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NOTE ON DRAGOMIR-AGARWAL INEQUALITIES, THE GENERAL EULER TWO-POINT FORMULAE AND CONVEX FUNCTIONS

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Abstract

The general Euler two-point formulae are used with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration.

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1. Introduction

One of the cornerstones of nonlinear analysis is the Hadamard inequality, which states that if $[a, b]$ ($a < b$) is a real interval and $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

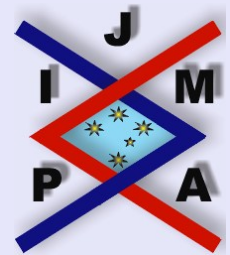
Recently, S.S. Dragomir and R.P. Agarwal [3] considered the trapezoid formula for numerical integration of functions f such that $|f'|^q$ is a convex function for some $q \geq 1$. Their approach was based on estimating the difference between the two sides of the right-hand inequality in (1.1). Improvements of their results were obtained in [5]. In particular, the following tool was established.

Suppose $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is differentiable on I^0 and such that $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, where $a, b \in I^0$ ($a < b$). Then

$$(1.2) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

Some generalizations to higher-order convexity and applications of these results are given in [1]. Related results for Euler midpoint, Euler-Simpson, Euler two-point, dual Euler-Simpson, Euler-Simpson 3/8 and Euler-Maclaurin formulae were considered in [7] and for Euler two-point formulae in [9] (see also [2] and [8]).

In the paper [4] Dah-Yan Hwang procured some new inequalities of this type and he applied the result to obtain a better estimate of the error in the trapezoidal formula.



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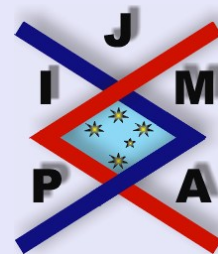
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In this paper we consider some related results using the general Euler two-point formulae. We will use the interval $[0, 1]$ because of simplicity and since it involves no loss in generality.



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2. The General Euler Two-point Formulae

In the recent paper [6] the following identities, named the general Euler two-point formulae, have been proved. Let $f \in C^n([0, 1], \mathbb{R})$ for some $n \geq 3$ and let $x \in [0, 1/2]$. If $n = 2r - 1$, $r \geq 2$, then

$$(2.1) \quad \int_0^1 f(t)dt = \frac{1}{2} [f(x) + f(1-x)] - T_{r-1}(f) + \frac{1}{2(2r-1)!} \int_0^1 f^{(2r-1)}(t) F_{2r-1}^x(t) dt,$$

while for $n = 2r$, $r \geq 2$ we have

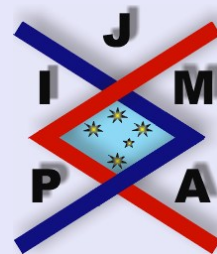
$$(2.2) \quad \int_0^1 f(t)dt = \frac{1}{2} [f(x) + f(1-x)] - T_{r-1}(f) + \frac{1}{2(2r)!} \int_0^1 f^{(2r)}(t) F_{2r}^x(t) dt$$

and

$$(2.3) \quad \int_0^1 f(t)dt = \frac{1}{2} [f(x) + f(1-x)] - T_r(f) + \frac{1}{2(2r)!} \int_0^1 f^{(2r)}(t) G_{2r}^x(t) dt.$$

Here we define $T_0(f) = 0$ and for $1 \leq m \leq \lfloor n/2 \rfloor$

$$T_m(f) = \sum_{k=1}^m \frac{B_{2k}(x)}{(2k)!} [f^{(2k-1)}(1) - f^{(2k-1)}(0)],$$



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$$G_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t)$$

and

$$F_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t) - B_n(x) - B_n(1-x),$$

where $B_k(\cdot), k \geq 0$, is the k -th Bernoulli polynomial and $B_k = B_k(0) = B_k(1) (k \geq 0)$ the k -th Bernoulli number. By $B_k^*(\cdot) (k \geq 0)$ we denote the function of period one such that $B_k^*(x) = B_k(x)$ for $0 \leq x \leq 1$.

