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NEW ČEBYŠEV TYPE INEQUALITIES VIA TRAPEZOIDAL-LIKE RULES

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Abstract

In this paper we establish new inequalities similar to the Čebyšev integral inequality involving functions and their derivatives via certain Trapezoidal like rules.

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Contents

1. Introduction

In 1882, P.L. Čebyšev [2] proved the following classical integral inequality (see also [10, p. 207]):

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f', g' are bounded and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

provided the integrals in (1.2) exist.

The inequality (1.1) has received considerable attention and a number of papers have appeared in the literature which deal with various generalizations, extensions and variants, see [5] – [10]. The aim of this paper is to establish new inequalities similar to (1.1) involving first and second order derivatives of the functions f, g. The analysis used in the proofs is based on certain trapezoidal like rules proved in [1, 3, 4].

2. Statement of Results

In what follows $\mathbb{R}$ and $'$ denote respectively the set of real numbers and the derivative of a function. Let $[a,b] \subset \mathbb{R}$; $a < b$. We use the following notations to simplify the detail of presentation. For suitable functions $f, g, m : [a,b] \to \mathbb{R}$, and the constants $\alpha, \beta \in \mathbb{R}$, we set:

$$L(f;a,b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(t-s)\, dt ds,$$

$$M(f;a,b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(m(t) - m(s))\, dt ds,$$

$$N(f', f''; a,b) = \frac{1}{2(b-a)} \int_a^b (t-a)(b-t)\{[f';a,b] - f''(t)\}\, dt,$$

$$P(\alpha, \beta, f, g) = \alpha\beta - \frac{1}{b-a}\left\{\alpha \int_a^b g(t)\, dt + \beta \int_a^b f(t)\, dt\right\} + \left(\frac{1}{b-a}\int_a^b f(t)\, dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\, dt\right),$$

$$[f;a,b] = \frac{f(b) - f(a)}{b-a},$$

$$F = \frac{f(a) + f(b)}{2}, \quad G = \frac{g(a) + g(b)}{2}, \quad A = f\left(\frac{a+b}{2}\right), \quad B = g\left(\frac{a+b}{2}\right),$$

$$\bar{F} = \frac{f(a)+f(b)}{2} - \frac{(b-a)^2}{12}[f';a,b], \quad \bar{G} = \frac{g(a)+g(b)}{2} - \frac{(b-a)^2}{12}[g';a,b],$$

and define

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)| < \infty, \quad \|f\|_p = \left(\int_a^b |f'(t)|^p\, dt\right)^{\frac{1}{p}} < \infty,$$

for $1 \le p < \infty$.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$ with $f', g' \in L_2[a,b]$, then,*

$$|P(F,G,f,g)| \le \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}} \times \left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}, \tag{2.1}$$

$$|P(A,B,f,g)| \le \frac{(b-a)^2}{12}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}} \times \left[\frac{1}{b-a}\|g'\|_2^2 - ([g;a,b])^2\right]^{\frac{1}{2}}. \tag{2.2}$$

Theorem 2.2. *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable functions so that f', g' are absolutely continuous on $[a,b]$, then,*

$$\left|P\left(\bar{F},\bar{G},f,g\right)\right| \le \frac{(b-a)^4}{144}\|f'' - [f';a,b]\|_\infty \|g'' - [g';a,b]\|_\infty. \tag{2.3}$$

3. Proofs of Theorems 2.1 and 2.2

From the hypotheses of Theorem 2.1, we have the following identities (see [3, p. 654]):

$$F - \frac{1}{b-a}\int_a^b f(t)\,dt = L(f;a,b), \tag{3.1}$$

$$G - \frac{1}{b-a}\int_a^b g(t)\,dt = L(g;a,b). \tag{3.2}$$

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$P(F,G,f,g) = L(f;a,b)\,L(g;a,b). \tag{3.3}$$

