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SOME APPLICATIONS OF A FIRST ORDER DIFFERENTIAL SUBORDINATION

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ABSTRACT. Let p and q be analytic functions in the unit disc $E = \{z : |z| < 1\}$, with $p(0) = q(0) = 1$. Assume that α and δ are real numbers such that $0 < \delta \leq 1$, $\alpha + \delta \geq 0$. Let β and γ be complex numbers with $\beta \neq 0$. In the present paper, we investigate the differential subordination

$$(p(z))^{\alpha}\left[p(z) + \frac{zp\prime(z)}{\beta p(z) + \gamma}\right]^{\delta} \prec (q(z))^{\alpha}\left[q(z) + \frac{zq\prime(z)}{\beta q(z) + \gamma}\right]^{\delta}, \quad z \in E,$$

and as applications, find several sufficient conditions for starlikeness and univalence of functions analytic in the unit disc E.

Key words and phrases: Univalent function, Starlike function, Convex function, Differential subordination.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions f, analytic in E, having the normalization $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{A}'$, the class of functions f which are analytic in E and satisfy $f(0) = 1$ and $f(z) \neq 0$, $z \in E$. Furthermore, let $St(\alpha)$, St and K denote the usual subclasses of $\mathcal{A}$ consisting of functions which are starlike of order $\alpha, 0 \leq \alpha < 1$, starlike (with respect to the origin) and convex in E, respectively.

For the analytic functions f and g, we say that f is subordinate to g in E, written as $f(z) \prec g(z)$ in E (or simply $f \prec g$), if g is univalent in E, $f(0) = g(0)$, and $f(E) \subset g(E)$.

Let $\psi : D \to \mathbb{C}$ ($\mathbb{C}$ is the complex plane) be an analytic function defined on a domain $D \subset \mathbb{C}^2$. Further, let p be a function analytic in E with $(p(z), zp'(z)) \in D$ for $z \in E$, and let h be a univalent function in E. Then p is said to satisfy first order differential subordination if

(1.1) $$\psi(p(z), zp'(z)) \prec h(z), \quad z \in E, \; \psi(p(0), 0) = h(0)$$

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A univalent function q is said to be the dominant of the differential subordination (1.1), if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ of (1.1) that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

The fascinating theory of differential subordination was put on sound footing by Miller and Mocanu [6] in 1981. Later, they used it (e.g. see [3], [7] and [10]) to obtain the best dominant for the Briot-Bouquet differential subordination of the type

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad z \in E, \tag{1.2}$$

which was first studied by Ruscheweyh and Singh [29]. The observation that the differential subordination (1.2) had several interesting applications to univalent functions led to its generalization in many ways (see [8], [16]).

In the present paper, we consider a generalization of the form

$$(p(z))^\alpha \left[p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}\right]^\delta \prec (q(z))^\alpha \left[q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}\right]^\delta = h(z), \quad z \in E, \tag{1.3}$$

where α, δ are suitably chosen real numbers, $\beta,\ \gamma \in \mathbb{C},\ \beta \neq 0$. Using the characterization of a subordination chain and a beautiful lemma of Miller and Mocanu [6], we determine conditions on q under which it becomes best dominant of differential subordination (1.3), Section 3. In Section 4, we apply our results to find several new sufficient conditions for starlikeness, strongly starlikeness and univalence of function $f \in \mathcal{A}$. Large number of known results also follow as particular cases from our results.

The motivation to study the differential subordination (1.3) was provided by the class $H(\gamma), \gamma \geq 0$, of γ-starlike functions defined by Lewandowski et al. [4], defined as under:

$$H(\gamma) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left[1 + \frac{zf''(z)}{f'(z)}\right]^\gamma > 0,\ z \in E \right\}.$$

Recently, Darus and Thomas [2], too, investigated the class $H(\gamma)$ and proved that the functions in this class are starlike.

Finally, we define few classes of analytic functions, which will be required by us in the present paper.

Miller, Mocanu and others studied extensively (e.g. see [11], [12], [14], [15] and [17]) the class of α-convex functions, $M_\alpha,\ \alpha$ real, defined as under

$$M_\alpha = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in E \right\}$$

where $\frac{f(z)f'(z)}{z} \neq 0$ in E. They proved that $M_\alpha \subset M_\beta \subset St$ for $0 \leq \alpha/\beta \leq 1$ and $M_\alpha \subset M_1 = K$ for $\alpha \geq 1$.

In [30], Silverman defined the following subclass of the class of starlike functions:

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ z \in E \right\}.$$

N. Tuneski ([22], [31], [32]) dedicated lot of his work to the study of this class and obtained some interesting conditions of starlikeness.

Another class which is of considerable interest and has been investigated in many articles ([23], [27]), consists of functions $f \in \mathcal{A}$ which satisfy

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad z \in E.$$

It is known that the functions in this class are starlike, too. We observe that all the classes mentioned above consist of algebraic expressions involving the analytic representations of convex and starlike functions.

2. PRELIMINARIES

We shall need the following definition and lemmas.