It was proved in [6] that $F_n^x(1-t) = (-1)^n F_n^x(t)$, $(-1)^{r-1} F_{2r-1}^x(t) \geq 0$, $(-1)^r F_{2r}^x(t) \geq 0$ for $x \in [0, \frac{1}{2} - \frac{1}{2\sqrt{3}}]$ and $t \in [0, 1/2]$, and $(-1)^r F_{2r-1}^x(t) \geq 0$, $(-1)^{r-1} F_{2r}^x(t) \geq 0$ for $x \in [\frac{1}{2\sqrt{3}}, \frac{1}{2}]$ and $t \in [0, 1/2]$. Also

$$\int_0^1 |F_{2r-1}^x(t)| dt = \frac{2}{r} \left| B_{2r} \left(\frac{1}{2} - x \right) - B_{2r}(x) \right|,$$

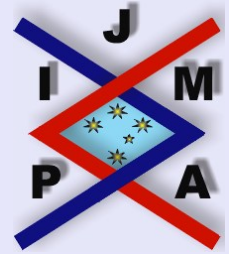
$$\int_0^1 |F_{2r}^x(t)| dt = 2|B_{2r}(x)|$$

and

$$\int_0^1 |G_{2r}^x(t)| dt \leq 4|B_{2r}(x)|.$$

With integration by parts, we have that the following identities hold:

$$\begin{aligned} (1) \quad C_1(x) &= \int_0^1 F_{2r-1}^x \left(\frac{y}{2} \right) dy \\ &= - \int_0^1 F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy = \frac{2}{r} \left[B_{2r}(x) - B_{2r} \left(\frac{1}{2} - x \right) \right], \end{aligned}$$



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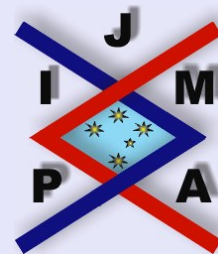
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$$\begin{aligned}
 (2) \quad C_2(x) &= \int_0^1 y F_{2r-1}^x \left(\frac{y}{2} \right) dy \\
 &= - \int_0^1 y F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \\
 &= - \frac{2}{r} B_{2r} \left(\frac{1}{2} - x \right),
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad C_3(x) &= \int_0^1 (1-y) F_{2r-1}^x \left(\frac{y}{2} \right) dy \\
 &= - \int_0^1 (1-y) F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \\
 &= \frac{2}{r} B_{2r}(x),
 \end{aligned}$$

$$(4) \quad C_4(x) = \int_0^1 F_{2r}^x \left(\frac{y}{2} \right) dy = \int_0^1 F_{2r}^x \left(1 - \frac{y}{2} \right) dy = -2B_{2r}(x),$$

$$\begin{aligned}
 (5) \quad C_5(x) &= \int_0^1 y F_{2r}^x \left(\frac{y}{2} \right) dy \\
 &= \int_0^1 y F_{2r}^x \left(1 - \frac{y}{2} \right) dy \\
 &= \frac{8}{(2r+1)(2r+2)} \left[B_{2r+2}(x) - B_{2r+2} \left(\frac{1}{2} - x \right) \right] - B_{2r}(x),
 \end{aligned}$$



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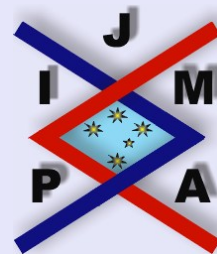


