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## NEW ČEBYŠEV TYPE INEQUALITIES VIA TRAPEZOIDAL-LIKE RULES

B.G. PACHPATTE

57 Shri Niketan Colony  
Near Abhinay Talkies  
Aurangabad 431 001 (Maharashtra)  
India.

*EMail:* [bgpachpatte@gmail.com](mailto:bgpachpatte@gmail.com)

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## Abstract

In this paper we establish new inequalities similar to the Čebyšev integral inequality involving functions and their derivatives via certain Trapezoidal like rules.

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*Key words:* Čebyšev type inequalities, Trapezoid-like rules, Absolutely continuous functions, Differentiable functions, Identities.

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# 1. Introduction

In 1882, P.L. Čebyšev [2] proved the following classical integral inequality (see also [10, p. 207]):

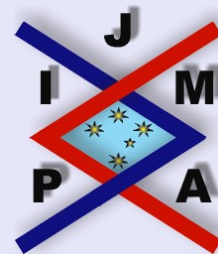
$$(1.1) \quad |T(f, g)| \leq \frac{1}{12} (b - a)^2 \|f'\|_{\infty} \|g'\|_{\infty},$$

where  $f, g : [a, b] \rightarrow \mathbb{R}$  are absolutely continuous functions, whose first derivatives  $f', g'$  are bounded and

$$(1.2) \quad T(f, g) = \frac{1}{b - a} \int_a^b f(x) g(x) dx - \left( \frac{1}{b - a} \int_a^b f(x) dx \right) \left( \frac{1}{b - a} \int_a^b g(x) dx \right),$$

provided the integrals in (1.2) exist.

The inequality (1.1) has received considerable attention and a number of papers have appeared in the literature which deal with various generalizations, extensions and variants, see [5] – [10]. The aim of this paper is to establish new inequalities similar to (1.1) involving first and second order derivatives of the functions  $f, g$ . The analysis used in the proofs is based on certain trapezoidal like rules proved in [1, 3, 4].



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## 2. Statement of Results

In what follows  $\mathbb{R}$  and  $'$  denote respectively the set of real numbers and the derivative of a function. Let  $[a, b] \subset \mathbb{R}$ ;  $a < b$ . We use the following notations to simplify the detail of presentation. For suitable functions  $f, g, m : [a, b] \rightarrow \mathbb{R}$ , and the constants  $\alpha, \beta \in \mathbb{R}$ , we set:

$$L(f; a, b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(t-s) dt ds,$$

$$M(f; a, b) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))(m(t) - m(s)) dt ds,$$

$$N(f', f''; a, b) = \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) \{[f'; a, b] - f''(t)\} dt,$$

$$P(\alpha, \beta, f, g) = \alpha\beta - \frac{1}{b-a} \left\{ \alpha \int_a^b g(t) dt + \beta \int_a^b f(t) dt \right\} \\ + \left( \frac{1}{b-a} \int_a^b f(t) dt \right) \left( \frac{1}{b-a} \int_a^b g(t) dt \right),$$

$$[f; a, b] = \frac{f(b) - f(a)}{b-a},$$

$$F = \frac{f(a) + f(b)}{2}, \quad G = \frac{g(a) + g(b)}{2}, \quad A = f\left(\frac{a+b}{2}\right), \quad B = g\left(\frac{a+b}{2}\right),$$



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$$\bar{F} = \frac{f(a) + f(b)}{2} - \frac{(b-a)^2}{12} [f'; a, b], \quad \bar{G} = \frac{g(a) + g(b)}{2} - \frac{(b-a)^2}{12} [g'; a, b],$$

and define

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)| < \infty, \quad \|f\|_p = \left( \int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty,$$

for  $1 \leq p < \infty$ .

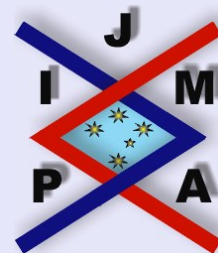
**Theorem 2.1.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous functions on  $[a, b]$  with  $f', g' \in L_2[a, b]$ , then,*

$$(2.1) \quad |P(F, G, f, g)| \leq \frac{(b-a)^2}{12} \left[ \frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \times \left[ \frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}},$$

$$(2.2) \quad |P(A, B, f, g)| \leq \frac{(b-a)^2}{12} \left[ \frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}} \times \left[ \frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}.$$

**Theorem 2.2.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be differentiable functions so that  $f', g'$  are absolutely continuous on  $[a, b]$ , then,*

$$(2.3) \quad |P(\bar{F}, \bar{G}, f, g)| \leq \frac{(b-a)^4}{144} \|f'' - [f'; a, b]\|_\infty \|g'' - [g'; a, b]\|_\infty.$$



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### 3. Proofs of Theorems 2.1 and 2.2

