{1}{4\alpha}\right)-$ *partially relaxed monotone.*

Proof. We include the proof for the sake of the completeness. Since S is α-cocoercive, it implies by Lemma 1.1, for all $x, y, z \in H$, that

$$\begin{aligned}
\langle S(x)-S(y), z-y\rangle &= \langle S(x)-S(y), x-y\rangle + \langle S(x)-S(y), z-x\rangle \\
&\geq \alpha \|S(x)-S(y)\|^2 + \langle S(x)-S(y), z-x\rangle \\
&= \alpha \left\{ \|S(x)-S(y)\|^2 + \left(\frac{1}{\alpha}\right) \langle S(x)-S(y), z-x\rangle \right\} \\
&\geq -\left(\frac{1}{4\alpha}\right)\|z-x\|^2 ,
\end{aligned}$$

that is, S is $\left(\frac{1}{4\alpha}\right)$ −partially relaxed monotone.

A mapping $T : H \to H$ is said to be μ*-co-Lipschitz continuous* if for each $x, y \in H$ and for a constant $\mu > 0$, we have

$$\|x-y\| \leq \mu \|T(x)-T(y)\| .$$

This clearly implies that

$$\langle T(x)-T(y), x-y\rangle \leq \mu \|T(x)-T(y)\|^2 .$$

Clearly, every μ-co-Lipschitz continuous mapping T is $\left(\frac{1}{\mu}\right)$ −expanding. □

2. GENERALIZED AUXILIARY PROBLEM PRINCIPLE

This section deals with the approximation-solvability of the NVI problem (1.1), based on the generalized auxiliary nonlinear variational inequality problem principle by Verma [18], which includes the auxiliary problem principle introduced by Cohen [3] and later applied and studied by others, including Zhu and Marcotte [22]. This generalized auxiliary nonlinear variational inequality (GANVI) problem is as follows: for a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$):

$$\begin{aligned}
\text{(2.1)} \quad & \left\langle \rho(S-T)\left(y^k\right) + h'\left(x^{k+1}\right) - h'\left(y^k\right), x - x^{k+1}\right\rangle \\
& \qquad + \rho\left(f(x) - f\left(x^{k+1}\right)\right) \geq 0 \quad \text{for all} \quad x \in K,
\end{aligned}$$

where

$$\left\langle \sigma(S-T)\left(x^{k}\right)+h'\left(y^{k}\right)-h'\left(x^{k}\right), x-y^{k}\right\rangle + \sigma\left(f(x)-f\left(y^{k}\right)\right) \geq 0 \quad \text{for all } x \in K, \tag{2.2}$$

and for a strongly convex function h on K (where h' denotes the derivative of h).

When $\sigma = \rho$ in the GANVI problem (2.1)-(2.2), we have GANVI problem as follows: for a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$):

$$\left\langle \rho(S-T)\left(y^{k}\right)+h'\left(x^{k+1}\right)-h'\left(y^{k}\right), x-x^{k+1}\right\rangle + \rho\left(f(x)-f\left(x^{k+1}\right)\right) \geq 0 \quad \text{for all } x \in K, \tag{2.3}$$

where

$$\left\langle \rho(S-T)\left(x^{k}\right)+h'\left(y^{k}\right)-h'\left(x^{k}\right), x-y^{k}\right\rangle + \rho\left(f(x)-f\left(y^{k}\right)\right) \geq 0 \quad \text{for all } x \in K. \tag{2.4}$$

For $\sigma = 0$ and $y^k = x^k$, the GANVI problem (2.1)-(2.2) reduces to: for a given iterate x^k , determine an x^{k+1} such that (for $k \geq 0$):

$$\left\langle \rho\left(S-T\right)\left(x^{k}\right)+h'\left(x^{k+1}\right)-h'\left(x^{k}\right), x-x^{k+1}\right\rangle + \rho\left(f(x)-f\left(x^{k+1}\right)\right) \geq 0 \quad \text{for all } x \in K. \tag{2.5}$$

Next, we recall some auxiliary results crucial to the approximation-solvability of the NVI problem (1.1).

Lemma 2.1. [23]. *Let $h : K \to \mathbb{R}$ be continuously differentiable on a convex subset K of H. Then we have the following conclusions:*

(i) *If h is b-strongly convex, then*

$$h(x)-h(y) \geq \langle h'(y), x-y\rangle + \left(\frac{b}{2}\right)\|x-y\|^{2} \quad \text{for all } x, y \in K.$$

(ii) *If the gradient h' is $p-$Lipschitz continuous, then*

$$h(x)-h(y) \leq \langle h'(y), x-y\rangle + \left(\frac{b}{2}\right)\|x-y\|^{2} \quad \text{for all } x, y \in K.$$

We are just about ready to present, based on the GANVI problem (2.1) – (2.2), the approximation-solvability of the NVI problem (1.1) involving $\gamma-$cocoercive mappings in a Hilbert space setting.