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$$\begin{aligned}
 (6) \quad C_6(x) &= \int_0^1 (1-y)F_{2r}^x\left(\frac{y}{2}\right) dy \\
 &= \int_0^1 (1-y)F_{2r}^x\left(1-\frac{y}{2}\right) dy \\
 &= \frac{8}{(2r+1)(2r+2)} \left[B_{2r+2}\left(\frac{1}{2}-x\right) - B_{2r+2}(x) \right] - B_{2r}(x),
 \end{aligned}$$

$$(7) \quad C_7(x) = \int_0^1 G_{2r}^x\left(\frac{y}{2}\right) dy = \int_0^1 G_{2r}^x\left(1-\frac{y}{2}\right) dy = 0,$$

$$\begin{aligned}
 (8) \quad C_8(x) &= \int_0^1 yG_{2r}^x\left(\frac{y}{2}\right) dy \\
 &= \int_0^1 yG_{2r}^x\left(1-\frac{y}{2}\right) dy \\
 &= -\int_0^1 (1-y)G_{2r}^x\left(\frac{y}{2}\right) dy \\
 &= \int_0^1 (1-y)G_{2r}^x\left(1-\frac{y}{2}\right) dy \\
 &= \frac{8}{(2r+1)(2r+2)} \left[B_{2r+2}(x) - B_{2r+2}\left(\frac{1}{2}-x\right) \right],
 \end{aligned}$$

Theorem 2.1. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is n -times differentiable and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \cup \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.

(a) If $|f^{(n)}|^q$ is convex for some $q \geq 1$, then for $n = 2r - 1$, $r \geq 2$, we have

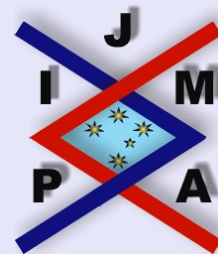
$$\begin{aligned}
 (2.4) \quad & \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f) \right| \\
 & \leq \frac{2}{(2r)!} \left| B_{2r} \left(\frac{1}{2} - x \right) - B_{2r}(x) \right|^{1-\frac{1}{q}} \\
 & \quad \times \left[\frac{r}{2} C_3(x) \right] \cdot \frac{|f^{(2r-1)}(0)|^q + |f^{(2r-1)}(1)|^q}{2} \\
 & \quad + \left[\frac{r}{2} C_2(x) \right] \cdot \left| f^{(2r-1)} \left(\frac{1}{2} \right) \right|^q \right]^{\frac{1}{q}}.
 \end{aligned}$$

If $n = 2r$, $r \geq 2$, then

$$\begin{aligned}
 (2.5) \quad & \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f) \right| \\
 & \leq \frac{|B_{2r}(x)|^{1-\frac{1}{q}}}{(2r)!} \cdot \left[\left| \frac{1}{2} C_6(x) \right| \frac{|f^{(2r)}(0)|^q + |f^{(2r)}(1)|^q}{2} \right. \\
 & \quad \left. + \left| \frac{1}{2} C_5(x) \right| \left| f^{(2r)} \left(\frac{1}{2} \right) \right|^q \right]^{\frac{1}{q}}
 \end{aligned}$$

and we also have

$$(2.6) \quad \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_r(f) \right|$$



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$$\leq \frac{2|B_{2r}(x)|^{1-\frac{1}{q}}}{(2r)!} \left[\left| \frac{1}{8}C_8(x) \right| \left(|f^{(2r)}(0)|^q + 2 \left| f^{(2r)}\left(\frac{1}{2}\right) \right|^q + |f^{(2r)}(1)|^q \right) \right]^{\frac{1}{q}}.$$

(b) If $|f^{(n)}|^q$ is concave, then for $n = 2r - 1$, $r \geq 2$, we have

$$(2.7) \quad \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f) \right| \\ \leq \frac{1}{(2r)!} \left| \frac{r}{2}C_1(x) \right| \cdot \left[\left| f^{(2r-1)}\left(\frac{|C_2(x)|}{2|C_1(x)|}\right) \right| \right. \\ \left. + \left| f^{(2r-1)}\left(\frac{|C_3(x) + \frac{1}{2}C_2(x)|}{|C_1(x)|}\right) \right| \right].$$

