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GENERALIZATIONS OF A CLASS OF INEQUALITIES FOR PRODUCTS

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ABSTRACT. In this paper, a class of inequalities for products of positive numbers are generalized.

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1. INTRODUCTION AND MAIN RESULTS

In 1987, H.-Sh. Huang [2] proved the following algebraic inequality for products:

$$\prod_{i=1}^{n}\left(\frac{1}{x_i}+x_i\right) \geq \left(n+\frac{1}{n}\right)^n, \tag{1.1}$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers with $\sum_{i=1}^{n} x_i = 1$.

In 2002, X.-Y. Yang [4] considered an analogous form of inequality (1.1) and posed an interesting open problem as follows.

Open Problem. *Assume $x_1, x_2, \ldots, x_n$ are positive real numbers with $\sum_{i=1}^{n} x_i = 1$ for $n \geq 3$. Then*

$$\prod_{i=1}^{n}\left(\frac{1}{x_i}-x_i\right) \geq \left(n-\frac{1}{n}\right)^n. \tag{1.2}$$

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In [1], Ch.-H. Dai and B.-H. Liu gave an affirmative answer to the above open problem.

In this article, by using the arithmetic-geometric mean inequality, inequalities (1.1) and (1.2) are refined and generalized as follows.

Theorem 1.1. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i = k$ and $k \leq n$, where k and n are natural numbers. Then we have for $m \in \mathbb{N}$*

$$\prod_{i=1}^n \left(\frac{1}{x_i^m} + x_i^m\right) \geq \left(\frac{n^m}{k^m} + \frac{k^m}{n^m}\right)^n \left(\prod_{i=1}^n \frac{nx_i}{k}\right)^{\frac{m(k^{2m}-n^{2m})}{k^{2m}+n^{2m}}} \geq \left(\frac{n^m}{k^m} + \frac{k^m}{n^m}\right)^n. \tag{1.3}$$

Theorem 1.2. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i = k$ for $k \leq 1$ and $n \geq 3$. Then for $m \in \mathbb{N}$ we have*

$$\prod_{i=1}^n \left(\frac{1}{x_i^m} - x_i^m\right) \geq \left(\frac{n^m}{k^m} - \frac{k^m}{n^m}\right)^n \left(\prod_{i=1}^n \frac{nx_i}{k}\right)^{\frac{m}{n}-\frac{m}{3}} \geq \left(\frac{n^m}{k^m} - \frac{k^m}{n^m}\right)^n. \tag{1.4}$$

Remark 1.3. Choosing $m = 1$ and $k = 1$ in Theorem 1.1 and Theorem 1.2, we can obtain inequalities (1.1) and (1.2) respectively.

2. LEMMAS

To prove Theorem 1.1 and Theorem 1.2, we will use following lemmas.

Lemma 2.1. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i = 1$ and $n \geq 3$. Then*

$$\prod_{i=1}^n \left(\frac{1}{x_i} - x_i\right) \geq \left(n - \frac{1}{n}\right)^n \left[\prod_{i=1}^n (nx_i)\right]^{\frac{1}{n}-\frac{1}{3}}. \tag{2.1}$$

Proof. From the conditions of Lemma 2.1 and by using the arithmetic-geometric mean inequality, we have for $1 \leq p, q \leq n$ and $p \neq q$

$$\begin{aligned}
(1-x_p)(1-x_q) &= 1 - x_p - x_q + x_p x_q \\
&= \sum_{k \neq p,q} x_k + x_p x_q \\
&= \sum_{k \neq p,q} \Bigg(\underbrace{\frac{x_k}{n} + \cdots + \frac{x_k}{n}}_{n}\Bigg) + x_p x_q \\
&\geq [n(n-2)+1] \left[\prod_{k \neq p,q} \left(\frac{x_k}{n}\right)^n x_p x_q\right]^{\frac{1}{n(n-2)+1}} \\
&= (n-1)^2 \left[\left(\frac{1}{n}\right)^{n(n-2)} \left(\prod_{k \neq p,q} x_k\right)^n x_p x_q\right]^{\frac{1}{(n-1)^2}} \\
&= (n-1)^2 \left(\frac{1}{n}\right)^{\frac{n(n-2)}{(n-1)^2}} \left(\prod_{k=1}^n x_i\right)^{\frac{n}{(n-1)^2}} (x_p x_q)^{\frac{1}{1-n}},
\end{aligned} \tag{2.2}$$

then

$$\prod_{i=1}^n (1-x_i) \geq (n-1)^n \left(\frac{1}{n}\right)^{\frac{n^2(n-2)}{2(n-1)^2}} \left(\prod_{i=1}^n x_i\right)^{\frac{n^2-2n+2}{2(n-1)^2}}. \tag{2.3}$$