Theorem 2.2. *Let H be a real Hilbert space and $S : K \to H$ a $\gamma-$cocoercive mapping from a nonempty closed convex subset K of H into H. Let $T : K \to H$ be a μ-co-Lipschitz continuous mapping. Suppose that $h : K \to \mathbb{R}$ is continuously differentiable and b-strongly convex, and h', the derivative of h, is $p-$Lipschitz continuous. Then x^{k+1} is a unique solution of (2.1) – (2.2). If in addition, if $x^* \in K$ is any fixed solution of the NVI problem (1.1), then x^k is bounded and converges to x^* for $0 < \rho < \frac{2b}{\gamma}$, $\rho + \sigma < b$ and $\left\langle x^{k+1} - x^k, x^k - y^k \right\rangle \geq 0$.*

Proof. Before we can show that the sequences $\left\{x^k\right\}$ converges to x^*, a solution of the NVI problem (1.1), we need to compute the estimates. Since h is $b-$strongly convex, it ensures the uniqueness of solution x^{k+1} of the GANVI problem (2.1) – (2.2). Let us define a function Λ^* by

$$\begin{aligned}\Lambda^*(x) &:= h(x^*) - h(x) - \langle h'(x), x^* - x\rangle \\ &\geq \left(\frac{b}{2}\right)\|x^* - x\|^2 \quad \text{for } x \in K,\end{aligned}$$

where x^* is any fixed solution of the NVI problem (1.1). It follows for $y^k \in K$ that

$$\begin{aligned}\Lambda^*\left(y^k\right) &= h(x^*) - h\left(y^k\right) - \left\langle h'\left(y^k\right), x^* - y^k\right\rangle \\ &= h(x^*) - h\left(y^k\right) - \left\langle h'\left(y^k\right), x^* - x^{k+1} + x^{k+1} - y^k\right\rangle.\end{aligned}$$

Similarly, we can have

$$\Lambda^*\left(x^{k+1}\right) = h(x^*) - h\left(x^{k+1}\right) - \left\langle h'\left(x^{k+1}\right), x^* - x^{k+1}\right\rangle.$$

Now we can write

$$\begin{aligned}(2.6)\quad & \Lambda^*\left(y^k\right) - \Lambda^*\left(x^{k+1}\right) \\ &= h\left(x^{k+1}\right) - h\left(y^k\right) - \left\langle h'\left(y^k\right), x^{k+1} - y^k\right\rangle + \left\langle h'\left(x^{k+1}\right) - h'\left(y^k\right), x^* - x^{k+1}\right\rangle \\ &\geq \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \left\langle h'\left(x^{k+1}\right) - h'\left(y^k\right), x^* - x^{k+1}\right\rangle \\ &\geq \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\left\langle (S-T)\left(y^k\right), x^{k+1} - x^*\right\rangle + \rho\left(f\left(x^{k+1}\right) - f(x^*)\right),\end{aligned}$$

for $x = x^*$ in (2.1).

If we replace x by x^{k+1} in (1.1) and combine with (2.6), we obtain

$$\begin{aligned}&\Lambda^*\left(y^k\right) - \Lambda^*\left(x^{k+1}\right) \\ &\geq \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\left\langle (S-T)\left(y^k\right), x^{k+1} - x^*\right\rangle - \rho\left\langle (S-T)(x^*), x^{k+1} - x^*\right\rangle \\ &= \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\left\langle (S-T)\left(y^k\right) - (S-T)(x^*), x^{k+1} - x^*\right\rangle \\ &= \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\left\langle (S-T)\left(y^k\right) - (S-T)(x^*), x^{k+1} - y^k + y^k - x^*\right\rangle \\ &= \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\left\langle (S-T)\left(y^k\right) - (S-T)(x^*), y^k - x^*\right\rangle \\ &\quad + \rho\left\langle (S-T)\left(y^k\right) - (S-T)(x^*), x^{k+1} - y^k\right\rangle.\end{aligned}$$