Definition 2.1. A function $L(z,t),\ z \in E$ and $t \geq 0$ is said to be a subordination chain if $L(\cdot,t)$ is analytic and univalent in E for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in E$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2.1 ([24, p. 159])**.** *The function $L(z,t) : E \times [0,\infty) \to \mathbb{C}$ of the form $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t\to\infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if there exist constants $r \in (0,1]$ and $M > 0$ such that*

(i) *$L(z,t)$ is analytic in $|z| < r$ for each $t \geq 0$, locally absolutely continuous in $t \geq 0$ for each $|z| < r$, and satisfies*

$$|L(z,t)| \leq M|a_1(t)|, \quad \text{for} \quad |z| < r,\ t \geq 0,$$

(ii) *there exists a function $p(z,t)$ analytic in E for all $t \in [0,\infty)$ and measurable in $[0,\infty)$ for each $z \in E$, such that $\operatorname{Re} p(z,t) > 0$ for $z \in E,\ t \geq 0$ and*

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t),$$

in $|z| < r$, and for almost all $t \in [0,\infty)$.

Lemma 2.2. *Let F be analytic in E and let G be analytic and univalent in $\overline{E}$ except for points ζ_0 such that $\lim_{z\to\zeta_0} F(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in E, then there is a point $z_0 \in E$ and $\zeta_0 \in \partial E$ (boundary of E) such that $F(|z| < |z_0|) \subset G(E)$, $F(z_0) = G(\zeta_0)$ and $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$ for $m \geq 1$.*

Lemma 2.2 is due to Miller and Mocanu [6].

Lemma 2.3. *Let $f \in \mathcal{A}$ be such that $f'(z) \prec (1 + az)$ in E, where $0 < a \leq 1$. Then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}, \quad z \in E,$$

where $0 < a \leq \frac{2\sin(\pi\mu/2)}{\sqrt{5+4\cos(\pi\mu/2)}}$.

Lemma 2.4. *Let $f \in \mathcal{A}$ be such that $f'(z) \prec (1 + az)$ in E, where $0 < a \leq \frac{1}{2}$. Then we have*

$$\frac{zf'(z)}{f(z)} \prec 1 + \left(\frac{3a}{2-a}\right) z.$$

Lemma 2.3 and Lemma 2.4 are due to Ponnusamy and Singh [26] and Ponnusamy [25], respectively.

3. MAIN RESULTS

Definition 3.1. Let α and δ be fixed real numbers, with $0 < \delta \leq 1, \alpha + \delta \geq 0$. Further, let β and γ be complex numbers such that $\beta \neq 0$. Then, by $R(\alpha,\beta,\gamma,\delta)$, we denote the class of functions $p \in \mathcal{A}'$, such that the function

$$P(z) = (p(z))^{\alpha}\left(p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}\right)^{\delta},\ P(0) = p(0), \quad z \in E,$$

is well defined in E.

For fixed α and δ, where $\alpha+\delta \geq 0, 0<\delta \leq 1$ and for given $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, the class $R(\alpha,\beta,\gamma,\delta)$ is non-empty, as the function $p(z)=1+p_n z^n$, where $z \in E$, $p_n \in \mathbb{C}$ and $n \in \mathbb{N}$, belongs to $R(\alpha,\beta,\gamma,\delta)$.

We are now in a position to state and prove our main theorem.

Theorem 3.1. *For a function $q \in R(\alpha,\beta,\gamma,\delta)$, analytic and univalent in E, set $\frac{zq'(z)}{\beta q(z)+\gamma}=Q(z)$. Suppose that q satisfies the following conditions:*

(i) $\operatorname{Re}(\beta q(z)+\gamma)>0, \quad z \in E$, *and*

(ii) $\operatorname{Re}\left[\frac{\alpha}{\delta}\frac{zq'(z)}{q(z)}+\frac{zQ'(z)}{Q(z)}\right]>0, \quad z \in E.$

If $p \in R(\alpha,\beta,\gamma,\delta)$ satisfies the first order differential subordination

$$(p(z))^{\alpha}\left(p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right)^{\delta} \prec (q(z))^{\alpha}\left(q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\delta}=h(z),\ h(0)=1, \tag{3.1}$$

for all $z \in E$. Then $p \prec q$ and q is the best dominant of differential subordination (3.1). All the powers chosen here are the principal ones.

Proof. Without any loss of generality, we assume that q is univalent on $\overline{E}$. If not, then we can replace $p, q,$ and h by $p_r(z)=p(rz), q_r(z)=q(rz)$ and $h_r(z)=h(rz)$, respectively, where $0<r<1$. These new functions satisfy the conditions of the theorem on $\overline{E}$. We would then prove that $p_r \prec q_r$, and by letting $r \to 1^-$, we obtain $p \prec q$.

We need to prove that p is subordinate to q in E. If possible, suppose that $p \not\prec q$ in E. Then by Lemma 2.2, there exist points $z_0 \in E$ and $\zeta_0 \in \partial E$ such that $p(|z|<|z_0|) \subset q(E)$, $p(z_0)=q(\zeta_0)$ and $z_0 p'(z_0)=m\zeta_0 q'(\zeta_0)$ for $m \geq 1$. Now

$$(p(z_0))^{\alpha}\left[p(z_0)+\frac{z_0 p'(z_0)}{\beta p(z_0)+\gamma}\right]^{\delta}=(q(\zeta_0))^{\alpha}\left[q(\zeta_0)+\frac{m\zeta_0 q'(\zeta_0)}{\beta q(\zeta_0)+\gamma}\right]^{\delta}. \tag{3.2}$$

