journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 1, Article 2, 6 pp.

UPPER AND LOWER BOUNDS FOR REGULARIZED DETERMINANTS

M. I. GIL'

DEPARTMENT OF MATHEMATICS
BEN GURION UNIVERSITY OF THE NEGEV
P.0. BOX 653, BEER-SHEVA 84105, ISRAEL
gilmi@cs.bgu.ac.il

Received 11 January, 2007; accepted 21 January, 2008
Communicated by F. Hansen

ABSTRACT. Let S_p be the von Neumann-Schatten ideal of compact operators in a separable Hilbert space. In the paper, upper and lower bounds for the regularized determinants of operators from S_p are established.

Key words and phrases: von Neumann-Schatten ideal, Regularized determinant.

2000 *Mathematics Subject Classification.* 47B10.

1. UPPER BOUNDS

For an integer $p \geq 2$, let S_p be the von Neumann-Schatten ideal of compact operators A in a separable Hilbert space with the finite norm $N_p(A) = [\text{Trace}(AA^*)^{p/2}]^{1/p}$ where A^* is the adjoint. Recall that for an $A \in S_p$ the regularized determinant is defined as

$$\det_p(A) := \prod_{j=1}^{\infty}(1 - \lambda_j(A)) \exp\left[\sum_{m=1}^{p-1} \frac{\lambda_j^m(A)}{m}\right]$$

where $\lambda_j(A)$ are the eigenvalues of A with their multiplicities arranged in decreasing order.

The inequality

$$\det_p(A) \leq \exp[q_p N_p^p(A)] \tag{1.1}$$

is well-known, cf. [2, p. 1106], [4, p. 194]. Recall that $|\det_2(A)| \leq e^{N_2^2(A)/2}$, cf. [5, Section IV.2]. However, to the best of our knowledge, the constant q_p for $p > 2$ is unknown in the available literature although it is very important, in particular, for perturbations of determinants. In the present paper we suggest bounds for q_p $(p > 2)$. In addition, we establish lower bounds for $\det_p(A)$. As far as we know, the lower bounds have not yet been investigated in the available literature.

Our results supplement the very interesting recent investigations of the von Neumann-Schatten operators [1, 3, 8, 9, 10]. In connection with the recent results on determinants, the paper [6]

This research was supported by the Kamea fund of the Israel.
021-07

should be mentioned. It is devoted to higher order asymptotics of Toeplitz determinants with symbols in weighted Wienar algebras.

To formulate the main result we need the algebraic equation

$$x^{p-2} = p(1-x)\left[1 + \sum_{m=1}^{p-3} \frac{x^m}{m+2}\right] \qquad (p > 2). \tag{1.2}$$

Below we prove that it has *a unique positive root* $x_0 < 1$. Moreover,

$$x_0 \le \sqrt[p-2]{\frac{p}{p+1}}. \tag{1.3}$$

Theorem 1.1. *Let* $A \in S_p \ \ (p = 3, 4, \dots)$. *Then inequality (1.1) holds with*

$$q_p = \frac{1}{p(1-x_0)}.$$

The proof of this theorem is divided into a series of lemmas presented below.

Lemma 1.2. *Equation (1.2) has a unique positive root* $x_0 < 1$.

Proof. Rewrite (1.2) as

$$g(x) := \frac{x^{p-2}}{p(1-x)} - \left(1 + \sum_{m=3}^{p-1} \frac{x^{m-2}}{m}\right) = 0.$$

Clearly, $g(0) = -1$, $g(x) \to +\infty$ as $x \to 1 - 0$. So (1.2) has at least one root from $(0, 1)$. But from (1.2) it follows that a root from $[1, \infty)$ is impossible. Moreover, (1.2) is equivalent to the equation

$$\frac{1}{p(1-x)} = \frac{1}{x^{p-2}} + \sum_{m=3}^{p-1} \frac{x^{m-p}}{m}.$$

The left part of this equation increases and the right part decreases on $(0, 1)$. So the positive root is unique. □

Furthermore, consider the function

$$f(z) := \operatorname{Re}\left[\ln(1-z) + \sum_{m=1}^{p-1} \frac{z^m}{m}\right] \qquad (z \in \mathbb{C};\ p > 2).$$

Clearly,

$$f(z) = -\operatorname{Re} \sum_{m=p}^{\infty} \frac{z^m}{m} \qquad (|z| < 1).$$