By the arithmetic-geometric mean inequality, we obtain

$$
\begin{aligned}
\prod_{i=1}^{n}(1+x_i) &= \prod_{i=1}^{n}\left(\underbrace{\frac{1}{n}+\cdots+\frac{1}{n}}_{n}+x_i\right) \\
&\geq \prod_{i=1}^{n}\left\{(n+1)\left[\left(\frac{1}{n}\right)^n x_i\right]^{\frac{1}{n+1}}\right\} \qquad (2.4)\\
&= (n+1)^n\left(\frac{1}{n}\right)^{\frac{n^2}{n+1}}\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n+1}}.
\end{aligned}
$$

Utilizing (2.3) and (2.4) yields

$$
\begin{aligned}
\prod_{i=1}^{n}\left(\frac{1}{x_i}-x_i\right) &= \left[\prod_{i=1}^{n}(1-x_i)\right]\left[\prod_{i=1}^{n}(1+x_i)\right]\prod_{i=1}^{n}\frac{1}{x_i} \qquad (2.5)\\
&\geq (n-1)^n(n+1)^n\left(\frac{1}{n}\right)^{\frac{n^2(n-2)}{2(n-1)^2}+\frac{n^2}{n+1}}\left(\prod_{i=1}^{n}x_i\right)^{\frac{n^2-2n+2}{2(n-1)^2}+\frac{1}{n+1}-1}\\
&= \left(n-\frac{1}{n}\right)^n\left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{-n^3+3n^2-2n+2}{2(n+1)(n-1)^2}}.
\end{aligned}
$$

From the arithmetic-geometric mean inequality and $\sum_{i=1}^{n}x_i=1$ for $n\geq 3$, we have

$$
0<\prod_{i=1}^{n}(nx_i)\leq\left(\sum_{i=1}^{n}x_i\right)^n=1. \qquad (2.6)
$$

Since $n\geq 3$, it follows that

$$
\begin{aligned}
\frac{-n^3+3n^2-2n+2}{2(n+1)(n-1)^2} &= \frac{1}{n}-\frac{1}{3}-\frac{n(n-3)\left(n^2+2n+8\right)+10n+6}{6n(n+1)(n-1)^2}\\
&< \frac{1}{n}-\frac{1}{3}\\
&\leq 0.
\end{aligned}
$$

Therefore, by the monotonicity of the exponential function, we obtain

$$
\left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{-n^3+3n^2-2n+2}{2(n+1)(n-1)^2}} \geq \left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{1}{n}-\frac{1}{3}}. \qquad (2.7)
$$

Combining inequalities (2.5) and (2.7) leads to inequality (2.1). □

Lemma 2.2. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n}x_i=1$ for $n\geq 3$ and m a natural number. Then*

$$
\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right)\geq\left(n^m-\frac{1}{n^m}\right)^n\left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{m}{n}-\frac{m}{3}}. \qquad (2.8)
$$

Proof. Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\sum_{j=0}^{m-1} x_i^{2j} &= \sum_{j=0}^{m-2}\left(\underbrace{\frac{x_i^{2j}}{n^{2(m-j-1)}}+\cdots+\frac{x_i^{2j}}{n^{2(m-j-1)}}}_{n^{2(m-j-1)}}\right)+x_i^{2m-2} \\ &\geq \left[\sum_{j=0}^{m-1} n^{2j}\right]\left[x_i^{2(m-1)}\prod_{j=0}^{m-2}\left(\frac{x_i^{2j}}{n^{2(m-j-1)}}\right)^{n^{2(m-j-1)}}\right]^{\frac{1}{\sum_{j=0}^{m-1} n^{2j}}} \\ &= \frac{n^{2m}-1}{n^{2(m-1)}(n^2-1)}(nx_i)^{\frac{\sum_{j=0}^{m-1}[2(m-j-1)]n^{2j}}{\sum_{j=0}^{m-1} n^{2j}}}. \end{aligned} \tag{2.9}$$