Consider the function

$$\begin{aligned} L(z,t) &= (q(z))^{\alpha}\left(q(z)+(1+t)\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\delta} \\ &= (q(z))^{\alpha}(q(z)+(1+t)Q(z))^{\delta} \\ &= 1+a_1(t)z+\cdots . \end{aligned}$$

As $q \in R(\alpha,\beta,\gamma,\delta)$, so $L(z,t)$ is analytic in E for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in E$.

$$a_1(t)=\left[\frac{\partial L(z,t)}{\partial z}\right]_{z=0}=q'(0)\left[\alpha+\delta\left(1+\frac{1+t}{\beta+\gamma}\right)\right].$$

Since q is univalent in E, $q'(0) \neq 0$ and in view of condition (i), $\beta+\gamma \neq 0$. Thus, $a_1(t) \neq 0$ and $\lim_{t\to\infty}|a_1(t)|=\infty$. Moreover,

$$\frac{L(z,t)}{a_1(t)}=\frac{(q(z))^{\alpha}\left(q(z)+(1+t)\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\delta}}{q'(0)\left[\alpha+\delta\left(1+\frac{1+t}{\beta+\gamma}\right)\right]}.$$

Taking limits on both sides

$$\lim_{t\to\infty}\frac{L(z,t)}{a_1(t)}=\lim_{t\to\infty}\frac{(q(z))^{\alpha}\left(\frac{q(z)}{t}+(\frac{1}{t}+1)\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\delta}}{q'(0)t^{1-\delta}\left[\frac{\alpha}{t}+\delta\left(\frac{1}{t}+\frac{\frac{1}{t}+1}{\beta+\gamma}\right)\right]}.$$

When $0<\delta<1$,

$$\lim_{t\to\infty}\frac{L(z,t)}{a_1(t)}=0.$$

Thus, there exists a positive constant ϵ such that

$$|L(z,t)|<\epsilon|a_1(t)|,\ t\geq 0.$$

When $\delta=1$, we have

$$\lim_{t\to\infty}\frac{L(z,t)}{a_1(t)}=\frac{(q(z))^{\alpha}Q(z)}{q'(0)/\beta+\gamma}=\phi(z),\ (\text{say}).$$

In view of condition (ii), it is obvious that ϕ is a normalized starlike (with respect to the origin), and hence, univalent function. By the Growth theorem for univalent functions [18, p. 217], the function ϕ is bounded for all $z,|z|<r$, where $r\in(0,1]$. Thus, we conclude that for $0<\delta\leq 1$, there exist constants $K_0>0$ and $r_0\in(0,1]$ such that

$$|L(z,t)|\leq K_0|a_1(t)|,\ |z|<r_0,\ t\in[0,\infty).$$

Now

$$\frac{\partial L(z,t)}{\partial t}=(q(z))^{\alpha}\delta(q(z)+(1+t)Q(z))^{\delta-1}Q(z).$$

(3.3)
$$\begin{aligned}\frac{z\partial L/\partial z}{\partial L/\partial t}&=\frac{\alpha}{\delta}\frac{zq'(z)}{q(z)}\frac{(q(z)+(1+t)Q(z))}{Q(z)}+\frac{z(q'(z)+(1+t)Q'(z))}{Q(z)}\\&=\left(\frac{\alpha}{\delta}+1\right)(\beta q(z)+\gamma)+(1+t)\left(\frac{\alpha}{\delta}\frac{zq'(z)}{q(z)}+\frac{zQ'(z)}{Q(z)}\right).\end{aligned}$$

In view of conditions (i)) and (ii), we obtain

$$\operatorname{Re}\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right)>0,\quad z\in E,$$

for $\alpha+\delta\geq 0$ and $0<\delta\leq 1$. Thus, all the conditions of Lemma 2.1 are satisfied. Therefore, we deduce that $L(z,t)$ is a subordination chain. In view of Definition 2.1, $L(z,t_1)\prec L(z,t_2)$ for $0\leq t_1\leq t_2$. Since $L(z,0)=h(z)$, we get $L(\zeta_0,t)\notin h(E)$ for $|\zeta_0|=1$ and $t\geq 0$. Moreover, h is univalent in E so that the subordination (3.1) is well-defined. In view of (3.2), we can write

$$(p(z_0))^{\alpha}\left[p\left(z_0\right)+\frac{z_0p\prime(z_0)}{\beta p(z_0)+\gamma}\right]^{\delta}=L(\zeta_0,m-1)\notin h(E),$$

which is a contradiction to (3.1). Hence $p\prec q$ in E. This completes the proof. □

Remark 3.2. From (3.3), we observe that the condition (i) in Theorem 3.1 can be omitted in case $\alpha+\delta=0$.

Setting $\alpha=0$ and $\delta=1$ in Theorem 3.1, we obtain the following result on Briot-Bouquet differential subordination (also see [7]):

Corollary 3.3. *Let $\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0$. Let $q\in R(0,\beta,\gamma,1)$, be a univalent function which satisfies the following conditions:*

(i) $\operatorname{Re}(\beta q(z)+\gamma)>0,\ z\in E$ *and*

(ii) $\log(\beta q(z) + \gamma)$ *is convex in* E.

If an analytic function $p \in R(0, \beta, \gamma, 1)$ *satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}, \quad z \in E,$$

then $p \prec q$ *in* E.