From (3.3) we have

$$|P(F,G,f,g)| = |L(f;a,b)|\,|L(g;a,b)|. \tag{3.4}$$

Using the Cauchy-Schwarz inequality for double integrals,

$$\begin{aligned} |L(f;a,b)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |(f'(t)-f'(s))(t-s)|\,dtds \\ &\leq \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

By simple computation,

$$\frac{1}{2\left(b-a\right)^{2}}\int_{a}^{b}\int_{a}^{b}\left(f'\left(t\right)-f'\left(s\right)\right)^{2}dtds = \frac{1}{b-a}\int_{a}^{b}\left(f'\left(t\right)\right)^{2}dt-\left(\frac{1}{b-a}\int_{a}^{b}f'\left(t\right)dt\right)^{2}, \tag{3.6}$$

and

$$\frac{1}{2\left(b-a\right)^{2}}\int_{a}^{b}\int_{a}^{b}\left(t-s\right)^{2}dtds=\frac{\left(b-a\right)^{2}}{12}. \tag{3.7}$$

Using (3.6), (3.7) in (3.5),

$$\left|L\left(f;a,b\right)\right|\leq\frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\left\|f'\right\|_{2}^{2}-\left(\left[f;a,b\right]\right)^{2}\right]^{\frac{1}{2}}. \tag{3.8}$$

Similarly,

$$\left|L\left(g;a,b\right)\right|\leq\frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\left\|g'\right\|_{2}^{2}-\left(\left[g;a,b\right]\right)^{2}\right]^{\frac{1}{2}}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.4), we obtain (2.1).

From the hypotheses of Theorem 2.1, we have (see [4, p. 238]):

$$A-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt=M\left(f;a,b\right), \tag{3.10}$$

$$B-\frac{1}{b-a}\int_a^b g(t)\,dt=M(g;a,b), \tag{3.11}$$

where $m(t)$ involved in the notation $M(\cdot;a,b)$ is given by

$$m(t)=\begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ t-b & \text{if } t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Multiplying the left sides and right sides of (3.10) and (3.11), we get

$$P(A,B,f,g)=M(f;a,b)\,M(g;a,b). \tag{3.12}$$

From (3.12),

$$|P(A,B,f,g)|=|M(f;a,b)|\,|M(g;a,b)|. \tag{3.13}$$

Again using the Cauchy-Schwarz inequality for double integrals, we have,

$$\begin{aligned} |M(f;a,b)| &\le \frac{1}{2(b-a)^2}\int_a^b\int_a^b |(f'(t)-f'(s))(m(t)-m(s))|\,dtds \\ &\le \left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds\right]^{\frac{1}{2}} \\ &\quad\times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b (m(t)-m(s))^2\,dtds\right]^{\frac{1}{2}}. \end{aligned} \tag{3.14}$$

By simple computation,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f'(t)-f'(s))^2\,dtds = \frac{1}{b-a}\int_a^b (f'(t))^2 - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2, \tag{3.15}$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (m(t)-m(s))^2\,dtds = \frac{1}{b-a}\int_a^b (m(t))^2 - \left(\frac{1}{b-a}\int_a^b m(t)\,dt\right)^2. \tag{3.16}$$

It is easy to observe that

$$\int_a^b m(t)\,dt = 0,$$

and

$$\frac{1}{b-a}\int_a^b m^2(t)\,dt = \frac{(b-a)^2}{12}.$$

Using (3.15), (3.16) and the above observations in (3.14) we get

$$|M(f;a,b)| \le \frac{b-a}{2\sqrt{3}}\left[\frac{1}{b-a}\|f'\|_2^2 - ([f;a,b])^2\right]^{\frac{1}{2}}. \tag{3.17}$$

Similarly ,

$$|M(g;a,b)| \leq \frac{b-a}{2\sqrt{3}} \left[\frac{1}{b-a} \|g'\|_2^2 - ([g;a,b])^2 \right]^{\frac{1}{2}} . \tag{3.18}$$

Using (3.17) and (3.18) in (3.13) we get (2.2).