Hence

$$\begin{aligned}\frac{1}{x_i^m}-x_i^m &= \left(\frac{1}{x_i}-x_i\right)x_i^{1-m}\sum_{j=0}^{m-1} x_i^{2j} \\ &\geq \left(\frac{1}{x_i}-x_i\right)x_i^{1-m}\frac{n^{2m}-1}{n^{2(m-1)}(n^2-1)}(nx_i)^{\frac{\sum_{j=0}^{m-1}[2(m-j-1)]n^{2j}}{\sum_{j=0}^{m-1} n^{2j}}} \\ &= \left(\frac{1}{x_i}-x_i\right)\frac{n^{2m}-1}{n^{m-1}(n^2-1)}(nx_i)^{\frac{(m-1)\sum_{j=0}^{m-1} n^{2j}-\sum_{j=1}^{m-1}2jn^{2j}}{\sum_{j=0}^{m-1} n^{2j}}}, \end{aligned} \tag{2.10}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right) \\ &\quad\geq n^{n(1-m)}\left[\frac{n^{2m}-1}{n^2-1}\right]^n\left[\prod_{i=1}^{n}\left(\frac{1}{x_i}-x_i\right)\right]\left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{(m-1)\sum_{j=0}^{m-1} n^{2j}-\sum_{j=1}^{m-1}2jn^{2j}}{\sum_{j=0}^{m-1} n^{2j}}}. \end{aligned} \tag{2.11}$$

In the following, we prove that for $n \geq 3$

$$\frac{(m-1)\sum_{j=0}^{m-1} n^{2j}-\sum_{j=1}^{m-1}2jn^{2j}}{\sum_{j=0}^{m-1} n^{2j}} \leq (m-1)\left(\frac{1}{n}-\frac{1}{3}\right). \tag{2.12}$$

For $m = 1$, the equality in (2.12) holds. For $m \geq 2$, we have

$$\begin{aligned}&\frac{(m-1)\sum_{j=0}^{m-1} n^{2j}-\sum_{j=1}^{m-1}2jn^{2j}}{\sum_{j=0}^{m-1} n^{2j}}-(m-1)\left(\frac{1}{n}-\frac{1}{3}\right) \\ &\quad= \frac{(m-1)\left(\frac{4}{3}-\frac{1}{n}\right)\sum_{j=0}^{m-1} n^{2j}-\sum_{j=1}^{m-1}2jn^{2j}}{\sum_{j=0}^{m-1} n^{2j}} \\ &\quad= \frac{(m-1)\left(\frac{4}{3}-\frac{1}{n}\right)\sum_{j=0}^{m-2} n^{2j}-\sum_{j=1}^{m-2}2jn^{2j}-(m-1)\left(\frac{1}{n}+\frac{2}{3}\right)n^{2(m-1)}}{\sum_{j=0}^{m-2} n^{2j}} \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
&= \frac{(m-1)\left[\frac{n^{2(m-1)}-1}{n^2-1}\left(\frac{4}{3}-\frac{1}{n}\right)-\left(\frac{1}{n}+\frac{2}{3}\right)n^{2(m-1)}\right]-\sum_{j=1}^{m-2}2jn^{2j}}{\sum_{j=0}^{m-2}n^{2j}} \\
&< \frac{(m-1)\left[\frac{1}{8}\left(\frac{4}{3}-\frac{1}{n}\right)n^{2(m-1)}-\left(\frac{1}{n}+\frac{2}{3}\right)n^{2(m-1)}\right]-\sum_{j=1}^{m-2}2jn^{2j}}{\sum_{j=0}^{m-2}n^{2j}} \\
&= \frac{(m-1)\left(-\frac{9}{8n}-\frac{1}{2}\right)n^{2(m-1)}-\sum_{j=1}^{m-2}2jn^{2j}}{\sum_{j=0}^{m-2}n^{2j}} \\
&< 0.
\end{aligned}
$$

Hence inequality (2.12) holds.