Taking $\alpha = \delta = 1$ and $\gamma = 0$, Theorem 3.1 provides the following result (also see [5] and [21]):

Corollary 3.4. *Let* β *be a complex number,* $\beta \neq 0$*. Let* $q \in R(1, \beta, 0, 1)$*, be convex univalent in* E *for which* $\operatorname{Re} \beta q(z) > 0,\ z \in E$*. If an analytic function* $p \in R(1, \beta, 0, 1)$ *satisfies the differential subordination*

$$\beta p^2(z) + zp'(z) \prec \beta q^2(z) + zq'(z), \quad z \in E,$$

then $p \prec q$ *in* E.

Writing $\alpha = -1, \beta = 1, \gamma = 0$ and $\delta = 1$ in Theorem 3.1, we obtain the following result of Ravichandran and Darus [28]:

Corollary 3.5. *Let* $q \in \mathcal{A}'$ *be univalent in* E *such that* $\frac{zq'(z)}{q^2(z)}$ *is starlike in* E*. If* $p \in \mathcal{A}'$ *satisfies the differential subordination*

$$\frac{zp'(z)}{p^2(z)} \prec \frac{zq'(z)}{q^2(z)}, \quad z \in E,$$

then $p(z) \prec q(z)$ *in* E.

4. APPLICATIONS TO UNIVALENT FUNCTIONS

In this section, we obtain sufficient conditions for a function to be starlike, strongly starlike and univalent in E.

Theorem 4.1. *Let* α *and* δ *be fixed real numbers, with* $0 < \delta \leq 1, \alpha + \delta \geq 0$*. Let* $\lambda, \operatorname{Re} \lambda > 0$*, be a complex number. For* f *and* $g \in \mathcal{A}$*, let* $G(z) = \frac{zg'(z)}{g(z)}$ *be a univalent function which satisfies*

(i) $\operatorname{Re} \frac{1}{\lambda} G(z) > 0$ *in* E*; and*

(ii) $\operatorname{Re} \left[\left(\frac{\alpha}{\delta} - 1 \right) \frac{zG'(z)}{G(z)} + 1 + \frac{zG''(z)}{G'(z)} \right] > 0,\ z \in E.$

Then, if $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\left(\frac{zf'(z)}{f(z)} \right)^{\alpha} \left[(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right]^{\delta} \prec \left(\frac{zg'(z)}{g(z)} \right)^{\alpha} \cdot \left[(1 - \lambda) \frac{zg'(z)}{g(z)} + \lambda \left(1 + \frac{zg''(z)}{g'(z)} \right) \right]^{\delta}, \quad z \in E,$$

then, $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ *in* E.

Proof. The proof of the theorem follows by writing $p(z) = \frac{zf'(z)}{f(z)}$, $q(z) = \frac{zg'(z)}{g(z)}$, $\beta = 1/\lambda$ and $\gamma = 0$ in Theorem 3.1. □

Letting $\lambda = 1$ and $\alpha = 1 - \delta$ in Theorem 4.1, we obtain the following general subordination theorem for the class of δ-starlike functions.

Theorem 4.2. *Let* $\delta,\ 0 < \delta \leq 1$*, be fixed and let* f *and* $g \in \mathcal{A}$*. Assume that* $G(z) = \frac{zg'(z)}{g(z)}$ *is univalent in* E *which satisfies the following conditions:*

(i) $\operatorname{Re} G(z) > 0$ *i.e.* g *is starlike in* E*; and*
(ii) $\operatorname{Re}\left[\left(\frac{1}{\delta}-2\right)\frac{zG'(z)}{G(z)}+1+\frac{zG''(z)}{G'(z)}\right] > 0,\ z \in E.$

Then the differential subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\delta} \prec \left(\frac{zg'(z)}{g(z)}\right)^{1-\delta}\left(1+\frac{zg''(z)}{g'(z)}\right)^{\delta}, \quad z \in E, \tag{4.1}$$

implies that $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ *in* E.

For $\delta = 1, \alpha = 0$, Theorem 4.1 gives the following subordination result.

Theorem 4.3. *Let* $\lambda, \operatorname{Re}\lambda > 0$*, be a complex number. For* $g \in \mathcal{A}$*, let* $G(z) = \frac{zg'(z)}{g(z)}$ *be univalent in* E *and satisfy the following conditions:*

(i) $\operatorname{Re}\frac{1}{\lambda}G(z) > 0,\ z \in E$*; and*
(ii) $\frac{zG'(z)}{G(z)}$ *is starlike in* E.

If an analytic function $f \in \mathcal{A}$ *satisfies the differential subordination*

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right) \prec (1-\lambda)\frac{zg'(z)}{g(z)}+\lambda\left(1+\frac{zg''(z)}{g'(z)}\right), \quad z \in E, \tag{4.2}$$

then $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ *in* E.

Remark 4.4. We observe that when λ is a positive real number, Theorem 4.3 provides a general subordination result for the class M_λ.

