Journal of Inequalities in Pure and Applied Mathematics

NOTE ON DRAGOMIR-AGARWAL INEQUALITIES, THE GENERAL EULER TWO-POINT FORMULAE AND CONVEX FUNCTIONS

A. VUKELIĆ

Faculty of Food Technology and Biotechnology
Mathematics Department
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia

EMail: avukelic@pbf.hr

volume 6, issue 1, article 19, 2005.

Received 12 January, 2005; accepted 20 January, 2005.

Communicated by: R.P. Agarwal

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
016-05

Abstract

The general Euler two-point formulae are used with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration.

2000 Mathematics Subject Classification: 26D15, 26D20, 26D99
Key words: Hadamard inequality, r-convexity, Integral inequalities, General Euler two-point formulae.

Contents

1. Introduction

One of the cornerstones of nonlinear analysis is the Hadamard inequality, which states that if $[a,b]$ $(a<b)$ is a real interval and $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Recently, S.S. Dragomir and R.P. Agarwal [3] considered the trapezoid formula for numerical integration of functions f such that $|f'|^q$ is a convex function for some $q\ge 1$. Their approach was based on estimating the difference between the two sides of the right-hand inequality in (1.1). Improvements of their results were obtained in [5]. In particular, the following tool was established.

Suppose $f:I^0\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable on I^0 and such that $|f'|^q$ is convex on $[a,b]$ for some $q\ge 1$, where $a,b\in I^0(a<b)$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.2}$$

Some generalizations to higher-order convexity and applications of these results are given in [1]. Related results for Euler midpoint, Euler-Simpson, Euler two-point, dual Euler-Simpson, Euler-Simpson $3/8$ and Euler-Maclaurin formulae were considered in [7] and for Euler two-point formulae in [9] (see also [2] and [8]).

In the paper [4] Dah-Yan Hwang procured some new inequalities of this type and he applied the result to obtain a better estimate of the error in the trapezoidal formula.

In this paper we consider some related results using the general Euler two-point formulae. We will use the interval $[0, 1]$ because of simplicity and since it involves no loss in generality.

2. The General Euler Two-point Formulae

In the recent paper [6] the following identities, named the general Euler two-point formulae, have been proved. Let $f \in C^n([0,1],\mathbb{R})$ for some $n \geq 3$ and let $x \in [0,1/2]$. If $n = 2r-1,\ r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f(x) + f(1-x)\right] - T_{r-1}(f) + \frac{1}{2(2r-1)!}\int_0^1 f^{(2r-1)}(t)F^x_{2r-1}(t)dt, \tag{2.1}$$

while for $n = 2r,\ r \geq 2$ we have

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f(x) + f(1-x)\right] - T_{r-1}(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)F^x_{2r}(t)dt \tag{2.2}$$

and

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f(x) + f(1-x)\right] - T_{r}(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)G^x_{2r}(t)dt. \tag{2.3}$$

Here we define $T_0(f) = 0$ and for $1 \leq m \leq \lfloor n/2 \rfloor$

$$T_m(f) = \sum_{k=1}^{m} \frac{B_{2k}(x)}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right],$$

$$G_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t)$$

and

$$F_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t) - B_n(x) - B_n(1-x),$$

where $B_k(\cdot), k \geq 0$, is the k-th Bernoulli polynomial and $B_k = B_k(0) = B_k(1) (k \geq 0)$ the k-th Bernoulli number. By $B_k^*(\cdot) (k \geq 0)$ we denote the function of period one such that $B_k^*(x) = B_k(x)$ for $0 \leq x \leq 1$.

It was proved in [6] that $F_n^x(1-t) = (-1)^n F_n^x(t)$, $(-1)^{r-1} F_{2r-1}^x(t) \geq 0$, $(-1)^r F_{2r}^x(t) \geq 0$ for $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right)$ and $t \in [0, 1/2]$, and $(-1)^r F_{2r-1}^x(t) \geq 0$, $(-1)^{r-1} F_{2r}^x(t) \geq 0$ for $x \in \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ and $t \in [0, 1/2]$. Also

$$\int_0^1 \left|F_{2r-1}^x(t)\right| dt = \frac{2}{r} \left| B_{2r}\left(\frac{1}{2} - x\right) - B_{2r}(x) \right|,$$