If $n = 2r$, $r \geq 2$, then

$$(2.8) \quad \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f) \right| \\ \leq \frac{|C_4(x)|}{4(2r)!} \left[\left| f^{(2r)}\left(\frac{|C_5(x)|}{2|C_4(x)|}\right) \right| \right. \\ \left. + \left| f^{(2r)}\left(\frac{|C_6(x) + \frac{1}{2}C_5(x)|}{|C_4(x)|}\right) \right| \right].$$

Proof. First, let $n = 2r - 1$ for some $r \geq 2$. Then by Hölder's inequality

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f) \right| \\ & \leq \frac{1}{2(2r-1)!} \int_0^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)| dt \\ & \leq \frac{1}{2(2r-1)!} \left(\int_0^1 |F_{2r-1}^x(t)| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)|^q dt \right)^{\frac{1}{q}} \\ & = \frac{1}{2(2r-1)!} \left(\frac{2}{r} \left| B_{2r} \left(\frac{1}{2} - x \right) - B_{2r}(x) \right| \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Now, by the convexity of $|f^{(2r-1)}|^q$ we have

$$\begin{aligned} & \int_0^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)|^q dt \\ & = \int_0^{\frac{1}{2}} |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)|^q dt + \int_{\frac{1}{2}}^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)|^q dt \\ & = \frac{1}{2} \int_0^1 \left| F_{2r-1}^x \left(\frac{y}{2} \right) \right| \cdot \left| f^{(2r-1)} \left((1-y) \cdot 0 + y \cdot \frac{1}{2} \right) \right|^q dy \\ & \quad + \frac{1}{2} \int_0^1 \left| F_{2r-1}^x \left(1 - \frac{y}{2} \right) \right| \cdot \left| f^{(2r-1)} \left((1-y) \cdot 1 + y \cdot \frac{1}{2} \right) \right|^q dy \end{aligned}$$



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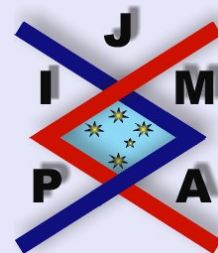


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$$\begin{aligned} &\leq \frac{1}{2} \left[\left| \int_0^1 (1-y) F_{2r-1}^x \left(\frac{y}{2} \right) dy \right| \cdot |f^{(2r-1)}(0)|^q \right. \\ &\quad + \left| \int_0^1 y F_{2r-1}^x \left(\frac{y}{2} \right) dy \right| \cdot \left| f^{(2r-1)} \left(\frac{1}{2} \right) \right|^q \\ &\quad + \left| \int_0^1 (1-y) F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \right| \cdot |f^{(2r-1)}(1)|^q \\ &\quad \left. + \left| \int_0^1 y F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \right| \cdot \left| f^{(2r-1)} \left(\frac{1}{2} \right) \right|^q \right]. \end{aligned}$$

On the other hand, if $|f^{(2r-1)}|^q$ is concave, then

$$\begin{aligned} &\left| \int_0^1 f(t) dt - \frac{1}{2} [f(x) + f(1-x)] + T_{r-1}(f) \right| \\ &\leq \frac{1}{2(2r-1)!} \int_0^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)| dt \\ &= \frac{1}{2(2r-1)!} \left[\int_0^{1/2} |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)| dt + \int_{1/2}^1 |F_{2r-1}^x(t)| \cdot |f^{(2r-1)}(t)| dt \right] \\ &= \frac{1}{2(2r-1)!} \left[\int_0^1 \left| F_{2r-1}^x \left(\frac{y}{2} \right) \right| \cdot \left| f^{(2r-1)} \left((1-y) \cdot 0 + y \cdot \frac{1}{2} \right) \right| dy \right. \\ &\quad \left. + \int_0^1 \left| F_{2r-1}^x \left(1 - \frac{y}{2} \right) \right| \cdot \left| f^{(2r-1)} \left((1-y) \cdot 1 + y \cdot \frac{1}{2} \right) \right| dy \right] \end{aligned}$$