Since S is $\gamma-$cocoercive and T is μ-co-Lipschitz continuous, it implies that

$$\begin{aligned}(2.7)\quad & \Lambda^*\left(y^k\right) - \Lambda^*\left(x^{k+1}\right) \\ &\geq \left(\frac{b}{2}\right)\left\|x^{k+1} - y^k\right\|^2 + \rho\gamma\left\|(S-T)\left(y^k\right) - (S-T)(x^*)\right\|^2 \\ &\quad + \rho\left\langle (S-T)\left(y^k\right) - (S-T)(x^*), x^{k+1} - y^k\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{b}{2}\right)\left\|x^{k+1}-y^{k}\right\|^{2}+\rho\gamma\left\{\left\|(S-T)\left(y^{k}\right)-(S-T)\left(x^{*}\right)\right\|^{2}\right.\\
&\qquad\left.+\left(\frac{1}{\gamma}\right)\left\langle (S-T)\left(y^{k}\right)-(S-T)\left(x^{*}\right), x^{k+1}-y^{k}\right\rangle\right\}\\
&\geq \left(\frac{b}{2}\right)\left\|x^{k+1}-y^{k}\right\|^{2}-\left(\frac{\rho}{4(\gamma-\mu)}\right)\left\|x^{k+1}-y^{k}\right\|^{2} \quad \text{(by Lemma 1.3)}\\
&= \frac{1}{2}\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\left\|x^{k+1}-y^{k}\right\|^{2} \quad \text{for } \gamma-\mu>0.
\end{aligned}
$$

Similarly, we can have

$$
\begin{aligned}
(2.8)\qquad &\Lambda^{*}\left(x^{k}\right)-\Lambda^{*}\left(y^{k}\right)\\
&= h\left(y^{k}\right)-h\left(x^{k}\right)-\left\langle h'\left(x^{k}\right), y^{k}-x^{k}\right\rangle+\left\langle h'\left(y^{k}\right)-h'\left(x^{k}\right), x^{*}-y^{k}\right\rangle\\
&\geq \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}+\left\langle h'\left(y^{k}\right)-h'\left(x^{k}\right), x^{*}-y^{k}\right\rangle\\
&\geq \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}+\sigma\left\langle T\left(x^{k}\right), y^{k}-x^{*}\right\rangle+\sigma\left(f\left(y^{k}\right)-f\left(x^{*}\right)\right),
\end{aligned}
$$

for $x = x^{*}$ in (2.2).

Again, if we replace x by y^k in (1.1) and combine with (2.8), we obtain

$$
\begin{aligned}
(2.9)\quad &\Lambda^{*}\left(x^{k}\right)-\Lambda^{*}\left(y^{k}\right)\\
&\geq \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}+\sigma\left\langle (S-T)\left(x^{k}\right), y^{k}-x^{*}\right\rangle-\sigma\left\langle (S-T)\left(x^{*}\right), y^{k}-x^{*}\right\rangle\\
&= \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}+\sigma\left\langle (S-T)\left(x^{k}\right)-(S-T)\left(x^{*}\right), y^{k}-x^{*}+x^{k}-x^{k}\right\rangle\\
&= \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}+\sigma\left\langle (S-T)\left(x^{k}\right)-(S-T)\left(x^{*}\right), x^{k}-x^{*}\right\rangle\\
&\quad+\sigma\left\langle (S-T)\left(x^{k}\right)-(S-T)\left(x^{*}\right), y^{k}-x^{k}\right\rangle\\
&\geq \left(\frac{b}{2}\right)\left\|y^{k}-x^{k}\right\|^{2}-\left(\frac{\sigma}{4(\gamma-\mu)}\right)\left\|y^{k}-x^{k}\right\|^{2}\\
&= \left(\frac{1}{2}\right)\left[b-\left(\frac{\sigma}{2(\gamma-\mu)}\right)\right]\left\|y^{k}-x^{k}\right\|^{2}.
\end{aligned}
$$

Finally, we move toward finding the required estimate

(2.10)

$$
\begin{aligned}
&\Lambda^{*}\left(x^{k}\right)-\Lambda^{*}\left(x^{k+1}\right)\\
&\quad= \Lambda^{*}\left(x^{k}\right)-\Lambda^{*}\left(y^{k}\right)+\Lambda^{*}\left(y^{k}\right)-\Lambda^{*}\left(x^{k+1}\right)\\
&\quad\geq \left(\frac{1}{2}\right)\left[b-\left(\frac{\sigma}{2(\gamma-\mu)}\right)\right]\left\|y^{k}-x^{k}\right\|^{2}+\left(\frac{1}{2}\right)\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\left\|x^{k+1}-y^{k}\right\|^{2}\\
&\quad= \left(\frac{1}{2}\right)\left[b-\left(\frac{\sigma}{2(\gamma-\mu)}\right)\right]\left\|y^{k}-x^{k}\right\|^{2}+\left(\frac{1}{2}\right)\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\\
&\quad\times\left\{\left\|x^{k+1}-x^{k}\right\|^{2}+\left\|x^{k}-y^{k}\right\|^{2}+2\left\langle x^{k+1}-x^{k}, x^{k}-y^{k}\right\rangle\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\right)\left\{\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\left\|x^{k+1}-x^{k}\right\|^{2}+\left[b-\left(\frac{\sigma+\rho}{4(\gamma-\mu)}\right)\right]\left\|y^{k}-x^{k}\right\|^{2}\right. \\
&\quad \left.+\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\left\langle x^{k+1}-x^{k},x^{k}-y^{k}\right\rangle\right\} \\
&\geq \left(\frac{1}{2}\right)\left[b-\left(\frac{\rho}{2(\gamma-\mu)}\right)\right]\left\|x^{k+1}-x^{k}\right\|^{2} \text{ for } b-\frac{\rho}{2(\gamma-\mu)}>0,
\end{aligned}
$$

