UPPER AND LOWER BOUNDS FOR REGULARIZED DETERMINANTS

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Abstract: Let S_p be the von Neumann-Schatten ideal of compact operators in a separable Hilbert space. In the paper, upper and lower bounds for the regularized determinants of operators from S_p are established.

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1. Upper bounds

For an integer $p \geq 2$, let S_p be the von Neumann-Schatten ideal of compact operators A in a separable Hilbert space with the finite norm $N_p(A) = [\mathrm{Trace}(AA^*)^{p/2}]^{1/p}$ where A^* is the adjoint. Recall that for an $A \in S_p$ the regularized determinant is defined as

$$\det_p(A) := \prod_{j=1}^{\infty}(1-\lambda_j(A)) \exp\left[\sum_{m=1}^{p-1} \frac{\lambda_j^m(A)}{m}\right]$$

where $\lambda_j(A)$ are the eigenvalues of A with their multiplicities arranged in decreasing order.

The inequality

$$\det_p(A) \leq \exp[q_p N_p^p(A)] \tag{1.1}$$

is well-known, cf. [2, p. 1106], [4, p. 194]. Recall that $|\det_2(A)| \leq e^{N_2^2(A)/2}$, cf. [5, Section IV.2]. However, to the best of our knowledge, the constant q_p for $p > 2$ is unknown in the available literature although it is very important, in particular, for perturbations of determinants. In the present paper we suggest bounds for q_p ($p > 2$). In addition, we establish lower bounds for $\det_p(A)$. As far as we know, the lower bounds have not yet been investigated in the available literature.

Our results supplement the very interesting recent investigations of the von Neumann-Schatten operators [1, 3, 8, 9, 10]. In connection with the recent results on determinants, the paper [6] should be mentioned. It is devoted to higher order asymptotics of Toeplitz determinants with symbols in weighted Wienar algebras.

To formulate the main result we need the algebraic equation

$$x^{p-2} = p(1-x)\left[1 + \sum_{m=1}^{p-3} \frac{x^m}{m+2}\right] \qquad (p > 2). \tag{1.2}$$

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Below we prove that it has *a unique positive root* $x_0 < 1$. Moreover,

$$x_0 \leq \sqrt[p-2]{\frac{p}{p+1}}. \tag{1.3}$$

Theorem 1.1. *Let* $A \in S_p \quad (p = 3, 4, \dots)$. *Then inequality (1.1) holds with*

$$q_p = \frac{1}{p(1-x_0)}.$$

The proof of this theorem is divided into a series of lemmas presented below.

Lemma 1.2. *Equation (1.2) has a unique positive root* $x_0 < 1$.

Proof. Rewrite (1.2) as

$$g(x) := \frac{x^{p-2}}{p(1-x)} - \left(1 + \sum_{m=3}^{p-1} \frac{x^{m-2}}{m}\right) = 0.$$

Clearly, $g(0) = -1$, $g(x) \to +\infty$ as $x \to 1-0$. So (1.2) has at least one root from $(0,1)$. But from (1.2) it follows that a root from $[1,\infty)$ is impossible. Moreover, (1.2) is equivalent to the equation

$$\frac{1}{p(1-x)} = \frac{1}{x^{p-2}} + \sum_{m=3}^{p-1} \frac{x^{m-p}}{m}.$$

The left part of this equation increases and the right part decreases on $(0,1)$. So the positive root is unique. □

Furthermore, consider the function

$$f(z) := \operatorname{Re}\left[\ln(1-z) + \sum_{m=1}^{p-1} \frac{z^m}{m}\right] \qquad (z \in \mathbb{C};\ p > 2).$$

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Clearly,

$$f(z) = -\operatorname{Re} \sum_{m=p}^{\infty} \frac{z^m}{m} \qquad (|z| < 1).$$

Lemma 1.3. *Let $w \in (0,1)$. Then*

$$|f(z)| \leq \frac{r^p}{p(1-w)} \qquad (r \equiv |z| < w).$$

Proof. Clearly,

$$|f(z)| \leq \sum_{m=p}^{\infty} \frac{r^m}{m} \qquad (r < 1).$$

Consequently,

$$|f(z)| \leq \int_0^r \sum_{m=p}^{\infty} s^{m-1} ds = \int_0^r s^{p-1} \sum_{k=0}^{\infty} s^k ds = \int_0^r \frac{s^{p-1} ds}{1-s}.$$

Hence we get the required result. □

Lemma 1.4. *For any $w \in (0,1)$ and all $z \in \mathbb{C}$ with $|z| \geq w$, the following inequality is valid:*

$$|f(z)| \leq h_p(w) r^p \quad \textit{where} \quad h_p(w) = w^{-p} \left[w^2 + \sum_{m=3}^{p-1} \frac{w^m}{m} \right] \qquad (p > 2).$$

Proof. Take into account that

$$|(1-z)e^z|^2 = (1 - 2\operatorname{Re} z + r^2) e^{2x} \leq e^{-2\operatorname{Re} z + r^2} e^{2\operatorname{Re} z} = e^{r^2} \qquad (z \in \mathbb{C}),$$

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since $1+x \le e^x$, $x \in \mathbb{R}$. So

$$\left|(1-z)\exp\left[\sum_{m=1}^{p-1}\frac{z^m}{m}\right]\right| \le \exp\left[r^2+\sum_{m=3}^{p-1}\frac{r^m}{m}\right].$$

Therefore,

$$|f(z)| \le r^2 + \sum_{m=3}^{p-1}\frac{r^m}{m} \quad (z \in \mathbb{C}).$$

But

$$\left[r^2+\sum_{m=3}^{p-1}\frac{r^m}{m}\right] r^{-p} \le h_p(w) \quad (r \ge w).$$