$$\leq \frac{1}{4(2r-1)!} \left[\left| \int_0^1 F_{2r-1}^x \left(\frac{y}{2} \right) dy \right| \right. \\ \times \left| f^{(2r-1)} \left(\frac{\left| \int_0^1 F_{2r-1}^x \left(\frac{y}{2} \right) ((1-y) \cdot 0 + y \cdot \frac{1}{2}) dy \right|}{\left| \int_0^1 F_{2r-1}^x \left(\frac{y}{2} \right) dy \right|} \right) \right| \\ + \left| \int_0^1 F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \right| \\ \times \left. \left| f^{(2r-1)} \left(\frac{\left| \int_0^1 F_{2r-1}^x \left(1 - \frac{y}{2} \right) ((1-y) \cdot 1 + y \cdot \frac{1}{2}) dy \right|}{\left| \int_0^1 F_{2r-1}^x \left(1 - \frac{y}{2} \right) dy \right|} \right) \right| \right],$$

so the inequality (2.4) and (2.7) are completely proved.

The proofs of the inequalities (2.5), (2.8) and (2.6) are similar. \square

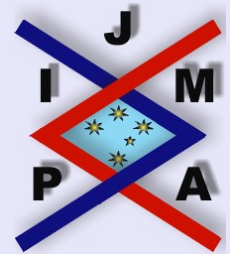
Remark 1. For (2.7) to be satisfied it is enough to suppose that $|f^{(2r-1)}|$ is a concave function. For if $|g|^q$ is concave and $[0, 1]$ for some $q \geq 1$, then for $x, y \in [0, 1]$ and $\lambda \in [0, 1]$

$$|g(\lambda x + (1-\lambda)y)|^q \geq \lambda|g(x)|^q + (1-\lambda)|g(y)|^q \geq (\lambda|g(x)| + (1-\lambda)|g(y)|)^q,$$

by the power-mean inequality. Therefore $|g|$ is also concave on $[0, 1]$.

Remark 2. If in Theorem 2.1 we chose $x = 0, 1/2, 1/3$, we get generalizations of the Dragomir-Agarwal inequality for Euler trapezoid (see [4]), Euler midpoint and Euler two-point Newton-Cotes formulae respectively.

The resultant formulae in Theorem 2.1 when $r = 2$ are of special interest, so we isolate it as corollary.



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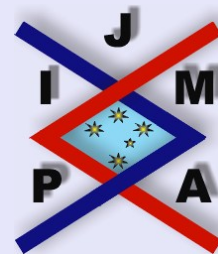
Corollary 2.2. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is 4-times differentiable and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \cup \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.

(a) If $|f^{(3)}|^q$ is convex for some $q \geq 1$, then

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2} [f(x) + f(1-x)] + \frac{1}{12} [f'(1) - f'(0)] \right| \\ & \leq \frac{1}{12} \left| 2x^3 - \frac{3}{2}x^2 + \frac{1}{16} \right|^{1-\frac{1}{q}} \\ & \quad \times \left[\left| x^4 - 2x^3 + x^2 - \frac{1}{30} \right| \frac{|f^{(3)}(0)|^q + |f^{(3)}(1)|^q}{2} \right. \\ & \quad \left. + \left| -x^4 + \frac{x^2}{2} - \frac{7}{240} \right| \left| f^{(3)}\left(\frac{1}{2}\right) \right|^q \right]^{\frac{1}{q}} \end{aligned}$$

and if $|f^{(4)}|^q$ is convex for some $q \geq 1$, then

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2} [f(x) + f(1-x)] + \frac{1}{12} [f'(1) - f'(0)] \right| \\ & \leq \frac{1}{24} \left| x^4 - 2x^3 + x^2 - \frac{1}{30} \right|^{1-\frac{1}{q}} \\ & \quad \times \left[\left| \frac{2x^5}{5} - x^4 + x^3 - \frac{3x^2}{8} + \frac{1}{96} \right| \frac{|f^{(4)}(0)|^q + |f^{(4)}(1)|^q}{2} \right. \end{aligned}$$



