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A q-ANALOGUE OF AN INEQUALITY DUE TO KONRAD KNOPP

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ABSTRACT. We derive a simple q-analogue of Konrad Knopp's inequality for Euler-Knopp means, using the finite and infinite q-binomial theorems.

Key words and phrases: Knopp's inequality, q-analogue.

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1. INTRODUCTION

Given a sequence (a_n) and a number t such that $0 < t < 1$, its Euler-Knopp mean $e_n(t)$ is defined by

$$e_n(t) = \sum_{m=0}^{n} \binom{n}{m} (1-t)^{n-m} t^m a_m.$$

Knopp's inequality [2, Section 3] states that, if $p > 1$ and $0 < t < 1$ then

$$\sum_{n=0}^{\infty} [e_n(t)]^p \leq \frac{1}{t} \sum_{m=0}^{\infty} [a_m]^p \tag{1.1}$$

if the series on the right hand side is convergent.

The aim of this short note is to prove the following q-extension of Knopp's inequality (1.1), valid if $p > 1$, $0 < t < 1$ and $0 < q < 1$:

$$\sum_{n=0}^{\infty} [e_n(t;q)]^p \leq \frac{1}{t} \sum_{m=0}^{\infty} q^{\frac{m(m-1)}{2}} [a_m]^p , \tag{1.2}$$

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provided the series on the right hand side of (1.2) converges. The sequence $(e_n(t;q))$ is

$$e_n(t;q)=\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}t^m a_m$$

with the q-analogue of the numbers $\binom{n}{m}_q$ defined by

$$\binom{n}{m}_q=\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}},$$

where

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n.$$

When $q\to 1$ then clearly $\binom{n}{m}_q\to\binom{n}{m}$ and both members in (1.2) tend to (1.1).

2. PROOF OF (1.2)

We will follow the notations in [1].

The proof of our result uses the q-binomial theorem and the finite q-binomial theorem. The finite q-binomial theorem states that

$$[x-a]_q^n=\sum_{m=0}^{n}(-1)^m\binom{n}{m}_q q^{\frac{m(m-1)}{2}}x^{n-m}a^m, \tag{2.1}$$

where

$$[x-a]_q^n=(x-a)(x-aq)...(x-aq^{n-1}).$$

The q-binomial theorem states that

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n,\qquad |z|<1. \tag{2.2}$$

Using Hölder's inequality in the form

$$\left[\sum a_k b_k\right]^p\le\sum b_k a_k^p\left[\sum b_k\right]^{p-1},\qquad p>1$$

we have, if $p>1$,

$$[e_n(t;q)]^p\le\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}t^m a_m^p\left[\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}t^m\right]^{p-1}.$$

The finite q-binomial theorem (2.1) gives

$$\begin{aligned}\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}t^m&=\sum_{m=0}^{n}(-1)^m\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}(-t)^m\\&=[(1-t)+t]_q^n\\&=(1-t+t)(1-t+qt)...(1-t+q^{n-1}t)<1\end{aligned}$$

Therefore,

$$[e_n(t;q)]^p\le\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}t^m a_m^p. \tag{2.3}$$

Now, if $0 < t < 1$, then for every positive k we have $(1-q^k)t < 1-q^k$. This implies $t < (1-(1-t)q^k)$ and consequently, $t^m < ((1-t)q;q)_m$. Using this last estimate on (2.3) and summing in n from 0 to ∞, the result is

$$\sum_{n=0}^{\infty}[e_n(t;q)]^p \leq \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}_q q^{\frac{m(m-1)}{2}}(1-t)^{n-m}\left((1-t)\,q;q\right)_m a_m^p$$

$$(2.4) \qquad = \sum_{m=0}^{\infty}\left((1-t)\,q;q\right)_m q^{\frac{m(m-1)}{2}} a_m^p \sum_{n\geq m}\binom{n}{m}_q (1-t)^{n-m}.$$

To evaluate the sum in n, observe that

$$\begin{aligned}\sum_{n>m}^{\infty}\binom{n}{m}_q (1-t)^{n-m} &= \sum_{n\geq m}^{\infty}\frac{(q;q)_n}{(q;q)_{n-m}(q;q)_m}(1-t)^{n-m}\\ &= \sum_{k=0}^{\infty}\frac{(q;q)_{m+k}}{(q;q)_k(q;q)_m}(1-t)^k\\ &= \sum_{k=0}^{\infty}\frac{(q^{m+1};q)_k}{(q;q)_k}(1-t)^k\\ &= \frac{(q^{m+1}(1-t);q)_\infty}{(q(1-t);q)_\infty}\\ &= \frac{1}{((1-t);q)_{m+1}},\end{aligned}$$

where the q-binomial formula (2.2) was used in the fourth identity. Substituting in (2.4) we finally have

$$\sum_{n=0}^{\infty}[e_n(t;q)]^p \leq \sum_{m=0}^{n}\frac{((1-t)\,q;q)_m}{((1-t);q)_{m+1}} q^{\frac{m(m-1)}{2}} a_m^p = \sum_{m=0}^{n}\frac{1}{t}\, q^{\frac{m(m-1)}{2}} a_m^p$$

for every n. Taking the limit as $n\to\infty$ gives (1.2).

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