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volume 4, issue 1, article 24,
2003.

*Received 15 October, 2002;
accepted 19 February, 2003.*

Communicated by: A.G. Babenko

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Abstract

The main purpose of this note is to characterize the operators $S \in \ker \Delta_{A,B} \cap C_1$ which are orthogonal to the range of elementary operators, where S is not a smooth point in C_1 by using the φ -directional derivative.

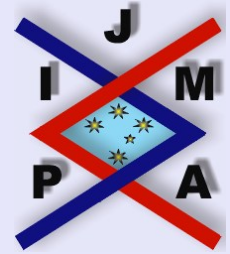
2000 Mathematics Subject Classification: Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words: Elementary operators, Schatten p -classes, Orthogonality.

This Work was supported by the research center project No. **Math/1422/10** and for the second author by the research center project No. **Math/1422/22**.

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1. Introduction

Let E be a complex Banach space. We first define orthogonality in E . We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$(1.1) \quad \|a + \lambda b\| \geq \|a\|.$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e, if and only if this complex line is a tangent one.

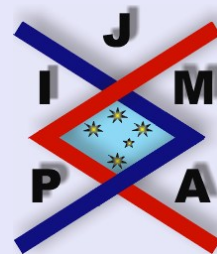
Note that if b is orthogonal to a , then a need not be orthogonal to b . If E is a Hilbert space, then from (1.1) follows $\langle a, b \rangle = 0$, i.e, orthogonality in the usual sense. This notion and first results concerning the orthogonality in linear metric space was given by G. Birkhoff [2].

Next we define the von Neumann-Schatten classes C_p ($1 \leq p < \infty$). Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq \dots \geq 0$ denote the singular values of T , i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p -classes C_p if

$$\|T\|_p = \left[\sum_{j=1}^{\infty} s_j(T)^p \right]^{\frac{1}{p}} = [tr(T)^p]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$



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denoting the usual operator norm. For the general theory of the Schatten p -classes the reader is referred to [13].

Recall that the norm $\|\cdot\|$ of the B -space V is said to be Gâteaux differentiable at non-zero elements $x \in V$ if

$$\lim_{\mathbb{R} \ni t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} = \operatorname{Re} D_x(y),$$

for all $y \in V$. Here \mathbb{R} denotes the set of all reals, Re denotes the real part and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gâteaux differentiability of the norm at x implies that x is a smooth point of the sphere of radius $\|x\|$. It is well known (see [7] and references therein) that for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of T is given by

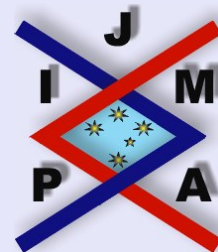
$$D_T(X) = \operatorname{tr} \left[\frac{|T|^{p-1} U X^*}{\|T\|_p^{p-1}} \right]$$

for all $X \in C_p$ ($1 < p < \infty$), where $T = U|T|$ is the polar decomposition of T .

In [1] Anderson proved that if A is a normal operator on Hilbert space H , then $AS = SA$ implies that for all bounded linear operator X there holds

$$(1.2) \quad \|S + AX - XA\| \geq \|S\|.$$

This means that the range of the derivation $\delta_A : B(H) \rightarrow B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized



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in two directions, by extending the class of elementary mappings

$$\tilde{E} : B(H) \rightarrow B(H); \tilde{E}(X) = \sum_{i=1}^n A_i X B_i$$

and

$$E : B(H) \rightarrow B(H); E(X) = \sum_{i=1}^n A_i X B_i - X,$$

where $(A_1, A_2, \dots, A_n), (B_1, B_2, \dots, B_n)$ are n -tuples of bounded operators on H and by extending the inequality (1.2) to C_p -classes with $1 < p < \infty$, see [3], [7], [10] and [11].

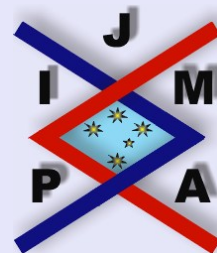
The Gâteaux derivative concept was used in [4], [5], [6], [8], [9] and [15] and, in order to characterize those operators for which the range of a derivation is orthogonal. In these papers, the attention was directed to C_p -classes for some $p > 1$.

