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ON THE GENERALIZED STRONGLY NONLINEAR IMPLICIT QUASIVARIATIONAL INEQUALITIES FOR SET-VALUED MAPPINGS

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ABSTRACT. In this paper, we introduce and study a new class of generalized strongly nonlinear implicit quasivariational inequalities for set-valued mappings and construct some new iterative algorithms for these kinds of generalized strongly nonlinear implicit quasivariational inequalities by using the projection method and Nadler's theorem. We prove some existence theorems of solutions for these kinds of generalized nonlinear strongly implicit quasivariational inequalities for set-valued mappings without compactness and convergence theorems of iterative sequences generated by the algorithms.

Key words and phrases: Nonlinear implicit quasivariational inequality, set-valued mapping, projection algorithm, Hausdorff metric, Hilbert space.

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1. INTRODUCTION

It is well known that variational inequality theory and complementarity problem theory are very powerful tools of current mathematical technology. In recent years, the classical variational inequality and complementarity problem have been extended and generalized in several directions to study a wide class of problems arising in mechanics, physics, optimization and control theory, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. For details, we refer the reader to [1] – [13], [15] – [27] and the references therein.

In 1991, Chang and Huang [4, 5] introduced and studied some new classes of quasi-(implicit) complementarity problems and quasi-(implicit) variational inequalities for set-valued mappings with compact values in Hilbert spaces, which included many kinds of complementarity problems and variational inequalities as special cases. In 1997, Huang [8] introduced and studied

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a new class of generalized nonlinear variational inequalities for set-valued mappings with noncompact values and constructed some new iterative algorithms for this class of generalized nonlinear variational inequalities. For the some recent results, see [3, 10, 20, 26] and the references therein.

Recently, Zeng [27] introduced and studied a class of general strongly quasi-variational inequalities for single-valued mappings which extends the general auxiliary variational inequality considered by Noor [18].

In this paper, we introduce and study a new class of generalized strongly nonlinear implicit quasivariational inequalities for set-valued mappings and construct some new iterative algorithms for this kind of generalized strongly nonlinear implicit quasivariational inequalities by using the projection method and Nadler's theorem [14]. We also show the existence of solutions for this class of generalized strongly nonlinear implicit quasivariational inequalities for set-valued mappings without compactness and the convergence of iterative sequences generated by the algorithms. Our results extend and improve the earlier and recent results of Noor [18], Stampacchia [24] and Zeng [27].

2. PRELIMINARIES

Let H be a real Hilbert space endowed with the norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$. Let K be a nonempty closed convex subset of H, P_K be the projection of H onto K and f be a linear continuous function on H.

Given single-valued mappings $g, T : H \to H$ and $N : H \times H \to H$ and set-valued mappings $F, G, S, K : H \to 2^H$, we consider the following problem:

Find $u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ such that $g(u) \in K(u)$ and

$$0 \geq \langle N(y, g(z)), v - g(u)\rangle - \rho\langle T(x) - f, v - g(u)\rangle \tag{2.1}$$

for all $v \in K(u)$, where $\rho > 0$ is a constant. The problem (2.1) is called the generalized strongly nonlinear implicit quasivariational inequality for set-valued mappings.

Example 2.1. To illustrate the applications and importance of the nonlinear implicit quasivariational inequality (2.1), we consider a elastoplasticity problem, which is mainly due to Panagiotopoulos and Stavroulakis [21]. For simplicity, it is assumed that a general hyperelastic material law holds for the elastic behaviour of the elastoplastic material under consideration. Moreover, a nonconvex yield function $\sigma \to F(\sigma)$ is introduced for the plasticity. For the basic definitions and concepts, see [21]. Let us assume the decomposition

$$E = E^e + E^p, \tag{2.2}$$

where E^e denotes the elastic and E^p the plastic deformation of the three-dimensional elastoplastic body. We write the complementary virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma\rangle + \langle E^p, \tau - \sigma\rangle = \langle f, \tau - \sigma\rangle \tag{2.3}$$

for all $\tau \in Z$. Here, we have assumed that the body on a part Γ_U of its boundary has given displacements, that is, $\mu_i = U_i$ on Γ_U, and that on the rest of its boundary $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on Γ_F, where

$$\langle E, \sigma\rangle = \int_\Omega \varepsilon_{ij}\sigma_{ij}\, d\Omega, \tag{2.4}$$

$$\langle f, \sigma\rangle = \int_{\Gamma_U} U_i S_i\, d\Gamma \qquad \text{and} \tag{2.5}$$