Putting $\delta = \lambda = 1, \alpha = -1$ in Theorem 4.1, we obtain the following general subordination result for the class $\mathcal{G}_b$:

Theorem 4.5. *For a function* $g \in \mathcal{A}$*, set* $\frac{zg'(z)}{g(z)} = G(z)$*. Suppose that* $G(z)$ *is a univalent function which satisfies*

$$\operatorname{Re}\left[1-\frac{2zG'(z)}{G(z)}+\frac{zG''(z)}{G(z)}\right] > 0,\ z \in E.$$

If an analytic function $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \frac{1+zg''(z)/g'(z)}{zg'(z)/g(z)}, \quad z \in E,$$

then $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ *in* E.

Taking $\delta = 1, \alpha = 0$ and $\lambda = 1$ in Theorem 4.1, we obtain the following Marx-Strohhäcker differential subordination theorem of first type (also see [9]):

Corollary 4.6. *Let* $g \in \mathcal{A}$ *be starlike in* E*. Suppose that* $\frac{zg'(z)}{g(z)} = G(z)$ *is univalent and* $\log G(z)$ *is convex in* E*. If an analytic function* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$1+\frac{zf''(z)}{f'(z)} \prec 1+\frac{zg''(z)}{g'(z)}, \quad z \in E,$$

then $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ *in* E.

Writing $\alpha = \delta = \lambda = 1$ in Theorem 4.1, we have the following subordination result:

Corollary 4.7. *Let $g \in \mathcal{A}$ be starlike in E. Set $\frac{zg'(z)}{g(z)} = G(z)$. Assume that G is convex univalent in E. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} \prec \frac{zg'(z)}{g(z)} + \frac{z^2 g''(z)}{g(z)}, \quad z \in E,$$

then $\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$ in E.

Setting $p(z) = f'(z),\ q(z) = g'(z)$ and $\beta = \frac{1}{\lambda}$ and $\gamma = 0$ in Theorem 3.1, we obtain

Theorem 4.8. *Let α and δ be fixed real numbers, with $0 < \delta \leq 1$ and $\alpha + \delta \geq 0$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$. Assume that $g \in \mathcal{A}$ satisfies the following conditions:*

(i) $\operatorname{Re} \frac{1}{\lambda}\, g'(z) > 0,\ z \in E$*; and*
(ii) $\operatorname{Re} \left[\left(\frac{\alpha}{\delta} - 1 \right) \frac{zg''(z)}{g'(z)} + 1 + \frac{zg'''(z)}{g''(z)} \right] > 0$ *in* E.

If $f \in \mathcal{A}$ satisfies the differential subordination

$$(f'(z))^{\alpha-\delta}[(f'(z))^2 + \lambda z f''(z)]^{\delta} \prec (g'(z))^{\alpha-\delta}[(g'(z))^2 + \lambda z g''(z)]^{\delta},$$

for all $z \in E$, then $f'(z) \prec g'(z)$ in E.

Writing $\alpha = \delta = 1$ in Theorem 4.8, we obtain the following result:

Corollary 4.9. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$. Let $g \in \mathcal{A}$ be such that*

(i) $\operatorname{Re} \frac{1}{\lambda} g'(z) > 0,\ z \in E$ *and*
(ii) g' *is convex in* E.

If $f \in \mathcal{A}$ satisfies the differential subordination

$$f'^2(z) + \lambda z f''(z) \prec g'^2(z) + \lambda z g''(z), \quad z \in E,$$

then $f'(z) \prec g'(z)$ in E.

Taking $\delta = \lambda = 1, \alpha = -1$ in Theorem 4.8, we obtain the following result:

Corollary 4.10. *Let $g \in \mathcal{A}$ be such that $\frac{zg''(z)}{g'^2(z)}$ is starlike in E. If an analytic function f, $f(0) = 0$, satisfies the differential subordination*

$$\frac{zf''(z)}{f'^2(z)} \prec \frac{zg''(z)}{g'^2(z)}, \quad z \in E,$$

then $f'(z) \prec g'(z)$ in E.

Setting $\alpha = 0$ and $\delta = 1$ in Theorem 4.8, we get:

Corollary 4.11. *Let λ be a complex number, $\operatorname{Re} \lambda > 0$. Let $g \in \mathcal{A}$ be such that*

(i) $\operatorname{Re} \frac{1}{\lambda} g'(z) > 0,\ z \in E$, *and*
(ii) $\log g'(z)$ *is convex in* E.

If $f \in \mathcal{A}$ satisfies the differential subordination

$$f'(z) + \lambda \frac{zf''(z)}{f'(z)} \prec g'(z) + \lambda \frac{zg''(z)}{g'(z)}, \quad z \in E,$$

then $f'(z) \prec g'(z)$ in E.

4.1. **Conditions for Strongly Starlikeness.** A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha,\ 0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in E.$$

Let the class of all such functions be denoted by $\tilde{S}(\alpha)$.

Consider $G(z) = (\frac{1+z}{1-z})^{\mu}, 0 < \mu < 1$. Let α, δ and λ be fixed real numbers such that $0 < \delta \leq 1, \alpha + \delta \geq 0$ and $\lambda > 0$. Then

(i) $\operatorname{Re} \frac{1}{\lambda} G(z) > 0$ in E; and

(ii) $\operatorname{Re}\left[\left(\frac{\alpha}{\delta} - 1\right)\frac{zG'(z)}{G(z)} + 1 + \frac{zG''(z)}{G'(z)}\right] = \operatorname{Re}\left[\frac{\alpha}{\delta}\frac{2\mu z}{1-z^2} + \frac{1+z^2}{1-z^2}\right] > 0,\ z \in E.$

Thus, all the conditions of Theorem 4.1 are satisfied. Therefore, we obtain the following result:

Theorem 4.12. *Let α, δ and λ be fixed real numbers such that $0 < \delta \leq 1,\ \alpha + \delta \geq 0$ and $\lambda > 0$. For a real number $\mu, 0 < \mu < 1$, if $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\alpha\mu}\left[\left(\frac{1+z}{1-z}\right)^{\mu} + \frac{2\lambda\mu z}{1-z^2}\right]^{\delta},$$

then $f \in \tilde{S}(\mu)$.