The main purpose of this note is to characterize the operators $S \in C_1$ which are orthogonal to the range of elementary operators, where S is not a smooth point in C_1 by using the φ -directional derivative.

Recall that the operator S is a smooth point of the corresponding sphere in C_1 if and only if either S is injective or S^* is injective.

It is very interesting to point out that this result has been done in C_p -classes with $1 < p < \infty$ but, at least to our acknowledge, it was not given, till now, for C_1 -classes.

It is well known see ([6]) that the norm $\|\cdot\|$ of the B -space V is said to be



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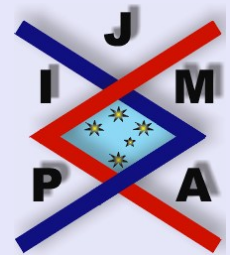
φ -directional differentiable at non-zero elements $x \in V$ if

$$\lim_{\mathbb{R} \ni t \rightarrow 0} \frac{\|x + te^{i\varphi}y\| - \|x\|}{t} = D_{x,\varphi}(y),$$

for all $y \in V$. Therefore for every non-zero $T \in C_1$ which is not a smooth point, the support functional of T is given by

$$D_{\varphi,T}(S) = \operatorname{Re} \{e^{i\varphi} \operatorname{tr}(U^*Y)\} + \|QYP\|_{C_1},$$

for all $X \in C_1$, where $S = U|S|$ is the polar decomposition of X , $P = P_{\ker X}$, $Q = Q_{\ker X^*}$.



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2. Main Results

Let $\phi : B(H) \rightarrow B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$, for all α, β, X, Y , and satisfying the following condition:

$$\text{tr}(X\phi(Y)) = \text{tr}(\phi(X)Y), \text{ for all } X, Y \in C_1.$$

Let $S \in C_1$ and put

$$\mathcal{U} = \{X \in B(H) : \phi(X) \in C_1\}.$$

Let $\psi : \mathcal{U} \rightarrow C_1$ defined by

$$\psi(X) = S + \phi(X).$$

Theorem 2.1. [12] Let $V \in C_1$. Then,

$$\|S + \phi(X)\|_{C_1} \geq \|\psi(S)\|_{C_1}, \text{ for all } X \in C_1,$$

if and only if $U^* \in \ker \phi$, where $\psi(V) = U|\psi(V)|$.

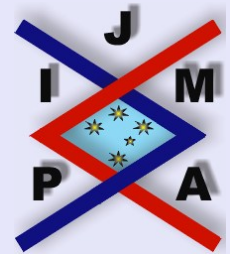
As a first consequence of this result we have the following theorem.

Theorem 2.2. Let $S \in C_1 \cap \ker \phi$. The following assertions are equivalent:

1.

$$\|S + \phi(X)\|_{C_1} \geq \|S\|_{C_1}, \text{ for all } X \in C_1,$$

2. $U^* \in \ker \phi$, where $S = U|S|$.



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Our main purpose in this paper is to use the general result in Theorem 2.1 in order to characterize all those operators $S \in C_1 \cap \ker \phi$ which are orthogonal to $Ran(\phi | C_1)$ (the range of $\phi | C_1$) when ϕ is one of the following elementary operators:

1. $E_{A,B} : B(H) \rightarrow B(H)$ defined by

$$E_{A,B}(X) = \sum_{i=1}^n A_i X B_i - X,$$

where $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ are n -tuples of operators in $B(H)$.

2. $\Delta_{A,B} : B(H) \rightarrow B(H)$ defined by

$$\Delta_{A,B}(X) = AXB - X,$$

where A and B are operators in $B(H)$.

3. $\delta_{A,B} : B(H) \rightarrow B(H)$ is defined by

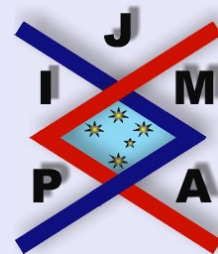
$$\delta_{A,B}(X) = AX - XB,$$

where A and B are operators in $B(H)$.