(2.6) $$Z = \{\tau : \tau_{i_j,j} + f_i = 0 \text{ on } \Omega,\ \ i,j = 1,2,3,\ \ T_i = F_i \text{ on } \Gamma_F,\ \ i = 1,2,3\}$$

is the set of statically admissible stresses and Ω is the structure of the body.

Let us assume that the material of the structure Ω is hyperelastic such that

(2.7) $$\langle E^e, \tau - \sigma\rangle \le \langle W_m'(\sigma), \tau - \sigma\rangle$$

for all $\tau \in \mathbb{R}^6$, where W_m is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasidifferentiable [21], that is, there exist convex and compact subsets $\underline{\partial}' W_m$ and $\overline{\partial}' W_m$ such that

(2.8) $$\langle W_m'(\sigma), \tau - \sigma\rangle = \max_{W_1^e \in \underline{\partial}' W_m} \langle W_1^e, \tau - \sigma\rangle + \min_{W_2^e \in \overline{\partial}' W_m} \langle W_2^e, \tau - \sigma\rangle.$$

We also introduce the generally nonconvex yield function $P \subset Z$, which is defined by means of the general quasidifferentiable function $F(\sigma)$, that is,

(2.9) $$P = \{\sigma \in Z : F(\sigma) \le 0\}.$$

Here W_m is a generally nonconvex and nonsmooth, but quasidifferentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining (2.2) – (2.9), Panagiotopoulos and Stavroulakis [21] have obtained the following multivalued variational inequality problem:

Find $\sigma \in P$ such that $W_1^e \in \underline{\partial}' W_m(\sigma)$, $W_2^e \in \overline{\partial}' W_m(\sigma)$ and

(2.10) $$\langle W_1^e + W_2^e, \tau - \sigma\rangle \ge \langle f, \tau - \sigma\rangle$$

for all $\tau \in P$, which is exactly the problem (2.1) with $u = \sigma$, $x = W_1^e$, $y = -W_2^e$, $S = T = g = I$, $\rho = 1$, $N(s,t) = s$ for all $s, t \in H$ and

$$F(u) = \underline{\partial}' W_m(\sigma), \quad G(u) = -\overline{\partial}' W_m(\sigma), \quad K(u) = P.$$

In a similar way, one can show that many problems in structural engineering can be studied in the general framework of the set-valued variational inequalities (2.1) following the ideas and techniques of quasidifferentiability (see [21]).

Special Cases of the problem (2.1):

(I) If $A, B : H \to H$ are both single-valued mappings and $N(s,t) = Bs - At$ for all $s, t \in H$, then the problem (2.1) is equivalent to finding $u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ such that $g(u) \in K(u)$ and

(2.11) $$\langle A(g(z)), v - g(u)\rangle \ge \langle B(y), v - g(u)\rangle - \rho\langle T(x) - f, v - g(u)\rangle$$

for all $v \in K(u)$, where $\rho > 0$ is a constant.

(II) If S is the identity mapping, then the problem (2.1) is equivalent to finding $u \in H$, $x \in Fu$ and $y \in Gu$ such that $g(u) \in K(u)$ and

(2.12) $$0 \ge \langle N(y, g(u)), v - g(u)\rangle - \rho\langle T(x) - f, v - g(u)\rangle$$

for all $v \in K(u)$, where $\rho > 0$ is a constant.

(III) If F and S are both the identity mappings, then the problem (2.1) is equivalent to finding $u \in H$ and $y \in Gu$ such that $g(u) \in K(u)$ and

(2.13) $$0 \ge \langle N(y, g(u)), v - g(u)\rangle - \rho\langle T(u) - f, v - g(u)\rangle$$

for all $v \in K(u)$, where $\rho > 0$ is a constant.