From the hypotheses of Theorem 2.1, we have the following identities (see [3, p. 654]):

$$(3.1) \quad F - \frac{1}{b-a} \int_a^b f(t) dt = L(f; a, b),$$

$$(3.2) \quad G - \frac{1}{b-a} \int_a^b g(t) dt = L(g; a, b).$$

Multiplying the left sides and right sides of (3.1) and (3.2) we get

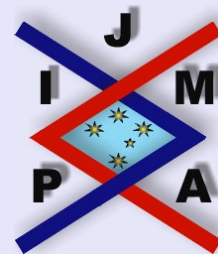
$$(3.3) \quad P(F, G, f, g) = L(f; a, b) L(g; a, b).$$

From (3.3) we have

$$(3.4) \quad |P(F, G, f, g)| = |L(f; a, b)| |L(g; a, b)|.$$

Using the Cauchy-Schwarz inequality for double integrals,

$$(3.5) \quad |L(f; a, b)| \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |(f'(t) - f'(s))(t-s)| dt ds \\ \leq \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 \right]^{\frac{1}{2}} \\ \times \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 \right]^{\frac{1}{2}}.$$



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By simple computation,

$$(3.6) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 dt ds \\ = \frac{1}{b-a} \int_a^b (f'(t))^2 dt - \left( \frac{1}{b-a} \int_a^b f'(t) dt \right)^2,$$

and

$$(3.7) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 dt ds = \frac{(b-a)^2}{12}.$$

Using (3.6), (3.7) in (3.5),

$$(3.8) \quad |L(f; a, b)| \leq \frac{b-a}{2\sqrt{3}} \left[ \frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}}.$$

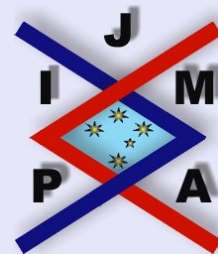
Similarly,

$$(3.9) \quad |L(g; a, b)| \leq \frac{b-a}{2\sqrt{3}} \left[ \frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}.$$

Using (3.8) and (3.9) in (3.4), we obtain (2.1).

From the hypotheses of Theorem 2.1, we have (see [4, p. 238]):

$$(3.10) \quad A - \frac{1}{b-a} \int_a^b f(t) dt = M(f; a, b),$$



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$$(3.11) \quad B - \frac{1}{b-a} \int_a^b g(t) dt = M(g; a, b),$$

where  $m(t)$  involved in the notation  $M(\cdot; a, b)$  is given by

$$m(t) = \begin{cases} t - a & \text{if } t \in [a, \frac{a+b}{2}] \\ t - b & \text{if } t \in (\frac{a+b}{2}, b]. \end{cases}$$

Multiplying the left sides and right sides of (3.10) and (3.11), we get

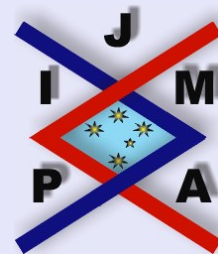
$$(3.12) \quad P(A, B, f, g) = M(f; a, b) M(g; a, b).$$

From (3.12),

$$(3.13) \quad |P(A, B, f, g)| = |M(f; a, b)| |M(g; a, b)|.$$

Again using the Cauchy-Schwarz inequality for double integrals, we have,

$$(3.14) \quad \begin{aligned} |M(f; a, b)| &\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |(f'(t) - f'(s))(m(t) - m(s))| dt ds \\ &\leq \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 dt ds \right]^{\frac{1}{2}} \\ &\quad \times \left[ \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s))^2 dt ds \right]^{\frac{1}{2}}. \end{aligned}$$



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By simple computation,

$$(3.15) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f'(t) - f'(s))^2 dt ds \\ = \frac{1}{b-a} \int_a^b (f'(t))^2 dt - \left( \frac{1}{b-a} \int_a^b f'(t) dt \right)^2,$$

and

$$(3.16) \quad \frac{1}{2(b-a)^2} \int_a^b \int_a^b (m(t) - m(s))^2 dt ds \\ = \frac{1}{b-a} \int_a^b (m(t))^2 dt - \left( \frac{1}{b-a} \int_a^b m(t) dt \right)^2.$$

It is easy to observe that

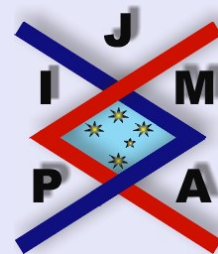
$$\int_a^b m(t) dt = 0,$$

and

$$\frac{1}{b-a} \int_a^b m^2(t) dt = \frac{(b-a)^2}{12}.$$

Using (3.15), (3.16) and the above observations in (3.14) we get

$$(3.17) \quad |M(f; a, b)| \leq \frac{b-a}{2\sqrt{3}} \left[ \frac{1}{b-a} \|f'\|_2^2 - ([f; a, b])^2 \right]^{\frac{1}{2}}.$$



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Similarly ,

$$(3.18) \quad |M(g; a, b)| \leq \frac{b-a}{2\sqrt{3}} \left[ \frac{1}{b-a} \|g'\|_2^2 - ([g; a, b])^2 \right]^{\frac{1}{2}}.$$

Using (3.17) and (3.18) in (3.13) we get (2.2).