Remark 4.13. Theorem 4.12 can also be written as:
Let α, δ, λ and μ be fixed real numbers such that $0 < \delta \leq 1, \alpha + \delta \geq 0$, $\lambda > 0$ and $0 < \mu < 1$. If an analytic function f in $\mathcal{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\delta}\right| < \frac{\nu\pi}{2},$$

then $f \in \tilde{S}(\mu)$, where

$$\nu = \alpha\mu + \frac{2}{\pi}\delta \arctan\left(\tan\frac{\mu\pi}{2} + \frac{\lambda\mu}{(1-\mu)^{(1-\mu)/2}(1+\mu)^{(1+\mu)/2}\cos\frac{\mu\pi}{2}}\right).$$

Deductions.

Writing $\lambda = 1$ and $\alpha = 1 - \delta$ in Theorem 4.12, we obtain the following result, essentially proved by Darus and Thomas [2]:

(i). Let $\delta, 0 < \delta \leq 1$, and $\mu, 0 < \mu < 1$, be fixed real numbers. Let $f \in \mathcal{A}$ satisfy

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\nu}, \quad z \in E,$$

then $f \in \tilde{S}(\mu)$, where

$$\nu = (1-\delta)\mu + \frac{2\delta}{\pi}\arctan\left[\tan\frac{\mu\pi}{2} + \frac{\mu}{(1-\mu)^{(1-\mu)/2}(1+\mu)^{(1+\mu)/2}\cos\frac{\mu\pi}{2}}\right].$$

Letting $\alpha = 0$ and $\delta = 1$ in Theorem 4.12, we get the following result of Mocanu [15]:

(ii). Let $\lambda, \lambda > 0$ and $\mu, 0 < \mu < 1$, be fixed real numbers. If $f \in \mathcal{A}$ satisfies the differential subordination

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\mu} + \frac{2\lambda\mu z}{1-z^2}, \quad z \in E,$$

then $f \in \tilde{S}(\mu)$.

For $\lambda = 0,\ \alpha = -1$ and $\delta = 1$, Theorem 4.12 gives the following result of Ravichandran and Darus [28]:

(iii). For $\mu, 0 < \mu < 1,$ a fixed real number, let $f \in \mathcal{A}$ satisfy

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{2\mu z(1-z)^{\mu-1}}{(1+z)^{\mu+1}}$$

for all $z \in E$. Then $f \in \tilde{S}(\mu)$.

Writing $\alpha = 0$ and $\delta = \lambda = 1$ in Theorem 4.12, we obtain the following result of Nunokawa and Thomas [20]:

(iv). For $\mu, 0 < \mu < 1,$ a fixed real number, let $f \in \mathcal{A}$ satisfy

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\nu},$$

for all $z \in E$. Then $f \in \tilde{S}(\mu)$, where

$$\nu = \frac{2}{\pi} \arctan\left[\tan\frac{\mu\pi}{2} + \frac{\mu}{(1-\mu)^{(1-\mu)/2}(1+\mu)^{(1+\mu)/2}\cos\frac{\mu\pi}{2}}\right].$$

Taking $\alpha = \delta = \lambda = 1$ in Theorem 4.12, we obtain the following result of Padmanabhan [23]:

(v). For $\mu, 0 < \mu < 1,$ a fixed real number, let $f \in \mathcal{A}$ satisfy

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\mu}\left[\left(\frac{1+z}{1-z}\right)^{\mu} + \frac{2\mu z}{1-z^2}\right],$$

for all $z \in E$. Then $f \in \tilde{S}(\mu)$.

4.2. **Conditions for Starlikeness.** Consider $G(z) = \frac{1+az}{1-z}, -1 < a \leq 1.$ Then, for $|\alpha| \leq \delta,\ 0 < \delta \leq 1$ and $\lambda > 0$, we have

(i) $\operatorname{Re} \frac{1}{\lambda}G(z) > \frac{1}{\lambda}(\frac{1-a}{2}) > 0$ in E; and

(ii) $\operatorname{Re}\left[\left(\frac{\alpha}{\delta} - 1\right)\frac{zG'(z)}{G(z)} + 1 + \frac{zG''(z)}{G'(z)}\right] > \operatorname{Re}\left[(1 - \frac{\alpha}{\delta})\left(\frac{1}{1+az} - \frac{1}{2}\right)\right] > 0,\ z \in E.$

Thus G satisfies all the conditions of Theorems 4.1. Therefore, we obtain:

Theorem 4.14. *Let $|\alpha| \leq \delta,\ 0 < \delta \leq 1$ be fixed and λ be a real number, $\lambda > 0$. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\delta} \prec \left(\frac{1+az}{1-z}\right)^{\alpha}\left[\frac{1+(\lambda+a)z}{1-z} + \frac{\lambda a z}{1+az}\right]^{\delta},$$

where $-1 < a \leq 1$, then $f \in St(\frac{1-a}{2})$.