(IV) If F, G and S are all the identity mappings, then the problem (2.1) is equivalent to finding $u \in H$ such that $g(u) \in K(u)$ and

$$0 \geq \langle N(u, g(u)), v - g(u)\rangle - \rho\langle T(u) - f, v - g(u)\rangle \tag{2.14}$$

for all $v \in K(u)$, where $\rho > 0$ is a constant.

(V) If F, G and S are the identity mappings, $N(u, v) = Au - Av$ for all $u, v \in H$ and $K(u) = m(u) + K$, then the problem (2.1) is equivalent to finding $u \in H$ such that $g(u) \in K(u)$ and

$$\langle A(g(u)), v - g(u)\rangle \geq \langle A(u), v - g(u)\rangle - \rho\langle T(u) - f, v - g(u)\rangle \tag{2.15}$$

for all $v \in K(u)$, where $\rho > 0$ is a constant, which is called the generalized strongly quasivariational inequality, considered by Zeng [27].

(VI) If $K(u) = K$ for all $u \in H$ and we denote $g(u)$ by w, then the problem (2.15) becomes the general auxiliary variational inequality considered by Noor [18], which is to find $w \in K$, for some $u \in K$ such that

$$\langle A(w), v - w\rangle \geq \langle A(u), v - w\rangle - \rho\langle T(u) - f, v - w\rangle \tag{2.16}$$

for all $v \in K(u)$, where $\rho > 0$ is a constant.

(VII) If F, g, G and S are all the identity mappings, $N = 0$ and $K(u) = K$ for all $u \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle T(u), v - u\rangle \geq \langle f, v - u\rangle$$

for all $v \in K(u)$, which is known as a variational inequality introduced by Stampacchia [24] and was also studied by Noor [18] by introducing the above auxiliary problem (2.16).

It is clear that the generalized strongly nonlinear implicit quasivariational inequality problem (2.1) includes many kinds of quasivariational inequalities, variational inequalities, complementarity and quasicomplementarity problems as special cases.

3. ITERATIVE ALGORITHMS

In this section, we construct some new iterative algorithms for finding approximate solutions of the generalized strongly nonlinear implicit quasivariational inequalities (2.1), (2.11) and (2.12) by using the projection method and Nadler's theorem [14]. We need the following lemmas:

Lemma 3.1. *$u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ are a solution of the generalized strongly nonlinear implicit quasivariational inequality (2.1) if and only if $u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ satisfy $g(u) \in K(u)$ and*

$$\langle u - \varphi(u, x, y, z), v - g(u)\rangle \geq 0$$

for all $v \in K(u)$, where $\varphi(u, x, y, z) \in H$ satisfies

$$\langle \varphi(u, x, y, z), v\rangle = \langle u, v\rangle + \langle N(y, g(z)), v\rangle - \rho\langle T(x) - f, v\rangle \tag{3.1}$$

for all $v \in H$.

Proof. The conclusion immediately follows from (2.1). □

Lemma 3.2. [13]. *If K is a closed convex subset of H and $z \in H$ is a given point, then $u \in K$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0$$

for all $v \in K$ *if and only if*

$$u = P_K z. \tag{3.2}$$

Lemma 3.3. [13]. *The mapping* P_K *defined by (3.2) is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\|$$

for all $u, v \in H$.

From Lemmas 3.1 and 3.2, we have the following lemma:

Lemma 3.4. *Let* $K : H \to 2^H$ *be a set-valued mapping such that, for each* $u \in H$, $K(u)$ *is a nonempty closed convex set of* H. *Then* $u \in H$, $x \in Fu$, $y \in Gu$ *and* $z \in Su$ *are the solution of the generalized strongly nonlinear implicit quasivariational inequality (2.1) if and only if* $u \in H$, $x \in Fu$, $y \in Gu$ *and* $z \in Su$ *satisfy the relation*

$$u = (1-\lambda)u + \lambda[u - g(u) + P_{K(u)}(g(u) - u + \varphi(u, x, y, z))], \tag{3.3}$$

where $0 < \lambda < 1$ *is a constant and* $\varphi(u, x, y, z)$ *is defined as in (3.1).*

Based on Lemma 3.4, we now propose some algorithms for the generalized strongly nonlinear implicit quasivariational inequality (2.1).

Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let $T, g : H \to H$, $N : H \times H \to H$ be mappings and $F, G, S : H \to CB(H)$ be set-valued mappings, where $CB(H)$ is the family of all nonempty bounded closed subsets of H. For given $u_0 \in H$, we take $x_0 \in Fu_0$, $y_0 \in Gu_0$ and $z_0 \in Su_0$, and let

$$u_1 = (1-\lambda)u_0 + \lambda[u_0 - g(u_0) + P_{K(u_0)}(g(u_0) - u_0 + \varphi(u_0, x_0, y_0, z_0))].$$

Since $Fu_0 \in CB(H)$, $Gu_0 \in CB(H)$ and $Su_0 \in CB(H)$, by Nadler's theorem [14], there exist $x_1 \in Fu_1$, $y_1 \in Gu_1$ and $z_1 \in Su_1$ such that

$$\begin{aligned} \|x_0 - x_1\| &\leq (1+1)H(Fu_0, Fu_1), \\ \|y_0 - y_1\| &\leq (1+1)H(Gu_0, Gu_1), \\ \|z_0 - z_1\| &\leq (1+1)H(Su_0, Su_1), \end{aligned}$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$. Let

$$u_2 = (1-\lambda)u_1 + \lambda[u_1 - g(u_1) + P_{K(u_1)}(g(u_1) - u_1 + \varphi(u_1, x_1, y_1, z_1))].$$

By induction, we can obtain the algorithm for the generalized strongly nonlinear implicit quasivariational inequality (2.1) as follows:

Algorithm 3.1. Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let $T, g : H \to H$, $N : H \times H \to H$ be mappings and $F, G, S : H \to CB(H)$ be set-valued mappings. For given $u_0 \in H$, we can get the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\begin{cases} x_n \in Fu_n, \quad \|x_n - x_{n-1}\| \leq \left(1 + \frac{1}{n}\right) H(Fu_n, Fu_{n-1}), \\ y_n \in Gu_n, \quad \|y_n - y_{n-1}\| \leq \left(1 + \frac{1}{n}\right) H(Gu_n, Gu_{n-1}), \\ z_n \in Su_n, \quad \|z_n - z_{n-1}\| \leq \left(1 + \frac{1}{n}\right) H(Su_n, Su_{n-1}), \\ u_{n+1} = (1-\lambda)u_n + \lambda[u_n - g(u_n) + P_{K(u_n)}(g(u_n) - u_n + \varphi(u_n, x_n, y_n, z_n))] \end{cases} \tag{3.4}$$

for $n = 0, 1, 2, \cdots$, where $0 < \lambda < 1$ is a constant and $\varphi(u, x, y, z)$ is defined as (3.1).

If $A, B : H \to H$ are both single-valued mappings and $N(s, t) = Bs - At$ for all $s, t \in H$, then, from Algorithm 3.1, we have the algorithm for the problem (2.11) as follows:

Algorithm 3.2. Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let $A, B, T, g : H \to H$ be mappings and $F, G, S : H \to CB(H)$ be set-valued mappings. For given $u_0 \in H$, we can obtain the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$(3.5)\qquad \begin{aligned} &x_n \in Fu_n, \quad \|x_n - x_{n-1}\| \le \left(1 + \tfrac{1}{n}\right) H(Fu_n, Fu_{n-1}),\\ &y_n \in Gu_n, \quad \|y_n - y_{n-1}\| \le \left(1 + \tfrac{1}{n}\right) H(Gu_n, Gu_{n-1}),\\ &z_n \in Su_n, \quad \|z_n - z_{n-1}\| \le \left(1 + \tfrac{1}{n}\right) H(Su_n, Su_{n-1}),\\ &u_{n+1} = (1-\lambda)u_n + \lambda[u_n - g(u_n) + P_{K(u_n)}(g(u_n) - u_n + \varphi(u_n, x_n, y_n, z_n))] \end{aligned}$$

for $n = 0, 1, 2, \cdots$, where $0 < \lambda < 1$ is a constant and $\varphi(u, x, y, z)$ is defined by

$$\langle \varphi(u, x, y, z), v\rangle = \langle u, v\rangle + \langle B(y) - A(g(z)), v\rangle - \rho\langle T(x) - f, v\rangle$$

for all $v \in H$.