4. $\tilde{E}_{A,B} : B(H) \rightarrow B(H)$ is defined by

$$\tilde{E}_{A,B}(X) = \sum_{i=1}^n A_i X B_i$$

where $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ are n -tuples of operators in $B(H)$.



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Note that all the elementary operators recalled above satisfy the assumptions assumed on our abstract general map ϕ .

Let us begin by proving our main results for the elementary operator E .

Theorem 2.3. *Let $A = (A_1, A_2, \dots, A_n)$, $B = (B_1, B_2, \dots, B_n)$ be n -tuples of operators in $B(H)$ such that*

$$\ker E_{A,B}|_{C_1} \subseteq \ker E_{A^*,B^*}|_{C_1}.$$

Assume that

$$(2.1) \quad \sum_{i=1}^n A_i A_i^* \leq 1, \quad \sum_{i=1}^n A_i^* A_i \leq 1, \quad \sum_{i=1}^n B_i B_i^* \leq 1 \text{ and } \sum_{i=1}^n B_i^* B_i \leq 1$$

and let $S = U|S| \in C_1$. Then $S \in \ker E_{A,B}$ if, and only if,

$$\|S + E_{A,B}(X)\|_1 \geq \|S\|_1,$$

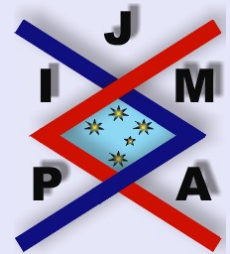
for all $X \in C_1$.

Proof. Let $S \in \ker E_{A,B}|_{C_1}$. Then it follows from Theorem 2.1 that

$$(2.2) \quad \|S + E_{A,B}(X)\|_1 \geq \|S\|_1,$$

for all $X \in C_1$ if and only if $U^* \in \ker E_{A,B}$. The hypothesis $\ker E_{A,B} \subseteq \ker E_{A^*,B^*}$, implies that $U^* \in \ker E_{A^*,B^*}$. Note that $U^* \in \ker E_{A,B} \subseteq \ker E_{A^*,B^*}$ if and only if

$$(2.3) \quad \text{tr}(U^* E_{A,B}(X)) = 0 = \text{tr}(U^* E_{A^*,B^*}(X)).$$



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Choosing $X \in C_1$ to be the rank one operator $x \otimes y$ it follows from (2.3) that if (2.2) holds then

$$\begin{aligned} &= \operatorname{tr} \left(\left(\sum_{i=1}^n B_i U^* A_i - U^* \right) (x \otimes y) \right) \\ &= \left(\sum_{i=1}^n B_i U^* A_i x, y \right) - (U^* x, y) = 0 \end{aligned}$$

and

$$\left(\sum_{i=1}^n B_i^* U^* A_i^* x, y \right) - (U^* x, y) = 0$$

for all $x, y \in H$ or

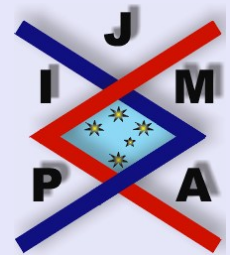
$$E_{A,B}(U) = 0 = E_{A^*,B^*}(U).$$

It is known that if $\sum_{i=1}^n B_i B_i^* \leq 1$, $\sum_{i=1}^n B_i^* B_i \leq 1$ and $E_{B,B}(S) = 0 = E_{B^*,B^*}^*(S)$, then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|S|^2$ reduces each B_i see ([3, Theorem 8], [15, Lemma 2.3]). In particular $|S|$ commutes with each B_i for all $1 \leq i \leq n$. Hence (2.2) holds if and only if,

$$E_{A,B}(S) = 0 = E_{A^*,B^*}^*(S).$$

□

Now, we prove a similar result for the operator $\Delta_{A,B}$. Note that in this case we don't need the condition (2.1).



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Theorem 2.4. Let A and B be two operators in $B(H)$ such that

$$\ker \Delta_{A,B}|_{C_1} \subseteq \ker \Delta_{A^*,B^*}|_{C_1}$$

and assume that $S = U|S| \in C_1$. Then $S \in \ker \Delta_{A,B}|_{C_1}$ if and only if,

$$(2.4) \quad \|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1,$$

for all $X \in C_1$.