$$\int_0^1 \left|F_{2r}^x(t)\right| dt = 2|B_{2r}(x)|$$

and

$$\int_0^1 \left|G_{2r}^x(t)\right| dt \leq 4|B_{2r}(x)|.$$

With integration by parts, we have that the following identities hold:

$$C_1(x) = \int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right) dy \tag{1}$$

$$= -\int_0^1 F_{2r-1}^x\left(1 - \frac{y}{2}\right) dy = \frac{2}{r}\left[B_{2r}(x) - B_{2r}\left(\frac{1}{2} - x\right)\right],$$

$$(2)\qquad C_2(x) = \int_0^1 yF_{2r-1}^x\left(\frac{y}{2}\right)dy$$
$$= -\int_0^1 yF_{2r-1}^x\left(1-\frac{y}{2}\right)dy$$
$$= -\frac{2}{r}B_{2r}\left(\frac{1}{2}-x\right),$$

$$(3)\qquad C_3(x) = \int_0^1 (1-y)F_{2r-1}^x\left(\frac{y}{2}\right)dy$$
$$= -\int_0^1 (1-y)F_{2r-1}^x\left(1-\frac{y}{2}\right)dy$$
$$= \frac{2}{r}B_{2r}(x),$$

$$(4)\qquad C_4(x) = \int_0^1 F_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 F_{2r}^x\left(1-\frac{y}{2}\right)dy = -2B_{2r}(x),$$

$$(5)\qquad C_5(x) = \int_0^1 yF_{2r}^x\left(\frac{y}{2}\right)dy$$
$$= \int_0^1 yF_{2r}^x\left(1-\frac{y}{2}\right)dy$$
$$= \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}(x) - B_{2r+2}\left(\frac{1}{2}-x\right)\right] - B_{2r}(x),$$

(6)
$$C_6(x) = \int_0^1 (1-y) F_{2r}^x\left(\frac{y}{2}\right) dy$$
$$= \int_0^1 (1-y) F_{2r}^x\left(1-\frac{y}{2}\right) dy$$
$$= \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}\left(\frac{1}{2}-x\right) - B_{2r+2}(x)\right] - B_{2r}(x),$$

(7)
$$C_7(x) = \int_0^1 G_{2r}^x\left(\frac{y}{2}\right) dy = \int_0^1 G_{2r}^x\left(1-\frac{y}{2}\right) dy = 0,$$

(8)
$$C_8(x) = \int_0^1 y G_{2r}^x\left(\frac{y}{2}\right) dy$$
$$= \int_0^1 y G_{2r}^x\left(1-\frac{y}{2}\right) dy$$
$$= -\int_0^1 (1-y) G_{2r}^x\left(\frac{y}{2}\right) dy$$
$$= \int_0^1 (1-y) G_{2r}^x\left(1-\frac{y}{2}\right) dy$$
$$= \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}(x) - B_{2r+2}\left(\frac{1}{2}-x\right)\right],$$

Theorem 2.1. *Suppose $f : [0,1] \to \mathbb{R}$ is n-times differentiable and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \cup \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.*

(a) If $\left|f^{(n)}\right|^q$ *is convex for some* $q \geq 1$*, then for* $n = 2r-1,\ r \geq 2$*, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \leq \frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|^{1-\frac{1}{q}} \times \left[\left|\frac{r}{2}C_3(x)\right| \cdot \frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2} + \left|\frac{r}{2}C_2(x)\right| \cdot \left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q\right]^{\frac{1}{q}}. \tag{2.4}$$

If $n = 2r,\ r \geq 2$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \leq \frac{|B_{2r}(x)|^{1-\frac{1}{q}}}{(2r)!} \cdot \left[\left|\frac{1}{2}C_6(x)\right| \frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2} + \left|\frac{1}{2}C_5(x)\right| \left|f^{(2r)}\left(\frac{1}{2}\right)\right|^q\right]^{\frac{1}{q}} \tag{2.5}$$

and we also have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_r(f)\right| \tag{2.6}$$

$$\leq \frac{2|B_{2r}(x)|^{1-\frac{1}{q}}}{(2r)!}\left[\left|\frac{1}{8}C_8(x)\right|\left(\left|f^{(2r)}(0)\right|^q+2\left|f^{(2r)}\left(\frac{1}{2}\right)\right|^q+\left|f^{(2r)}(1)\right|^q\right)\right]^{\frac{1}{q}}.$$