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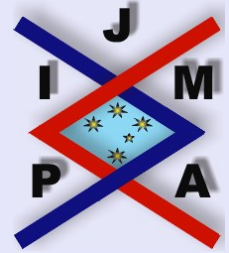
$$+ \left| -\frac{2x^5}{5} + x^3 - \frac{5x^2}{8} + \frac{11}{480} \right| \left| f^{(4)} \left(\frac{1}{2} \right) \right|^q \Bigg|^{\frac{1}{q}}.$$

(b) If $|f^{(3)}|$ is concave, then

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2} [f(x) + f(1-x)] + \frac{1}{12} [f'(1) - f'(0)] \right| \\ & \leq \frac{1}{24} \left| 2x^3 - \frac{3}{2}x^2 + \frac{1}{16} \right| \left[\left| f^{(3)} \left(\frac{-x^4 + \frac{x^2}{2} - \frac{7}{240}}{-4x^3 + 3x^2 - \frac{1}{8}} \right) \right| \right. \\ & \quad \left. + \left| f^{(3)} \left(\frac{\frac{x^4}{2} - 2x^3 + \frac{5x^2}{4} - \frac{23}{480}}{-2x^3 + \frac{3x^2}{2} - \frac{1}{16}} \right) \right| \right] \end{aligned}$$

and if $|f^{(4)}|$ is concave, then

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2} [f(x) + f(1-x)] + \frac{1}{12} [f'(1) - f'(0)] \right| \\ & \leq \frac{1}{48} \left| x^4 - 2x^3 + x^2 - \frac{1}{30} \right| \left[\left| f^{(4)} \left(\frac{-\frac{4x^5}{5} + 2x^3 - \frac{5x^2}{4} + \frac{11}{240}}{-4x^4 + 8x^3 - 4x^2 + \frac{2}{15}} \right) \right| \right. \\ & \quad \left. + \left| f^{(4)} \left(\frac{\frac{2x^5}{5} - 2x^4 + 3x^3 - \frac{11x^2}{8} + \frac{7}{160}}{-2x^4 + 4x^3 - 2x^2 + \frac{1}{15}} \right) \right| \right]. \end{aligned}$$



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Now, we will give some results of the same type in the case when $r = 1$.

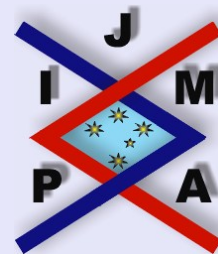
Theorem 2.3. *Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is 2-times differentiable.*

(a) *If $|f'|^q$ is convex for some $q \geq 1$, then for $x \in [0, 1/2]$ we have*

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ & \leq \frac{|8x^2 - 4x + 1|^{1-\frac{1}{q}}}{4} \cdot \left[\left| 2x^2 - 2x + \frac{2}{3} \right| \frac{|f'(0)|^q + |f'(1)|^q}{2} \right. \\ & \quad \left. + \left| -2x^2 + 2x + \frac{1}{3} \right| \left| f' \left(\frac{1}{2} \right) \right|^q \right]^{\frac{1}{q}}. \end{aligned}$$

If $|f''|^q$ is convex for some $q \geq 1$ and $x \in [0, 1/4]$, then

$$\begin{aligned} & \left| \int_0^1 f(t) dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ & \leq \frac{\left| \frac{-6x^2+6x-1}{3} + \frac{2}{3}(1-4x)^{3/2} \right|^{1-\frac{1}{q}}}{4} \left[-x^2 + x - \frac{1}{8} \left| \frac{|f''(0)|^q + |f''(1)|^q}{2} \right. \right. \\ & \quad \left. \left. + \left| -2x^2 + 2x - \frac{5}{24} \right| \left| f'' \left(\frac{1}{2} \right) \right|^q \right]^{\frac{1}{q}}, \end{aligned}$$