Deductions.

Taking $\lambda = 1$, $\alpha = 1 - \delta$ and $a = 0$ in Theorem 4.14, we obtain the following result:

(i). Let $f \in \mathcal{A}$ satisfy

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\delta} \prec \frac{(1+z)^{\delta}}{1-z},\ \frac{1}{2} \leq \delta \leq 1.$$

Then $f \in St(1/2)$.

Remark 4.15. For $\delta = 1$, Deduction **(i)** yields the well-known result: $K \subset St(1/2)$.

Writing $\alpha = 0$ and $\delta = 1$ in Theorem 4.14, we obtain the following form of the result of Mocanu [14]:

(ii). Let λ be a real number, such that $\lambda > 0$. If $f \in \mathcal{A}$ satisfies the differential subordination

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(a+\lambda)z}{1-z} + \frac{\lambda az}{1+az},$$

where $-1 < a \leq 1$ and $z \in E$, then $zf'(z)/f(z) \prec (1+az)/(1-z)$.

Taking $\lambda = \delta = 1$ and $\alpha = -1$ in Theorem 4.14, we have the following result:

(iii). If $f \in \mathcal{A}$ satisfies

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{(1+a)z}{(1+az)^2}, \ -1 < a \leq 1, \quad z \in E,$$

then $zf'(z)/f(z) \prec (1+az)/(1-z)$.

Remark 4.16. For $a = 1$, Deduction **(iii)** gives Theorem 3 in [22] and Corollary 8 in [28]. Theorem 3 (in case $\gamma = 0$) in [31] is improved by writing $a = 1 - 2\alpha$ in above deduction.

For $\lambda = \delta = 1$ and $\alpha = 0$, we obtain from Theorem 4.14:

(iv). Let $f \in \mathcal{A}$ satisfy

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(a+1)z}{1-z} + \frac{az}{1+az}, \ -1 < a \leq 1, \quad z \in E.$$

Then $zf'(z)/f(z) \prec (1+az)/(1-z)$.

Taking $\alpha = \delta = \lambda = 1$ in Theorem 4.14, we obtain the following result:

(v). If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+az}{1-z}\right)^2 + \frac{(1+a)z}{(1-z)^2},$$

for all z in E and $-1 < a \leq 1$, then $zf'(z)/f(z) \prec (1+az)/(1-z)$.

Remark 4.17. Theorem 3 of Padmanabhan [23] corresponds to $a = 0$ in deduction (v).

Writing $a = 1$ and $\alpha = 1 - \delta$ in Theorem 4.14, we obtain:

Theorem 4.18. *Let $\delta, 1/2 < \delta \leq 1$, be fixed. For $\lambda, \lambda > 0$, let $f \in \mathcal{A}$ satisfy*

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\delta} \neq it,$$

where t is a real number with $|t| \geq \delta^{\delta}\sqrt{\lambda}\left(\frac{2+\lambda}{2\delta-1}\right)^{\delta-1/2}$. Then $f \in St$.

Proof. As the proof runs on the same lines as in [1], so we omit it. □

Deductions.

For $\lambda = 1$ in Theorem 4.18, we obtain the following result which improves the Theorem 2 of Darus and Thomas [2]:

(i). Let $\delta, 1/2 < \delta \leq 1$, be fixed. Let $f \in \mathcal{A}$ satisfy

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\delta}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{\delta} \neq it,$$

where t is a real number with $|t| \geq \delta^{\delta}\left(\frac{3}{2\delta-1}\right)^{\delta-1/2}$. Then $f \in St$.

Writing $\delta = 1$ in Theorem 4.18, we get the the following result obtained by Nunokawa [19]:

(ii). Let $\lambda, \lambda > 0$ be a real number. If $f \in \mathcal{A}$ satisfies the differential subordination

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq it,$$

where t is a real number and $|t| \geq \sqrt{\lambda(\lambda+2)}$. Then f is starlike in E.

Taking $\lambda = \delta = 1$ in Theorem 4.18, we obtain the following result of Mocanu [13]:

(iii). If $f \in \mathcal{A}$ satisfies

$$1 + \frac{zf''(z)}{f'(z)} \neq it, \quad z \in E,$$

where t is a real number and $|t| \geq \sqrt{3}$, then $f \in St$.

4.3. **Conditions for Univalence.** Consider $g'(z) = \frac{1+az}{1-z}$, $-1 < a \leq 1$. It can be easily checked that all the conditions of Theorem 4.8 are satisfied. Thus, we obtain the following result:

Theorem 4.19. *Let $|\alpha| \leq \delta, 0 < \delta \leq 1$, be fixed real numbers. Let λ be a positive real number. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$(f'(z))^{\alpha-\delta}[(f'(z))^2 + \lambda z f''(z)]^{\delta} \prec \left(\frac{1+az}{1-z}\right)^{\alpha-\delta}\left[\left(\frac{1+az}{1-z}\right)^2 + \lambda\frac{(1+a)z}{(1-z)^2}\right]^{\delta},$$

for all $z \in E$ and $-1 < a \leq 1$, then $f'(z) \prec (1+az)/(1-z)$ in E.

Deductions.