(b) If $\left|f^{(n)}\right|^q$ is concave, then for $n=2r-1,\ r\geq 2$, we have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right| \leq \frac{1}{(2r)!}\left|\frac{r}{2}C_1(x)\right|\cdot\left[\left|f^{(2r-1)}\left(\frac{|C_2(x)|}{2\,|C_1(x)|}\right)\right| + \left|f^{(2r-1)}\left(\frac{\left|C_3(x)+\frac{1}{2}C_2(x)\right|}{|C_1(x)|}\right)\right|\right]. \tag{2.7}$$

If $n=2r,\ r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right| \leq \frac{|C_4(x)|}{4(2r)!}\left[\left|f^{(2r)}\left(\frac{|C_5(x)|}{2|C_4(x)|}\right)\right| + \left|f^{(2r)}\left(\frac{\left|C_6(x)+\frac{1}{2}C_5(x)\right|}{|C_4(x)|}\right)\right|\right]. \tag{2.8}$$

Proof. First, let $n = 2r - 1$ for some $r \geq 2$. Then by Hölder's inequality

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] + T_{r-1}(f)\right|$$
$$\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right| dt$$
$$\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{2(2r-1)!}\left(\frac{2}{r}\left|B_{2r}\left(\frac{1}{2} - x\right) - B_{2r}(x)\right|\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right|^q dt\right)^{\frac{1}{q}}.$$

Now, by the convexity of $|f^{(2r-1)}|^q$ we have

$$\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right|^q dt$$
$$= \int_0^{\frac{1}{2}} \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right|^q dt + \int_{\frac{1}{2}}^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right|^q dt$$
$$= \frac{1}{2}\int_0^1 \left|F_{2r-1}^x\left(\frac{y}{2}\right)\right| \cdot \left|f^{(2r-1)}\left((1-y)\cdot 0 + y \cdot \frac{1}{2}\right)\right|^q dy$$
$$+ \frac{1}{2}\int_0^1 \left|F_{2r-1}^x\left(1 - \frac{y}{2}\right)\right| \cdot \left|f^{(2r-1)}\left((1-y)\cdot 1 + y \cdot \frac{1}{2}\right)\right|^q dy$$

$$\leq \frac{1}{2}\left[\left|\int_0^1 (1-y)F_{2r-1}^x\left(\frac{y}{2}\right)dy\right| \cdot \left|f^{(2r-1)}(0)\right|^q \right.$$
$$+\left|\int_0^1 yF_{2r-1}^x\left(\frac{y}{2}\right)dy\right| \cdot \left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q$$
$$+\left|\int_0^1 (1-y)F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right| \cdot \left|f^{(2r-1)}(1)\right|^q$$
$$\left. +\left|\int_0^1 yF_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right| \cdot \left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q\right].$$

On the other hand, if $\left|f^{(2r-1)}\right|^q$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right|$$
$$\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right| dt$$
$$= \frac{1}{2(2r-1)!}\left[\int_0^{1/2} \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right| dt + \int_{1/2}^1 \left|F_{2r-1}^x(t)\right| \cdot \left|f^{(2r-1)}(t)\right| dt\right]$$
$$= \frac{1}{2(2r-1)!}\left[\int_0^1 \left|F_{2r-1}^x\left(\frac{y}{2}\right)\right| \cdot \left|f^{(2r-1)}\left((1-y)\cdot 0 + y\cdot\frac{1}{2}\right)\right| dy \right.$$
$$\left. + \int_0^1 \left|F_{2r-1}^x\left(1-\frac{y}{2}\right)\right| \cdot \left|f^{(2r-1)}\left((1-y)\cdot 1 + y\cdot\frac{1}{2}\right)\right| dy\right]$$

$$\leq \frac{1}{4(2r-1)!}\left[\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)dy\right|\right.$$
$$\times\left|f^{(2r-1)}\left(\frac{\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)((1-y)\cdot 0+y\cdot\frac{1}{2})dy\right|}{\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)dy\right|}\right)\right|$$
$$+\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|$$
$$\left.\times\left|f^{(2r-1)}\left(\frac{\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)((1-y)\cdot 1+y\cdot\frac{1}{2})dy\right|}{\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|}\right)\right|\right],$$

so the inequality (2.4) and (2.7) are completely proved.