Lemma 1.3. *Let* $w \in (0, 1)$. *Then*

$$|f(z)| \le \frac{r^p}{p(1-w)} \qquad (r \equiv |z| < w).$$

Proof. Clearly,

$$|f(z)| \le \sum_{m=p}^{\infty} \frac{r^m}{m} \qquad (r < 1).$$

Consequently,

$$|f(z)| \le \int_0^r \sum_{m=p}^{\infty} s^{m-1} ds = \int_0^r s^{p-1} \sum_{k=0}^{\infty} s^k ds = \int_0^r \frac{s^{p-1} ds}{1-s}.$$

Hence we get the required result. □

Lemma 1.4. *For any $w \in (0,1)$ and all $z \in \mathbb{C}$ with $|z| \geq w$, the following inequality is valid:*

$$|f(z)| \leq h_p(w) r^p \quad \textit{where} \quad h_p(w) = w^{-p}\left[w^2 + \sum_{m=3}^{p-1} \frac{w^m}{m}\right] \quad (p > 2).$$

Proof. Take into account that

$$|(1-z)e^z|^2 = (1 - 2\operatorname{Re} z + r^2)e^{2x} \leq e^{-2\operatorname{Re} z + r^2} e^{2\operatorname{Re} z} = e^{r^2} \quad (z \in \mathbb{C}),$$

since $1 + x \leq e^x$, $x \in \mathbb{R}$. So

$$\left|(1-z)\exp\left[\sum_{m=1}^{p-1} \frac{z^m}{m}\right]\right| \leq \exp\left[r^2 + \sum_{m=3}^{p-1} \frac{r^m}{m}\right].$$

Therefore,

$$|f(z)| \leq r^2 + \sum_{m=3}^{p-1} \frac{r^m}{m} \quad (z \in \mathbb{C}).$$

But

$$\left[r^2 + \sum_{m=3}^{p-1} \frac{r^m}{m}\right] r^{-p} \leq h_p(w) \quad (r \geq w).$$

This proves the lemma. □

Lemmas 1.3 and 1.4 imply

Corollary 1.5. *One has*

$$|f(z)| \leq \tilde{q}_p r^p \ (z \in \mathbb{C}, p > 2) \quad \textit{where} \quad \tilde{q}_p := \min_{w \in (0,1)} \max\left\{h_p(w), \frac{1}{p(1-w)}\right\}.$$

However, function $h_p(w)$ decreases in $w \in (0,1)$ and $\frac{1}{p(1-w)}$ increases. So the minimum in the previous corollary is attained when

$$h_p(w) = \frac{1}{p(1-w)}.$$

This equation is equivalent to (1.2). So $\tilde{q}_p = q_p$ and we thus get the inequality

$$|f(z)| \leq q_p r^p \quad (z \in \mathbb{C}). \tag{1.4}$$

Lemma 1.6. *Let $A \in S_p, p > 2$. Then $\det_p(A) \leq \exp[q_p w_p(A)]$ where*

$$w_p(A) := \sum_{k=1}^{\infty} |\lambda_k(A)|^p.$$

Proof. Due to (1.4),

$$\det_p(A) \leq \prod_{j=1}^{\infty} e^{q_p |\lambda_j(A)|^p} \leq \exp\left[\sum_{k=1}^{\infty} q_p |\lambda_j(A)|^p\right].$$

As claimed. □

Proof of Theorem 1.1. The assertion of Theorem 1.1 follows from the previous lemma and the inequality

$$\sum_{k=1}^{\infty} |\lambda_j(A)|^p \le N_p^p(A)$$

cf. [5]. □

Furthermore, from (1.2) it follows that

$$x_0^{p-2} \le p(1-x_0) \sum_{m=0}^{p-3} x_0^m = p(1-x_0^{p-2})$$

since

$$\sum_{m=0}^{p-3} x_0^m = \frac{1-x_0^{p-2}}{1-x_0}.$$

This proves inequality (1.3). Thus

$$q_p \le \frac{1}{p\left(1-\sqrt[p-2]{\frac{p}{p+1}}\right)}.$$

Note that if the spectral radius $r_s(A)$ of A is less than one, then according to Lemma 1.3 one can take

$$q_p = \frac{1}{p(1-r_s(A))}.$$

Corollary 1.7. *Let* $A, B \in S_p \ \ (p > 2)$. *Then*

$$|\det_p(A) - \det_p(B)| \le N_p(A-B)\exp[q_p(1+N_p(A)+N_p(B))^p].$$

Indeed, this result is due to Theorem 1.1 and the theorem by Seiler and Simon [7] (see also [4, p. 32]).