Proof. Let $S \in \ker \Delta_{A,B}|_{C_1}$. Then it follows from Theorem 2.1 that

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1,$$

for all $X \in C_1$ if and only if $U^* \in \ker \Delta_{A,B}$. By the same arguments as in the proof of the above theorem, it follows that (2.4) holds if and only if

$$AUB = U = A^*UB^* \quad \text{or} \quad B^*U^*A^* = U^* = BU^*A.$$

Multiplying at right by $|S|$ we get

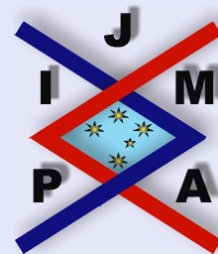
$$(2.5) \quad AUB|S| = U|S| = A^*UB^*|S|.$$

Now as $S \in \ker \Delta_{A,B}|_{C_1} \subseteq \ker \Delta_{A^*,B^*}|_{C_1}$, i.e.,

$$ASB = S = A^*SB^*A \quad \text{or} \quad B^*S^*A^* = S^* = BS^*A,$$

then

$$BS^*S = BS^*ASB = S^*SB, \text{ i.e., } B|S| = |S|B.$$



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We also get $A|S| = |S|A$, that is, both operators A and B commute with $|S|$. Thus, (2.5) is equivalent to

$$AU|S|B = U|S| = A^*U|S|B^*, \quad \text{i.e., } ASB = S = A^*SB^*.$$

Thus $S \in \ker \Delta_{A,B}$. □

Remark 2.1. *The above theorem is still true if we consider instead of $\Delta_{A,B}$ the generalized derivation $\delta_{A,B}(X) = AX - XB$. It is still possible to characterize the operators $S \in \ker \phi_{A,B} \cap C_1$ which are orthogonal to $\text{Ran}(\phi_{A,B})$, where $\phi_{A,B} = AXB + CXD$. In [13] Shulman stated that there exists a normally represented elementary operator of the form $\sum_{i=1}^n A_i X B_i$ with $n > 2$ such that $\text{asc} E > 1$, i.e. the range and the kernel have non trivial intersection. Hence Theorem 2.1 does not hold in the case where $E_{A,B}$ is replaced by $\phi_{A,B} = \sum_{i=3}^n A_i X B_i$*

Corollary 2.5. *Let A, B be normal operators in $B(H)$ and let $S = U|S| \in C_1$. Then $S \in \ker \Delta_{A,B}$, if and only if,*

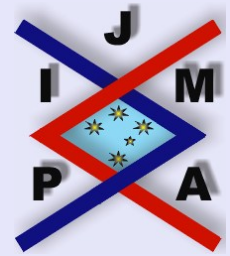
$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1,$$

for all $X \in C_1$.

Proof. If A, B are normal operators the Putnam-Fuglede theorem ensures that $\ker \Delta_{A,B} \subseteq \ker \Delta_{A,B}^*$ □

Corollary 2.6. *Let A, B in $B(H)$ be contractions and let $S = U|S| \in C_1$. Then $S \in \ker \Delta_{A,B}$, if and only if,*

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1,$$



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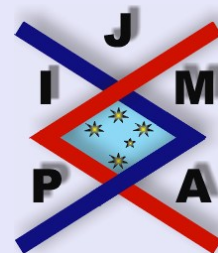
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for all $X \in C_1$.

Proof. It is known [14, Theorem 2.2] that if A and B are contractions and $S \in C_1$, then $\ker \Delta_{A,B} \subseteq \ker \Delta_{A,B}^*$ and the result holds by the above theorem. \square

Remark 2.2. *The above corollaries still hold true when we consider $\delta_{A,B}$ instead of $\Delta_{A,B}$.*



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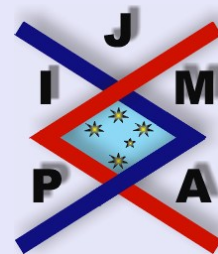
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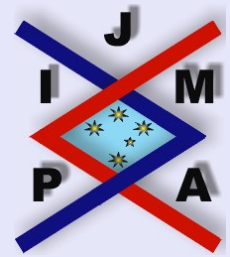
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