

Journal of Inequalities in Pure and Applied Mathematics

A q -ANALOGUE OF AN INEQUALITY DUE TO KONRAD KNOPP

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©2000 Victoria University
ISSN (electronic): 1443-5756
023-06



volume 7, issue 3, article 110,
2006.

Received 25 January, 2006;
accepted 22 March, 2006.

Communicated by: H.M. Srivastava

Abstract

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Abstract

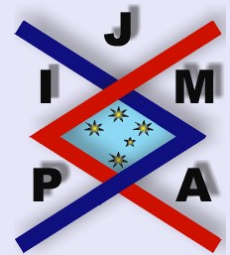
We derive a simple q -analogue of Konrad Knopp's inequality for Euler-Knopp means, using the finite and infinite q -binomial theorems.

2000 Mathematics Subject Classification: 05A30.

Key words: Knopp's inequality, q -analogue.

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1. Introduction

Given a sequence (a_n) and a number t such that $0 < t < 1$, its Euler-Knopp mean $e_n(t)$ is defined by

$$e_n(t) = \sum_{m=0}^n \binom{n}{m} (1-t)^{n-m} t^m a_m.$$

Knopp's inequality [2, Section 3] states that, if $p > 1$ and $0 < t < 1$ then

$$(1.1) \quad \sum_{n=0}^{\infty} [e_n(t)]^p \leq \frac{1}{t} \sum_{m=0}^{\infty} [a_m]^p$$

if the series on the right hand side is convergent.

The aim of this short note is to prove the following q -extension of Knopp's inequality (1.1), valid if $p > 1$, $0 < t < 1$ and $0 < q < 1$:

$$(1.2) \quad \sum_{n=0}^{\infty} [e_n(t; q)]^p \leq \frac{1}{t} \sum_{m=0}^{\infty} q^{\frac{m(m-1)}{2}} [a_m]^p,$$

provided the series on the right hand side of (1.2) converges. The sequence $(e_n(t; q))$ is

$$e_n(t; q) = \sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} t^m a_m$$

with the q -analogue of the numbers $\binom{n}{m}_q$ defined by

$$\binom{n}{m}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}},$$



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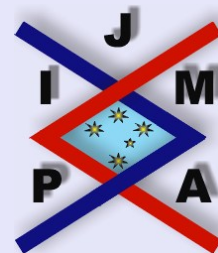
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where

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^n (1 - aq^{k-1}),$$

$$(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n.$$

When $q \rightarrow 1$ then clearly $\binom{n}{m}_q \rightarrow \binom{n}{m}$ and both members in (1.2) tend to (1.1).



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2. Proof of (1.2)

We will follow the notations in [1].

The proof of our result uses the q -binomial theorem and the finite q -binomial theorem. The finite q -binomial theorem states that

$$(2.1) \quad [x - a]_q^n = \sum_{m=0}^n (-1)^m \binom{n}{m}_q q^{\frac{m(m-1)}{2}} x^{n-m} a^m,$$

where

$$[x - a]_q^n = (x - a)(x - aq) \dots (x - aq^{n-1}).$$

The q -binomial theorem states that

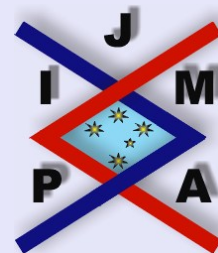
$$(2.2) \quad \frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n, \quad |z| < 1.$$

Using Hölder's inequality in the form

$$\left[\sum a_k b_k \right]^p \leq \sum b_k a_k^p \left[\sum b_k \right]^{p-1}, \quad p > 1$$

we have, if $p > 1$,

$$\begin{aligned} & [e_n(t; q)]^p \\ & \leq \sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} t^m a_m^p \left[\sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} t^m \right]^{p-1}. \end{aligned}$$



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The finite q -binomial theorem (2.1) gives

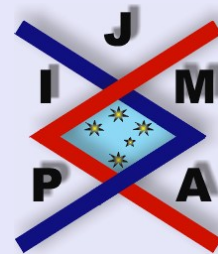
$$\begin{aligned} \sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} t^m &= \sum_{m=0}^n (-1)^m \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} (-t)^m \\ &= [(1-t) + t]_q^n \\ &= (1-t+t)(1-t+qt)\dots(1-t+q^{n-1}t) < 1 \end{aligned}$$

Therefore,

$$(2.3) \quad [e_n(t; q)]^p \leq \sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} t^m a_m^p.$$

Now, if $0 < t < 1$, then for every positive k we have $(1 - q^k)t < 1 - q^k$. This implies $t < (1 - (1 - t)q^k)$ and consequently, $t^m < ((1 - t)q; q)_m$. Using this last estimate on (2.3) and summing in n from 0 to ∞ , the result is

$$\begin{aligned} &\sum_{n=0}^{\infty} [e_n(t; q)]^p \\ &\leq \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m}_q q^{\frac{m(m-1)}{2}} (1-t)^{n-m} ((1-t)q; q)_m a_m^p \\ (2.4) \quad &= \sum_{m=0}^{\infty} ((1-t)q; q)_m q^{\frac{m(m-1)}{2}} a_m^p \sum_{n \geq m}^{\infty} \binom{n}{m}_q (1-t)^{n-m}. \end{aligned}$$



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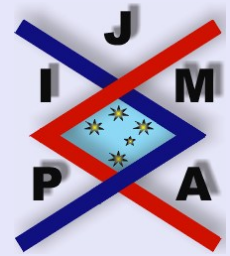
To evaluate the sum in n , observe that

$$\begin{aligned}
 \sum_{n>m}^{\infty} \binom{n}{m}_q (1-t)^{n-m} &= \sum_{n\geq m}^{\infty} \frac{(q; q)_n}{(q; q)_{n-m}(q; q)_m} (1-t)^{n-m} \\
 &= \sum_{k=0}^{\infty} \frac{(q; q)_{m+k}}{(q; q)_k (q; q)_m} (1-t)^k \\
 &= \sum_{k=0}^{\infty} \frac{(q^{m+1}; q)_k}{(q; q)_k} (1-t)^k \\
 &= \frac{(q^{m+1}(1-t); q)_{\infty}}{(q(1-t); q)_{\infty}} \\
 &= \frac{1}{((1-t); q)_{m+1}},
 \end{aligned}$$

where the q -binomial formula (2.2) was used in the fourth identity. Substituting in (2.4) we finally have

$$\sum_{n=0}^{\infty} [e_n(t; q)]^p \leq \sum_{m=0}^n \frac{((1-t)q; q)_m}{((1-t); q)_{m+1}} q^{\frac{m(m-1)}{2}} a_m^p = \sum_{m=0}^n \frac{1}{t} q^{\frac{m(m-1)}{2}} a_m^p$$

for every n . Taking the limit as $n \rightarrow \infty$ gives (1.2).



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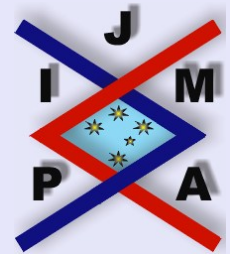
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