Considering inequality (2.6) and the monotonicity of the exponential function and combining inequality (2.11) with (2.12) reveals

$$
\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right) \ge n^{n(1-m)}\left(\frac{n^{2m}-1}{n^2-1}\right)^n\left[\prod_{i=1}^{n}\left(\frac{1}{x_i}-x_i\right)\right]\left[\prod_{i=1}^{n}(nx_i)\right]^{(m-1)\left(\frac{1}{n}-\frac{1}{3}\right)}. \tag{2.14}
$$

Substituting inequality (2.1) into (2.14) produces

$$
\begin{aligned}
&\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right) \\
&\ge n^{n(1-m)}\left(\frac{n^{2m}-1}{n^2-1}\right)^n\left(n-\frac{1}{n}\right)^n\left[\prod_{i=1}^{n}(nx_i)\right]^{\frac{1}{n}-\frac{1}{3}}\left[\prod_{i=1}^{n}(nx_i)\right]^{(m-1)\left(\frac{1}{n}-\frac{1}{3}\right)} \\
&= \left(n^m-\frac{1}{n^m}\right)^n\left[\prod_{i=1}^{n}(nx_i)\right]^{m\left(\frac{1}{n}-\frac{1}{3}\right)}.
\end{aligned} \tag{2.15}
$$

The proof is complete. □

Lemma 2.3. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = k \le 1$ for $n \ge 3$. Then for any natural number m, we have*

$$
\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right) \ge \left(n^m-\frac{1}{n^m}\right)^{-n}\left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n\prod_{i=1}^{n}\left(\frac{k^m}{x_i^m}-\frac{x_i^m}{k^m}\right). \tag{2.16}
$$

Proof. It is easy to see that

$$
\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right)\prod_{i=1}^{n}\left(\frac{k^m}{x_i^m}-\frac{x_i^m}{k^m}\right)^{-1} = k^{nm}\prod_{i=1}^{n}\frac{1-x_i^{2m}}{k^{2m}-x_i^{2m}}. \tag{2.17}
$$

Define

$$
f(x) = \ln\frac{1-x^{2m}}{k^{2m}-x^{2m}} \tag{2.18}
$$

for $x \in (0,k)$, $m \ge 1$ and $k \le 1$. Direct calculation shows that

$$
f'(x) = \frac{2m(1-k^{2m})x^{2m-1}}{(1-x^{2m})(k^{2m}-x^{2m})}, \tag{2.19}
$$

$$(2.20)\qquad f''(x) = \frac{2mx^{2(m-1)}(1-k^{2m})}{(1-x^{2m})^2(k^{2m}-x^{2m})^2}[(2m-1)(1-x^{2m})(k^{2m}-x^{2m}) + 2mx^{2m}(k^{2m}-x^{2m}+1-x^{2m})] \geq 0.$$

This means that f is convex in the interval $(0,k)$. Using Jensen's inequality [3], we obtain

$$(2.21)\qquad \frac{1}{n}\sum_{i=1}^{n}\ln\frac{1-x_i^{2m}}{k^{2m}-x_i^{2m}} \geq \ln\frac{1-[\frac{1}{n}\sum_{i=1}^{n}x_i]^{2m}}{k^{2m}-\left[\frac{1}{n}\sum_{i=1}^{n}x_i\right]^{2m}}$$

for any $0<x_i<k\leq 1$ and $i\in\mathbb{N}$. Using $\sum_{i=1}^{n}x_i=k$ in (2.21), it follows that

$$(2.22)\qquad \prod_{i=1}^{n}\frac{1-x_i^{2m}}{k^{2m}-x_i^{2m}} \geq \left(\frac{1-\frac{k^{2m}}{n^{2m}}}{k^{2m}-\frac{k^{2m}}{n^{2m}}}\right)^n,$$

therefore

$$(2.23)\qquad k^{nm}\prod_{i=1}^{n}\frac{1-x_i^{2m}}{k^{2m}-x_i^{2m}} \geq \left(n^m-\frac{1}{n^m}\right)^{-n}\left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n.$$