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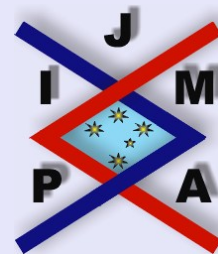


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while for $x \in [1/4, 1/2]$ we have

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ & \leq \frac{\left| \frac{-6x^2+6x-1}{3} \right|^{1-\frac{1}{q}}}{4} \left[\left| -x^2 + x - \frac{1}{8} \right| \frac{|f''(0)|^q + |f''(1)|^q}{2} \right. \\ & \quad \left. + \left| -2x^2 + 2x - \frac{5}{24} \right| \left| f'' \left(\frac{1}{2} \right) \right|^q \right]^{\frac{1}{q}}. \end{aligned}$$

(b) If $|f'|$ is concave for some $q \geq 1$, then for $x \in [0, 1/2]$ we have

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ & \leq \frac{1}{8} \left[\left| f' \left(-x^2 + x + \frac{1}{6} \right) \right| + \left| f' \left(x^2 - x + \frac{5}{6} \right) \right| \right]. \end{aligned}$$

If $|f''|$ is concave for some $q \geq 1$ and $x \in [0, 1/2]$, then

$$\begin{aligned} & \left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ & \leq \frac{1}{8} \left| -3x^2 + 3x - \frac{1}{3} \right| \left[\left| f'' \left(\frac{|-2x^2 + 2x - \frac{5}{24}|}{|-6x^2 + 6x - \frac{2}{3}|} \right) \right| \right. \\ & \quad \left. + \left| f'' \left(\frac{|-2x^2 + 2x - \frac{11}{48}|}{|-3x^2 + 3x - \frac{1}{3}|} \right) \right| \right]. \end{aligned}$$

Proof. It was proved in [6] that for $x \in [0, 1/2]$

$$\int_0^1 |F_1^x(t)| dt = \frac{8x^2 - 4x + 1}{2},$$

for $x \in [0, 1/4]$

$$\int_0^1 |F_2^x(t)| dt = \frac{-6x^2 + 6x - 1}{3} + \frac{2}{3}(1 - 4x)^{3/2},$$

and for $x \in [1/4, 1/2]$

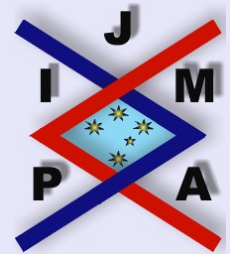
$$\int_0^1 |F_2^x(t)| dt = \frac{-6x^2 + 6x - 1}{3}.$$

So, using identities (2.1) and (2.2) with calculation of $C_1(x), C_2(x), C_3(x), C_4(x), C_5(x)$ and $C_6(x)$ similar to that in Theorem 2.1 we get the inequalities in (a) and (b). \square

Remark 3. For $x = 0$ in the above theorem we have the trapezoid formula and for $|f''|^q$ a convex function and $|f''|$ a concave function we get the results from [4].

If $|f'|^q$ is convex for some $q \geq 1$, then

$$\left| \int_0^1 f(t) dt - \frac{1}{2}[f(0) + f(1)] \right| \leq \frac{1}{4} \left[\frac{|f'(0)|^q + |f'(\frac{1}{2})|^q + |f'(1)|^q}{3} \right]^{\frac{1}{q}}$$



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and if $|f'|$ is concave, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(0) + f(1)] \right| \leq \frac{1}{8} \left[\left| f' \left(\frac{1}{6} \right) \right| + \left| f' \left(\frac{5}{6} \right) \right| \right].$$