This proves the lemma. □

Lemmas 1.3 and 1.4 imply

Corollary 1.5. *One has*

$$|f(z)| \le \tilde{q}_p r^p \ (z \in \mathbb{C}, p > 2) \quad \textit{where} \quad \tilde{q}_p := \min_{w\in(0,1)} \max\left\{h_p(w), \frac{1}{p(1-w)}\right\}.$$

However, function $h_p(w)$ decreases in $w \in (0,1)$ and $\frac{1}{p(1-w)}$ increases. So the minimum in the previous corollary is attained when

$$h_p(w) = \frac{1}{p(1-w)}.$$

This equation is equivalent to (1.2). So $\tilde{q}_p = q_p$ and we thus get the inequality

$$|f(z)| \le q_p r^p \quad (z \in \mathbb{C}). \tag{1.4}$$

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Lemma 1.6. *Let* $A \in S_p, p > 2$. *Then* $\det_p(A) \le \exp[q_p w_p(A)]$ *where*

$$w_p(A) := \sum_{k=1}^{\infty} |\lambda_k(A)|^p.$$

Proof. Due to (1.4),

$$\det_p(A) \le \prod_{j=1}^{\infty} e^{q_p|\lambda_j(A)|^p} \le \exp\left[\sum_{k=1}^{\infty} q_p|\lambda_j(A)|^p\right].$$

As claimed. □

Proof of Theorem 1.1. The assertion of Theorem 1.1 follows from the previous lemma and the inequality

$$\sum_{k=1}^{\infty} |\lambda_j(A)|^p \le N_p^p(A)$$

cf. [5]. □

Furthermore, from (1.2) it follows that

$$x_0^{p-2} \le p(1-x_0)\sum_{m=0}^{p-3} x_0^m = p(1-x_0^{p-2})$$

since

$$\sum_{m=0}^{p-3} x_0^m = \frac{1-x_0^{p-2}}{1-x_0}.$$

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This proves inequality (1.3). Thus

$$q_p \leq \frac{1}{p\left(1 - \sqrt[p-2]{\frac{p}{p+1}}\right)}.$$

Note that if the spectral radius $r_s(A)$ of A is less than one, then according to Lemma 1.3 one can take

$$q_p = \frac{1}{p(1 - r_s(A))}.$$

Corollary 1.7. *Let* $A, B \in S_p \ \ (p > 2)$. *Then*

$$|\det_p(A) - \det_p(B)| \leq N_p(A - B) \exp[q_p(1 + N_p(A) + N_p(B))^p].$$

Indeed, this result is due to Theorem 1.1 and the theorem by Seiler and Simon [7] (see also [4, p. 32]).

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2. Lower Bounds

In this section for brevity we put $\lambda_j(A) = \lambda_j$. Denote by L *a Jordan contour connecting* 0 *and* 1, *lying in the disc* $\{z \in \mathbb{C} : |z| \le 1\}$, *not containing the points* $1/\lambda_j$ for any eigenvalue λ_j, such that

(2.1) $$\phi_A := \inf_{s \in L;\ k=1,2,\dots} |1 - s\lambda_k| > 0.$$

Let $l = |L|$ be the length of L. For example, if A does not have eigenvalues on $[1, \infty)$, then one can take $L = [0, 1]$. In this case $l = 1$ and $\phi_A = \inf_{k,s\in[0,1]} |1-s\lambda_k|$. If $r_s(A) < 1$, then $l = 1$, $\phi_A \ge 1 - r_s(A)$.

Theorem 2.1. *Let* $A \in S_p$ $(p = 2, 3, \dots)$, $1 \notin \sigma(A)$ *and condition (2.1) hold. Then*

$$|\det_p(A)| \ge e^{-\frac{lN_p^p(A)}{\phi_A}}.$$

Proof. Consider the function

$$D(z) = \prod_{j=1}^{\infty} G_j(z) \quad \text{where} \quad G_j(z) := (1 - z\lambda_j) \exp\left[\sum_{m=1}^{p-1} \frac{z^m \lambda_j^m}{m}\right].$$

Clearly,

$$D'(z) = \sum_{k=1}^{\infty} G_k'(z) \prod_{j=1, j\ne k}^{\infty} G_j(z)$$

and

$$G_j'(z) = \left[-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1}\right] \exp\left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m}\right].$$

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But

$$-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1} = -z^{p-1}\lambda_j^p,$$

since

$$\sum_{m=0}^{p-2} z^m z_j^m = \frac{1 - (z\lambda_j)^{p-1}}{1 - z\lambda_j}.$$

So

$$G_j'(z) = -z^{p-1}\lambda_j^p \exp\left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m}\right] = -\frac{z^{p-1}\lambda_j^p}{1 - z\lambda_j} G_j(z).$$

Hence, $D'(z) = h(z)D(z)$, where

$$h(z) := -z^{p-1} \sum_{k=1}^{\infty} \frac{\lambda_k^p}{1 - z\lambda_k}.$$

Consequently,

$$D(1) = \det_p(A) = \exp\left[\int_L h(s)ds\right].$$

But $|s| \leq 1$ for any $s \in L$ and thus

$$\left|\int_L h(s)ds\right| \leq \sum_{k=1}^{\infty} \lambda_k^p \int_L \frac{|s|^{p-1}|ds|}{|1 - s\lambda_k|} \leq w_p(A) l \phi_A^{-1}.$$

Therefore,

$$|\det_p(A)| = \left|\exp\left[\int_L h(s)ds\right]\right| \geq \exp\left[-\left|\int_L h(s)ds\right|\right] \geq \exp[-w_p(A) l \phi_A^{-1}].$$

This proves the theorem. □

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