From the hypotheses of Theorem 2.2, we have the following identities (see [1, p. 197]):

$$(3.19) \quad \frac{1}{b-a} \int_a^b f(t) dt - \bar{F} = N(f', f''; a, b),$$

$$(3.20) \quad \frac{1}{b-a} \int_a^b g(t) dt - \bar{G} = N(g', g''; a, b).$$

Multiplying the left sides and right sides of (3.19) and (3.20), we get

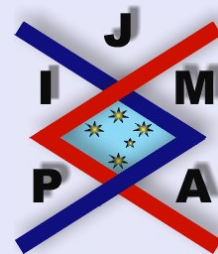
$$(3.21) \quad P(\bar{F}, \bar{G}, f, g) = N(f', f''; a, b) N(g', g''; a, b).$$

From (3.21),

$$(3.22) \quad |P(\bar{F}, \bar{G}, f, g)| = |N(f', f''; a, b)| |N(g', g''; a, b)|.$$

By simple computation, we have,

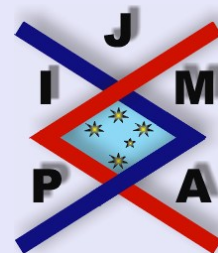
$$(3.23) \quad \begin{aligned} |N(f', f''; a, b)| &\leq \frac{1}{2(b-a)} \int_a^b (t-a)(b-t) |[f'; a, b] - f''(t)| dt \\ &\leq \frac{1}{2(b-a)} \|f''(t) - [f'; a, b]\|_{\infty} \int_a^b (t-a)(b-t) dt \\ &= \frac{(b-a)^2}{12} \|f''(t) - [f'; a, b]\|_{\infty}. \end{aligned}$$



Similarly,

$$(3.24) \quad |N(g', g''; a, b)| \leq \frac{(b-a)^2}{12} \|g''(t) - [g'; a, b]\|_\infty.$$

Using (3.23) and (3.24) in (3.22), we get the required inequality in (2.3).



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## 4. Applications

In this section we present applications of the inequalities established in Theorem 2.1, to obtain results which are of independent interest.

Let  $X$  be a continuous random variable having the probability density function (p.d.f.)  $h : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$  and  $E(X) = \int_a^b th(t) dt$  its expectation and the cumulative density function  $H : [a, b] \rightarrow [0, 1]$ , i.e.  $H(x) = \int_a^x h(t) dt$ ,  $x \in [a, b]$ . Then  $H(a) = 0, H(b) = 1$  and  $\frac{H(a)+H(b)}{2} = \frac{1}{2}$ ,  $\int_a^b H(x) dx = b - E(X)$ .

Let  $f = g = h$  and choose in (2.1)  $H$  instead of  $f$  and  $g$  and  $\frac{1}{2}$  instead of  $F$  and  $G$ . By simple computation, we have,

$$P\left(\frac{1}{2}, \frac{1}{2}, H, H\right) = \frac{1}{4} - \frac{1}{b-a} (b - E(X)) \left[1 - \frac{b - E(X)}{b-a}\right],$$

and the right hand side in (2.1) is equal to

$$\frac{1}{12} [(b-a) \|h\|_2^2 - 1],$$

and hence the following inequality holds:

$$\left| \frac{1}{4} - \frac{1}{b-a} (b - E(X)) \left[1 - \frac{b - E(X)}{b-a}\right] \right| \leq \frac{1}{12} [(b-a) \|h\|_2^2 - 1].$$

Let  $a, b > 0$  and consider the function  $f : (0, \infty) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{x}$ , then  $f\left(\frac{a+b}{2}\right) = g\left(\frac{a+b}{2}\right) = \frac{2}{a+b}$ .



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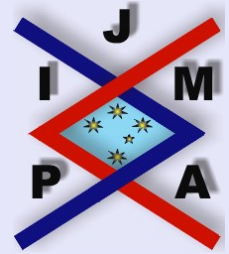
Let  $g = f$  and choose in (2.2)  $\frac{1}{x}$  instead of  $f$  and  $g$  and  $\frac{2}{a+b}$  instead of  $A$  and  $B$ . By simple computation, we have,

$$P\left(\frac{2}{a+b}, \frac{2}{a+b}, \frac{1}{x}, \frac{1}{x}\right) = \left(\frac{2}{a+b} - \frac{\log b - \log a}{b-a}\right)^2,$$

$$\frac{1}{b-a} \left\| \left(\frac{1}{x}\right)' \right\|_2^2 - \left( \left[\frac{1}{x}; a, b\right] \right)^2 = \frac{(b-a)^2}{3a^3b^3}.$$

Using the above facts in (2.2), the following inequality holds:

$$\left(\frac{2}{a+b} - \frac{\log b - \log a}{b-a}\right)^2 \leq \frac{(b-a)^4}{36a^3b^3}.$$



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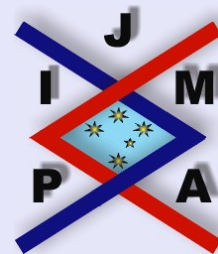
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