For $x = 1/4$ we get two-point Maclaurin formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

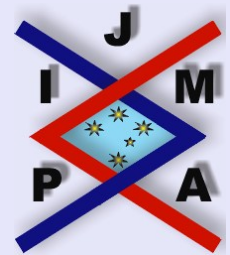
$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f \left(\frac{1}{4} \right) + f \left(\frac{3}{4} \right) \right] \right| \leq \frac{1}{8} \left[\frac{7|f'(0)|^q + 34|f' \left(\frac{1}{2} \right)|^q + 7|f'(1)|^q}{24} \right]^{\frac{1}{q}}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f \left(\frac{1}{4} \right) + f \left(\frac{3}{4} \right) \right] \right| \leq \frac{1}{96} \left[\frac{3|f''(0)|^q + 16|f'' \left(\frac{1}{2} \right)|^q + 3|f''(1)|^q}{4} \right]^{\frac{1}{q}}.$$

If $|f'|$ is concave, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f \left(\frac{1}{4} \right) + f \left(\frac{3}{4} \right) \right] \right| \leq \frac{1}{8} \left[\left| f' \left(\frac{17}{48} \right) \right| + \left| f' \left(\frac{31}{48} \right) \right| \right]$$



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and if $|f''|$ is concave for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) \right] \right| \leq \frac{11}{384} \left[\left| f''\left(\frac{4}{11}\right) \right| + \left| f''\left(\frac{7}{11}\right) \right| \right].$$

For $x = 1/3$ we get two-point Newton-Cotes formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] \right| \leq \frac{5}{36} \left[\frac{|f'(0)|^q + 7|f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{5} \right]^{\frac{1}{q}}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

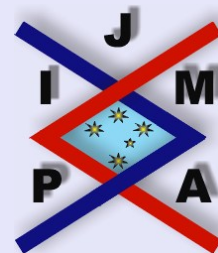
$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] \right| \leq \frac{1}{36} \left[\frac{7|f''(0)|^q + 34|f''\left(\frac{1}{2}\right)|^q + 7|f''(1)|^q}{16} \right]^{\frac{1}{q}}.$$

If $|f'|$ is concave, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] \right| \leq \frac{1}{8} \left[\left| f'\left(\frac{7}{18}\right) \right| + \left| f'\left(\frac{11}{18}\right) \right| \right]$$

and if $|f''|$ is concave for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2} \left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] \right| \leq \frac{1}{24} \left[\left| f''\left(\frac{17}{48}\right) \right| + \left| f''\left(\frac{31}{48}\right) \right| \right].$$



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For $x = 1/2$ we get midpoint formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \right| \leq \frac{1}{4} \left[\frac{|f'(0)|^q + 10|f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{12} \right]^{\frac{1}{q}}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

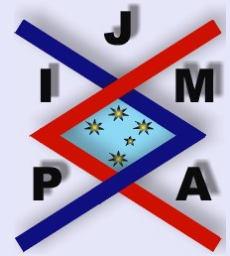
$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \right| \leq \frac{1}{24} \left[\frac{3|f''(0)|^q + 14|f''\left(\frac{1}{2}\right)|^q + 3|f''(1)|^q}{8} \right]^{\frac{1}{q}}.$$

If $|f'|$ is concave, then

$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \right| \leq \frac{1}{8} \left[\left| f'\left(\frac{5}{12}\right) \right| + \left| f'\left(\frac{7}{12}\right) \right| \right]$$

and if $|f''|$ is concave for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \right| \leq \frac{5}{96} \left[\left| f''\left(\frac{7}{20}\right) \right| + \left| f''\left(\frac{13}{20}\right) \right| \right].$$



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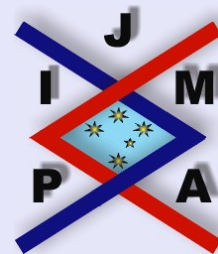
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