From the hypotheses of Theorem 2.2, we have the following identities (see [1, p. 197]):

$$\frac{1}{b-a} \int_a^b f(t)\,dt - \bar{F} = N(f', f''; a, b), \tag{3.19}$$

$$\frac{1}{b-a} \int_a^b g(t)\,dt - \bar{G} = N(g', g''; a, b). \tag{3.20}$$

Multiplying the left sides and right sides of (3.19) and (3.20), we get

$$P\left(\bar{F}, \bar{G}, f, g\right) = N(f', f''; a, b)\, N(g', g''; a, b). \tag{3.21}$$

From (3.21),

$$\left|P\left(\bar{F}, \bar{G}, f, g\right)\right| = |N(f', f''; a, b)|\, |N(g', g''; a, b)|. \tag{3.22}$$

By simple computation, we have,

$$\begin{aligned} |N(f', f''; a, b)| &\leq \frac{1}{2(b-a)} \int_a^b (t-a)(b-t)\, |[f';a,b] - f''(t)|dt \\ &\leq \frac{1}{2(b-a)} \|f''(t) - [f';a,b]\|_\infty \int_a^b (t-a)(b-t)dt \\ &= \frac{(b-a)^2}{12} \|f''(t) - [f';a,b]\|_\infty . \end{aligned} \tag{3.23}$$

Similarly,

$$|N(g', g''; a, b)| \leq \frac{(b-a)^2}{12} \|g''(t) - [g'; a, b]\|_\infty . \tag{3.24}$$

Using (3.23) and (3.24) in (3.22), we get the required inequality in (2.3).

4. Applications

In this section we present applications of the inequalities established in Theorem 2.1, to obtain results which are of independent interest.

Let X be a continuous random variable having the probability density function (p.d.f.) $h:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ and $E(x)=\int_a^b th(t)\,dt$ its expectation and the cumulative density function $H:[a,b]\to[0,1]$, i.e. $H(x)=\int_a^x h(t)\,dt$, $x\in[a,b]$. Then $H(a)=0, H(b)=1$ and $\frac{H(a)+H(b)}{2}=\frac{1}{2}$, $\int_a^b H(x)\,dx$ $=b-E(X)$.

Let $f=g=h$ and choose in (2.1) H instead of f and g and $\frac{1}{2}$ instead of F and G . By simple computation, we have,

$$P\left(\frac{1}{2},\frac{1}{2},H,H\right)=\frac{1}{4}-\frac{1}{b-a}(b-E(X))\left[1-\frac{b-E(X)}{b-a}\right],$$

and the right hand side in (2.1) is equal to

$$\frac{1}{12}\left[(b-a)\|h\|_2^2-1\right],$$

and hence the following inequality holds:

$$\left|\frac{1}{4}-\frac{1}{b-a}(b-E(X))\left[1-\frac{b-E(X)}{b-a}\right]\right|\le\frac{1}{12}\left[(b-a)\|h\|_2^2-1\right].$$

Let $a,b>0$ and consider the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}$, then $f\left(\frac{a+b}{2}\right)=g\left(\frac{a+b}{2}\right)=\frac{2}{a+b}$.

Let $g = f$ and choose in (2.2) $\frac{1}{x}$ instead of f and g and $\frac{2}{a+b}$ instead of A and B. By simple computation, we have,

$$P\left(\frac{2}{a+b}, \frac{2}{a+b}, \frac{1}{x}, \frac{1}{x}\right) = \left(\frac{2}{a+b} - \frac{\log b - \log a}{b-a}\right)^2,$$

$$\frac{1}{b-a}\left\|\left(\frac{1}{x}\right)'\right\|_2^2 - \left(\left[\frac{1}{x}; a, b\right]\right)^2 = \frac{(b-a)^2}{3a^3b^3}.$$

Using the above facts in (2.2), the following inequality holds:

$$\left(\frac{2}{a+b} - \frac{\log b - \log a}{b-a}\right)^2 \le \frac{(b-a)^4}{36a^3b^3}.$$

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