2. Lower Bounds

In this section for brevity we put $\lambda_j(A) = \lambda_j$. Denote by L *a Jordan contour connecting* 0 *and* 1, *lying in the disc* $\{z \in \mathbb{C} : |z| \le 1\}$, *not containing the points* $1/\lambda_j$ for any eigenvalue λ_j, such that

(2.1) $$\phi_A := \inf_{s\in L;\ k=1,2,\dots} |1 - s\lambda_k| > 0.$$

Let $l = |L|$ be the length of L. For example, if A does not have eigenvalues on $[1, \infty)$, then one can take $L = [0,1]$. In this case $l = 1$ and $\phi_A = \inf_{k,s\in[0,1]} |1 - s\lambda_k|$. If $r_s(A) < 1$, then $l = 1, \phi_A \ge 1 - r_s(A)$.

Theorem 2.1. *Let* $A \in S_p$ $(p = 2,3,\dots)$, $1 \notin \sigma(A)$ *and condition (2.1) hold. Then*

$$|\det_p(A)| \ge e^{-\frac{lN_p^p(A)}{\phi_A}}.$$

Proof. Consider the function

$$D(z) = \prod_{j=1}^{\infty} G_j(z) \quad \text{where} \quad G_j(z) := (1 - z\lambda_j)\exp\left[\sum_{m=1}^{p-1} \frac{z^m \lambda_j^m}{m}\right].$$

Clearly,

$$D'(z) = \sum_{k=1}^{\infty} G_k'(z) \prod_{j=1, j\ne k}^{\infty} G_j(z)$$

and

$$G'_j(z) = \left[-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1} \right] \exp \left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m} \right].$$

But

$$-\lambda_j + (1 - z\lambda_j) \sum_{m=0}^{p-2} z^m \lambda_j^{m+1} = -z^{p-1} \lambda_j^p,$$

since

$$\sum_{m=0}^{p-2} z^m z_j^m = \frac{1 - (z\lambda_j)^{p-1}}{1 - z\lambda_j}.$$

So

$$G'_j(z) = -z^{p-1} \lambda_j^p \exp \left[\sum_{m=1}^{p} \frac{z^m \lambda_j^m}{m} \right] = -\frac{z^{p-1} \lambda_j^p}{1 - z\lambda_j} G_j(z).$$

Hence, $D'(z) = h(z)D(z)$, where

$$h(z) := -z^{p-1} \sum_{k=1}^{\infty} \frac{\lambda_k^p}{1 - z\lambda_k}.$$

Consequently,

$$D(1) = \det_p(A) = \exp \left[\int_L h(s) ds \right].$$

But $|s| \le 1$ for any $s \in L$ and thus

$$\left| \int_L h(s) ds \right| \le \sum_{k=1}^{\infty} \lambda_k^p \int_L \frac{|s|^{p-1} |ds|}{|1 - s\lambda_k|} \le w_p(A) l \phi_A^{-1}.$$

Therefore,

$$|\det_p(A)| = \left| \exp \left[\int_L h(s) ds \right] \right| \ge \exp \left[- \left| \int_L h(s) ds \right| \right] \ge \exp[-w_p(A) l \phi_A^{-1}].$$

This proves the theorem. □

REFERENCES

[1] P. CHAISURIYA AND SING-CHEONG ONG, Schatten's theorems on functionally defined Schur algebras, *Int. J. Math. Math. Sci.*, (14) 2005, 2175–2193.

[2] N. DUNFORD AND J.T. SCHWARTZ, *Linear Operators, Part II. Spectral Theory*. Interscience Publishers, New York, London, 1963.

[3] L.G. GHEORGHE, Hankel operators in Schatten ideals, *Ann. Mat. Pura Appl.*, IV Ser., **180**(2) (2001), 203–210.

[4] I. GOHBERG, S. GOLBERG AND N. KRUPNIK, *Traces and Determinants of Linear Operators*, Birkháuser Verlag, Basel, 2000.

[5] I.C. GOHBERG AND M.G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, v. 18, Amer. Math. Soc., Providence, R. I., 1969.

[6] A. Yu KARLOVICH, Higher order asymptotics of Toeplitz determinants with symbols in weighted Wienar algebras, *Math. Analysis Appl.*, **320** (2006), 944–963

[7] E. SEILER AND B. SIMON, An inequality for determinants, *Proc. Nat. Acad. Sci.*, **72** (1975), 3277–3278.

[8] M. SIGG, A Minkowski-type inequality for the Schatten norm, *J. Inequal. Pure Appl. Math.*, **6**(3) (2005), Art. 87. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=560`].

[9] M.W. WONG, Schatten-von Neumann norms of localization operators, *Arch. Inequal. Appl.*, **2**(4) (2004), 391–396.

[10] JINGBO XIA, On the Schatten class membership of Hankel operators on the unit ball, *Ill. J. Math.*, **46**(3) (2002), 913–928.