The proofs of the inequalities (2.5), (2.8) and (2.6) are similar. □

Remark 1. *For (2.7) to be satisfied it is enough to suppose that* $|f^{(2r-1)}|$ *is a concave function. For if* $|g|^q$ *is concave and* $[0,1]$ *for some* $q \geq 1$, *then for* $x, y \in [0,1]$ *and* $\lambda \in [0,1]$

$$|g(\lambda x+(1-\lambda)y)|^q \geq \lambda|g(x)|^q+(1-\lambda)|g(y)|^q \geq (\lambda|g(x)|+(1-\lambda)|g(y)|)^q,$$

by the power-mean inequality. Therefore $|g|$ *is also concave on* $[0,1]$.

Remark 2. *If in Theorem 2.1 we chose* $x = 0, 1/2, 1/3$, *we get generalizations of the Dragomir-Agarwal inequality for Euler trapezoid (see [4]) , Euler midpoint and Euler two-point Newton-Cotes formulae respectively.*

The resultant formulae in Theorem 2.1 when $r = 2$ are of special interest, so we isolate it as corollary.

Corollary 2.2. *Suppose $f:[0,1]\to\mathbb{R}$ is 4-times differentiable and $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)\cup\left(\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$.*

(a) If $\left|f^{(3)}\right|^q$ is convex for some $q\geq 1$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(x\right)+f\left(1-x\right)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|$$
$$\leq\frac{1}{12}\left|2x^3-\frac{3}{2}x^2+\frac{1}{16}\right|^{1-\frac{1}{q}}$$
$$\times\left[\left|x^4-2x^3+x^2-\frac{1}{30}\right|\frac{\left|f^{(3)}(0)\right|^q+\left|f^{(3)}(1)\right|^q}{2}\right.$$
$$\left.+\left|-x^4+\frac{x^2}{2}-\frac{7}{240}\right|\left|f^{(3)}\left(\frac{1}{2}\right)\right|^q\right]^{\frac{1}{q}}$$

and if $\left|f^{(4)}\right|^q$ is convex for some $q\geq 1$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(x\right)+f\left(1-x\right)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|$$
$$\leq\frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|^{1-\frac{1}{q}}$$
$$\times\left[\left|\frac{2x^5}{5}-x^4+x^3-\frac{3x^2}{8}+\frac{1}{96}\right|\frac{\left|f^{(4)}(0)\right|^q+\left|f^{(4)}(1)\right|^q}{2}\right.$$

$$+\left|-\frac{2x^5}{5}+x^3-\frac{5x^2}{8}+\frac{11}{480}\right|\left|f^{(4)}\left(\frac{1}{2}\right)\right|^q\Bigg]^{\frac{1}{q}}.$$

(b) If $\left|f^{(3)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|$$
$$\leq\frac{1}{24}\left|2x^3-\frac{3}{2}x^2+\frac{1}{16}\right|\left[\left|f^{(3)}\left(\frac{\left|-x^4+\frac{x^2}{2}-\frac{7}{240}\right|}{\left|-4x^3+3x^2-\frac{1}{8}\right|}\right)\right|\right.$$
$$\left.+\left|f^{(3)}\left(\frac{\left|\frac{x^4}{2}-2x^3+\frac{5x^2}{4}-\frac{23}{480}\right|}{\left|-2x^3+\frac{3x^2}{2}-\frac{1}{16}\right|}\right)\right|\right]$$

and if $\left|f^{(4)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|$$
$$\leq\frac{1}{48}\left|x^4-2x^3+x^2-\frac{1}{30}\right|\left[\left|f^{(4)}\left(\frac{\left|-\frac{4x^5}{5}+2x^3-\frac{5x^2}{4}+\frac{11}{240}\right|}{\left|-4x^4+8x^3-4x^2+\frac{2}{15}\right|}\right)\right|\right.$$
$$\left.+\left|f^{(4)}\left(\frac{\left|\frac{2x^5}{5}-2x^4+3x^3-\frac{11x^2}{8}+\frac{7}{160}\right|}{\left|-2x^4+4x^3-2x^2+\frac{1}{15}\right|}\right)\right|\right].$$

Now, we will give some results of the same type in the case when $r = 1$.