Substituting (2.17) into (2.23) leads to (2.16). The proof is complete. □

3. PROOFS OF THEOREMS

Proof of Theorem 1.1. Using the arithmetic-geometric mean inequality, we obtain

$$(3.1)\qquad \begin{aligned}\frac{1}{x_i^m}+x_i^m &= \underbrace{\frac{1}{n^{2m}x_i^m}+\cdots+\frac{1}{n^{2m}x_i^m}}_{n^{2m}}+\underbrace{\frac{x_i^m}{k^{2m}}+\cdots+\frac{x_i^m}{k^{2m}}}_{k^{2m}}\\ &\geq (n^{2m}+k^{2m})\left[\left(\frac{1}{n^{2m}x_i^m}\right)^{n^{2m}}\left(\frac{x_i^m}{k^{2m}}\right)^{k^{2m}}\right]^{\frac{1}{k^{2m}+n^{2m}}}\\ &= (n^{2m}+k^{2m})\left(k^{-2mk^{2m}}n^{-2mn^{2m}}x_i^{mk^{2m}-mn^{2m}}\right)^{\frac{1}{k^{2m}+n^{2m}}},\end{aligned}$$

therefore

$$(3.2)\qquad \prod_{i=1}^{n}\left(\frac{1}{x_i^m}+x_i^m\right) \geq \left(n^{2m}+k^{2m}\right)^n\left(k^{-2mk^{2m}}n^{-2mn^{2m}}\right)^{\frac{n}{k^{2m}+n^{2m}}}\left(\prod_{i=1}^{n}x_i\right)^{\frac{m(k^{2m}-n^{2m})}{k^{2m}+n^{2m}}},$$

that is

$$(3.3)\qquad \prod_{i=1}^{n}\left(\frac{1}{x_i^m}+x_i^m\right) \geq \left(\frac{n^m}{k^m}+\frac{k^m}{n^m}\right)^n\left(\prod_{i=1}^{n}\frac{nx_i}{k}\right)^{\frac{m(k^{2m}-n^{2m})}{k^{2m}+n^{2m}}}.$$

From $\sum_{i=1}^{n}x_i=k$ and the arithmetic-geometric mean inequality, it follows that

$$(3.4)\qquad \prod_{i=1}^{n}\frac{nx_i}{k} \leq \left(\sum_{i=1}^{n}\frac{x_i}{k}\right)^n = 1,$$

and then, considering $k\leq n$, we have

$$(3.5)\qquad \left(\frac{n^m}{k^m}+\frac{k^m}{n^m}\right)^n\left(\prod_{i=1}^{n}\frac{nx_i}{k}\right)^{\frac{m(k^{2m}-n^{2m})}{k^{2m}+n^{2m}}} \geq \left(\frac{n^m}{k^m}+\frac{k^m}{n^m}\right)^n.$$

Inequality (1.3) is then deduced by combining (3.3) and (3.5). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Applying $\sum_{i=1}^{n} \frac{x_i}{k} = 1$ to Lemma 2.2, we have

$$\prod_{i=1}^{n}\left(\frac{k^m}{x_i^m}-\frac{x_i^m}{k^m}\right) \geq \left(n^m-\frac{1}{n^m}\right)^n \left(\prod_{i=1}^{n}\frac{nx_i}{k}\right)^{\frac{m}{n}-\frac{m}{3}}. \tag{3.6}$$

Substituting inequality (3.6) into Lemma 2.3 gives

$$\begin{aligned}\prod_{i=1}^{n}\left(\frac{1}{x_i^m}-x_i^m\right) &\geq \left(n^m-\frac{1}{n^m}\right)^{-n}\left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n \prod_{i=1}^{n}\left(\frac{k^m}{x_i^m}-\frac{x_i^m}{k^m}\right)\\ &\geq \left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n \left(\prod_{i=1}^{n}\frac{nx_i}{k}\right)^{\frac{m}{n}-\frac{m}{3}}.\end{aligned} \tag{3.7}$$

Since

$$0 < \prod_{i=1}^{n}\frac{nx_i}{k} \leq \left(\sum_{i=1}^{n}\frac{x_i}{k}\right)^n = 1 \tag{3.8}$$

and $\frac{m}{n}-\frac{m}{3} \leq 0$, we have

$$\left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n \left(\prod_{i=1}^{n}\frac{nx_i}{k}\right)^{\frac{m}{n}-\frac{m}{3}} \geq \left(\frac{n^m}{k^m}-\frac{k^m}{n^m}\right)^n. \tag{3.9}$$

Combining (3.7) and (3.9), we immediately obtain inequality (1.4). This completes the proof of Theorem 1.2. □

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