$b-\frac{\sigma+\rho}{4(\gamma-\mu)}>0$ and $\left\langle x^{k+1}-x^{k},x^{k}-y^{k}\right\rangle \geq 0.$

□

It follows from (2.10) that for $x^{k+1}=y^k=x^k$ that x^k is a solution of the variational inequality. If not, the conditions $b-\frac{\rho}{2(\gamma-\mu)}>0$, $b-\frac{\sigma+\rho}{4(\gamma-\mu)}>0$ and $\left\langle x^{k+1}-x^{k},x^{k}-y^{k}\right\rangle \geq 0$ ensure that the sequence $\left\{\Lambda^*(x^k)-\Lambda^*(x^{k+1})\right\}$ is nonnegative and, as a result, we have

$$\lim_{k\to\infty}\left\|x^{k+1}-x^{k}\right\|=0.$$

On the top of that, $\left\|x^*-x^k\right\|^2 \leq \left(\frac{2}{b}\right)\Lambda^*\left(x^k\right)$ and the sequence $\left\{\Lambda^*\left(x^k\right)\right\}$ is decreasing , that means $\left\{x^k\right\}$ is a bounded sequence. Assume that x' is a cluster point of $\left\{x^k\right\}$. Then as $k\to\infty$ in (2.1) – (2.2), x' is a solution of the variational inequality because there is no loss generality if x^* is replaced by x'. If we associate x' to Λ' and define Λ' by

$$
\begin{aligned}
\Lambda'\left(x^k\right) &= h\left(x'\right)-h\left(x^k\right)-\left\langle h'\left(x^k\right),x'-x^k\right\rangle \\
&\leq \left(\frac{p}{2}\right)\left\|x'-x^k\right\|^2 \quad \text{(by Lemma 2.1)},
\end{aligned}
$$

then we have

$$\Lambda'\left(x^k\right)\leq\left(\frac{p}{2}\right)\left\|x'-x^k\right\|^2.$$

Since the sequence $\left\{\Lambda'\left(x^k\right)\right\}$ is strictly decreasing, it follows that $\Lambda'\left(x^k\right)\to 0$. On the other hand, we already have

$$\Lambda'\left(x^k\right)\geq\left(\frac{b}{2}\right)\left\|x'-x^k\right\|^2.$$

Thus, we can conclude that the entire sequence $\left\{x^k\right\}$ converges to x', and this completes the proof. For $\sigma=\rho$ in Theorem 2.2, we find:

Theorem 2.3. *Let H be a real Hilbert space and $T:K\to H$ a $\gamma-$cocoercive mapping from a nonempty closed convex subset K of H into H. Let $h:K\to\mathbb{R}$ be continuously differentiable and $b-$strongly convex, and h', the derivative of h, is $p-$Lipschitz continuous. Then x^{k+1} is a unique solution of (2.3) – (2.4).*

If in addition, $x^*\in K$ is any fixed solution of the NVI problem (1.1), then $\left\{x^k\right\}$ is bounded and converges to x^* for $0<\rho<\frac{2b}{\gamma}$ and $\left\langle x^{k+1}-x^{k},x^{k}-y^{k}\right\rangle \geq 0$.

When $\sigma=0$ and $y^k=x^k$, Theorem 2. reduces to:

Theorem 2.4. [23]. *Let H be a real Hilbert space and $T:K\to H$ a $\gamma-$cocoercive mapping from a nonempty closed convex subset K of H into H. Let $h:K\to\mathbb{R}$ be continuously differentiable and $b-$strongly convex, and h', the derivative of h, is $p-$Lipschitz continuous. Then x^{k+1} is a unique solution of (2.5).*

If in addition, $x^*\in K$ is any fixed solution of the NVI problem (1.1), then $\left\{x^k\right\}$ is bounded and converges to x^* for $0<\rho<\frac{2b}{\gamma}$.

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