Theorem 2.3. *Suppose $f : [0,1] \to \mathbb{R}$ is 2-times differentiable.*

(a) If $|f'|^q$ is convex for some $q \geq 1$, then for $x \in [0, 1/2]$ we have

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \frac{|8x^2 - 4x + 1|^{1-\frac{1}{q}}}{4} \cdot \left[\left| 2x^2 - 2x + \frac{2}{3} \right| \frac{|f'(0)|^q + |f'(1)|^q}{2} + \left| -2x^2 + 2x + \frac{1}{3} \right| \left| f'\left(\frac{1}{2}\right) \right|^q \right]^{\frac{1}{q}}.$$

If $|f''|^q$ is convex for some $q \geq 1$ and $x \in [0, 1/4]$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \frac{\left| \frac{-6x^2+6x-1}{3} + \frac{2}{3}(1-4x)^{3/2} \right|^{1-\frac{1}{q}}}{4} \left[\left| -x^2 + x - \frac{1}{8} \right| \frac{|f''(0)|^q + |f''(1)|^q}{2} + \left| -2x^2 + 2x - \frac{5}{24} \right| \left| f''\left(\frac{1}{2}\right) \right|^q \right]^{\frac{1}{q}},$$

while for $x \in [1/4, 1/2]$ we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right|$$
$$\le \frac{\left|\frac{-6x^2+6x-1}{3}\right|^{1-\frac{1}{q}}}{4}\left[\left|-x^2+x-\frac{1}{8}\right|\frac{|f''(0)|^q+|f''(1)|^q}{2} + \left|-2x^2+2x-\frac{5}{24}\right|\left|f''\left(\frac{1}{2}\right)\right|^q\right]^{\frac{1}{q}}.$$

(b) If $|f'|$ is concave for some $q \ge 1$, then for $x \in [0, 1/2]$ we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right|$$
$$\le \frac{1}{8}\left[\left|f'\left(\left|-x^2+x+\frac{1}{6}\right|\right)\right| + \left|f'\left(\left|x^2-x+\frac{5}{6}\right|\right)\right|\right].$$

If $|f''|$ is concave for some $q \ge 1$ and $x \in [0, 1/2]$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)]\right|$$
$$\le \frac{1}{8}\left|-3x^2+3x-\frac{1}{3}\right|\left[\left|f''\left(\frac{\left|-2x^2+2x-\frac{5}{24}\right|}{\left|-6x^2+6x-\frac{2}{3}\right|}\right)\right| + \left|f''\left(\frac{\left|-2x^2+2x-\frac{11}{48}\right|}{\left|-3x^2+3x-\frac{1}{3}\right|}\right)\right|\right].$$

Proof. It was proved in [6] that for $x \in [0, 1/2]$

$$\int_0^1 |F_1^x(t)|dt = \frac{8x^2 - 4x + 1}{2},$$

for $x \in [0, 1/4]$

$$\int_0^1 |F_2^x(t)|dt = \frac{-6x^2 + 6x - 1}{3} + \frac{2}{3}(1 - 4x)^{3/2},$$

and for $x \in [1/4, 1/2]$

$$\int_0^1 |F_2^x(t)|dt = \frac{-6x^2 + 6x - 1}{3}.$$

So, using identities (2.1) and (2.2) with calculation of $C_1(x), C_2(x), C_3(x)$, $C_4(x)$, $C_5(x)$ and $C_6(x)$ similar to that in Theorem 2.1 we get the inequalities in (a) and (b). □

Remark 3. *For $x = 0$ in the above theorem we have the trapezoid formula and for $|f''|^q$ a convex function and $|f''|$ a concave function we get the results from [4].*
If $|f'|^q$ is convex for some $q \geq 1$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(0) + f(1)] \right| \leq \frac{1}{4} \left[\frac{|f'(0)|^q + |f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{3} \right]^{\frac{1}{q}}$$

and if $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]\right| \le \frac{1}{8}\left[\left|f'\left(\frac{1}{6}\right)\right| + \left|f'\left(\frac{5}{6}\right)\right|\right].$$