For $\alpha = \delta = 1$, Theorem 4.19 gives the following result:

(i). Let λ be a positive real number. If $f \in \mathcal{A}$ satisfies the differential subordination

$$(f'(z))^2 + \lambda z f''(z) \prec \left(\frac{1+az}{1-z}\right)^2 + \lambda\frac{(1+a)z}{(1-z)^2}, \quad z \in E,$$

for all $z \in E$ and $-1 < a \leq 1$, then $f'(z) \prec (1+az)/(1-z)$ in E.

Taking $\alpha = -1$ and $\delta = \lambda = 1$ in Theorem 4.19, we obtain:

(ii). If $f \in \mathcal{A}$ satisfies

$$\frac{zf''(z)}{(f'(z))^2} \prec \frac{(1+a)z}{(1+az)^2}, \qquad z \in E,$$

then $f'(z) \prec (1+az)/(1-z)$.

In particular, $a = 0$ gives the following interesting result:

(iii). If $f \in \mathcal{A}$ satisfies

$$\frac{zf''(z)}{(f'(z))^2} \prec z, \quad z \in E,$$

then $\operatorname{Re} f'(z) > 1/2$ for all z in E.

Writing $\alpha = 0$ and $\delta = 1$ in Theorem 4.19, we get the following result:

(iv). Let λ be a positive real number. If $f \in \mathcal{A}$ satisfies

$$f'(z) + \lambda\frac{zf''(z)}{f'(z)} \prec \frac{1+az}{1-z} + \frac{\lambda(1+a)z}{(1+az)(1-z)}, \quad z \in E,$$

where $-1 < a \leq 1$, then $f'(z) \prec (1+az)/(1-z)$. For $a = 1$, this result can be expressed as (also see [1]):

(v). Let λ be a positive real number.If $f \in \mathcal{A}$ satisfies

$$f'(z) + \lambda \frac{zf''(z)}{f'(z)} \neq it,$$

where t is a real number and $|t| \geq \sqrt{\lambda(\lambda+2)}$, then $\operatorname{Re} f'(z) > 0$ in E.

Writing $g'(z) = 1 + az$, $0 < a \leq 1$, in Theorem 4.8, then, for $|\alpha| \leq \delta$, $0 < \delta \leq 1$ and $\lambda > 0$, we obtain:

(i) $\operatorname{Re} \frac{1}{\lambda} g'(z) > 0$ in E; and

(ii) $\operatorname{Re} \left[\left(\frac{\alpha}{\delta} - 1\right) \frac{zg''(z)}{g'(z)} + 1 + \frac{zg'''(z)}{g''(z)} \right] = \operatorname{Re} \left[(1 - \frac{\alpha}{\delta}) \frac{1}{1+az} + \frac{\alpha}{\delta} \right] > 0,\ z \in E.$

Thus, all the conditions of Theorem 4.8 are satisfied. Therefore, we obtain:

Theorem 4.20. *Let $|\alpha| \leq \delta, 0 < \delta \leq 1$, be fixed real numbers. Let λ be a positive real number. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$(f'(z))^{\alpha-\delta}[(f'(z))^2 + \lambda z f''(z)]^{\delta} \prec (1+az)^{\alpha-\delta}[(1+az)^2 + \lambda az]^{\delta}, \tag{4.3}$$

where $0 < a \leq 1$, then $|f'(z) - 1| < a$.

Remark 4.21. Let $f \in \mathcal{A}$ satisfy (4.3). Then by Lemma 2.3, $f \in \tilde{S}(\mu)$, where μ is given by

$$0 < a \leq \frac{2\sin(\pi\mu/2)}{\sqrt{5 + 4\cos(\pi\mu/2)}}.$$

Remark 4.22. For $0 < a \leq 1/2$, let $f \in \mathcal{A}$ satisfy (4.3). Then by Lemma 2.4,

$$\frac{zf'(z)}{f(z)} \prec 1 + \left(\frac{3a}{2-a}\right) z, \quad z \in E.$$

Deductions.

For $\alpha = \delta = 1$, Theorem 4.20 gives the following result:

(i). For $\lambda, \lambda > 0$, if $f \in \mathcal{A}$ satisfies

$$(f'(z))^2 + \lambda z f''(z) \prec (1+az)^2 + \lambda az,\ 0 < a \leq 1, \quad z \in E,$$

then $|f'(z) - 1| < a$ in E.

Writing $\alpha = -1$ and $\delta = \lambda = 1$ in Theorem 4.20, we obtain:

(ii). If $f \in \mathcal{A}$ satisfies

$$\frac{zf''(z)}{(f'(z))^2} \prec \frac{az}{(1+az)^2},\ 0 < a \leq 1, \quad z \in E,$$

then $|f'(z) - 1| < a$ in E.

Taking $\alpha = 0$ and $\delta = 1$ in Theorem 4.20, we obtain:

(iii). Let $f \in \mathcal{A}$ satisfy

$$f'(z) + \lambda \frac{zf''(z)}{f'(z)} \prec (1+az) + \frac{\lambda az}{1+az}, \quad z \in E,$$

where $0 < a \leq 1$ and λ is a positive real number. Then $|f'(z) - 1| < a$ in E.

We note that results similar to the ones proved in Remarks 4.21 and 4.22 can also be derived in the case of deductions (i), (ii) and (iii).

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