For $x = 1/4$ we get two-point Maclaurin formula and then if $|f'|^q$ is convex for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \le \frac{1}{8}\left[\frac{7|f'(0)|^q + 34\left|f'\left(\frac{1}{2}\right)\right|^q + 7|f'(1)|^q}{24}\right]^{\frac{1}{q}}$$

and if $|f''|^q$ is convex for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \le \frac{1}{96}\left[\frac{3|f''(0)|^q + 16\left|f''\left(\frac{1}{2}\right)\right|^q + 3|f''(1)|^q}{4}\right]^{\frac{1}{q}}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \le \frac{1}{8}\left[\left|f'\left(\frac{17}{48}\right)\right| + \left|f'\left(\frac{31}{48}\right)\right|\right]$$

and if $|f''|$ *is concave for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{11}{384}\left[\left|f''\left(\frac{4}{11}\right)\right| + \left|f''\left(\frac{7}{11}\right)\right|\right].$$

For $x = 1/3$ *we get two-point Newton-Cotes formula and then if* $|f'|^q$ *is convex for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{5}{36}\left[\frac{|f'(0)|^q + 7\left|f'\left(\frac{1}{2}\right)\right|^q + |f'(1)|^q}{5}\right]^{\frac{1}{q}}$$

and if $|f''|^q$ *is convex for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{1}{36}\left[\frac{7|f''(0)|^q + 34\left|f''\left(\frac{1}{2}\right)\right|^q + 7|f''(1)|^q}{16}\right]^{\frac{1}{q}}.$$

If $|f'|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{1}{8}\left[\left|f'\left(\frac{7}{18}\right)\right| + \left|f'\left(\frac{11}{18}\right)\right|\right]$$

and if $|f''|$ *is concave for some* $q \geq 1$*, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{1}{24}\left[\left|f''\left(\frac{17}{48}\right)\right| + \left|f''\left(\frac{31}{48}\right)\right|\right].$$

For $x = 1/2$ we get midpoint formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{1}{4}\left[\frac{|f'(0)|^q + 10|f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{12}\right]^{\frac{1}{q}}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{1}{24}\left[\frac{3|f''(0)|^q + 14|f''\left(\frac{1}{2}\right)|^q + 3|f''(1)|^q}{8}\right]^{\frac{1}{q}}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{1}{8}\left[\left|f'\left(\frac{5}{12}\right)\right| + \left|f'\left(\frac{7}{12}\right)\right|\right]$$

and if $|f''|$ is concave for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{5}{96}\left[\left|f''\left(\frac{7}{20}\right)\right| + \left|f''\left(\frac{13}{20}\right)\right|\right].$$

References

[1] Lj. DEDIĆ, C.E.M. PEARCE AND J. PEČARIĆ, Hadamard and Dragomir-Agarwal inequalities, higher-order convexity and the Euler formula, *J. Korean Math. Soc.*, **38** (2001), 1235–1243.

[2] Lj. DEDIĆ, C.E.M. PEARCE AND J. PEČARIĆ, The Euler formulae and convex functions, *Math. Inequal. Appl.*, **3**(2) (2000), 211–221.

[3] S.S. DRAGOMIR AND R.P. AGARWAL, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Mat. Lett.*, **11**(5) (1998), 91–95.

[4] D.Y. HWANG, Improvements of Euler-trapezodial type inequalities with higher-order convexity and applications, *J. of Inequal. in Pure and Appl. Math.*, **5**(3) (2004), Art. 66. [ONLINE: http://jipam.vu.edu.au/article.php?sid=412]

[5] C.E.M. PEARCE AND J. PEČARIĆ, Inequalities for differentiable mappings and applications to special means and quadrature formulas, *Appl. Math. Lett.*, **13** (2000), 51–55.

[6] J. PEČARIĆ, I. PERIĆ AND A. VUKELIĆ, On general Euler two-point formulae, to appear in *ANZIAM J.*

[7] J. PEČARIĆ AND A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions, *Functional Equations, Inequalities and Applications*, Th.M. Rassias (ed.)., Dordrecht, Kluwer Academic Publishers, 2003.

[8] J. PEČARIĆ AND A. VUKELIĆ, On generalizations of Dragomir-Agarwal inequality via some Euler-type identities, *Bulletin de la Société des Mathématiciens de R. Macédoine*, **26 (LII)** (2002), 17–42.

[9] J. PEČARIĆ AND A. VUKELIĆ, Hadamard and Dragomir-Agrawal inequalities, the general Euler two-point formulae and convex function, to appear in *Rad HAZU, Matematičke Znanosti.*