If S is the identity mapping, then, from Algorithm 3.1, we have the algorithm for the problem (2.12) as follows:

Algorithm 3.3. Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let $T, g : H \to H$, $N : H \times H \to H$ be mappings and $F, G : H \to CB(H)$ be set-valued mappings. For given $u_0 \in H$, we can obtain the sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ such that

$$(3.6)\qquad \begin{aligned} &x_n \in Fu_n, \quad \|x_n - x_{n-1}\| \le \left(1 + \tfrac{1}{n}\right) H(Fu_n, Fu_{n-1}),\\ &y_n \in Gu_n, \quad \|y_n - y_{n-1}\| \le \left(1 + \tfrac{1}{n}\right) H(Gu_n, Gu_{n-1}),\\ &u_{n+1} = (1-\lambda)u_n + \lambda[u_n - g(u_n) + P_{K(u_n)}(g(u_n) - u_n + \varphi(u_n, x_n, y_n))] \end{aligned}$$

for $n = 0, 1, 2, \cdots$, where $0 < \lambda < 1$ is a constant and $\varphi(u, x, y)$ is defined by

$$\langle \varphi(u, x, y), v\rangle = \langle u, v\rangle + \langle N(y, (g(u)), v\rangle - \rho\langle T(x) - f, v\rangle$$

for all $v \in H$.

Remark 3.1.

(i) For appropriate and suitable choices of the mappings K, g, F, G, S, T and N, a number of algorithms for variational inequality, quasivariational inequality, complementarity and quasicomplementarity problems can be obtained as special cases of Algorithm 3.1.

(ii) Algorithms 3.2 and 3.3 include several known algorithms of Noor [18] and Zeng [27] as special cases.

4. EXISTENCE AND CONVERGENCE THEOREMS

In this section, we prove some existence theorems for solutions of the generalized strongly nonlinear implicit quasivariational inequalities (2.1), (2.11) and (2.12) without compactness and the convergence of iterative sequences generated by the algorithms.

Definition 4.1. Let $g : H \to H$ be a single-valued mapping and $G : H \to 2^H$ be a set-valued mapping. Then

(i) g is called strongly monotone if there exists a number $r > 0$ such that

$$\langle gu_1 - gu_2, u_1 - u_2\rangle \ge r\|u_1 - u_2\|^2$$

for all $u_i \in H$, $i = 1, 2$.

(ii) g is called Lipschitz continuous if there exists a number $s > 0$ such that

$$\|gu_1 - gu_2\| \leq s\|u_1 - u_2\|$$

for all $u_i \in H$, $i = 1, 2$.

(iii) G is called H-Lipschitz continuous if there exists a number $\delta > 0$ such that

$$H(G(u_1), G(u_2)) \leq \delta\|u_1 - u_2\|$$

for all $u_i \in H$, $i = 1, 2$.

(iv) G is called strongly monotone with respect to g if there exists a number $\gamma > 0$ such that

$$\langle gw_1 - gw_2, u_1 - u_2\rangle \geq \gamma\|u_1 - u_2\|^2$$

for all $u_i \in H$ and $w_i \in Gu_i$, $i = 1, 2$.

Definition 4.2. The mapping $N : H \times H \to H$ is called Lipschitz continuous with respect to the first argument if there exists a number $\beta > 0$ such that

$$\|N(u, \cdot) - N(v, \cdot)\| \leq \beta\|u - v\|$$

for all $u, v \in H$.

In a similar way, we can define Lipschitz continuity of N with respect to the second argument.

Definition 4.3. Let $K : H \to 2^H$ be a set-valued mapping such that, for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. The projection $P_{K(x)}$ is said to be Lipschitz continuous if there exists a number $\eta > 0$ such that

$$\|P_{K(x)}z - P_{K(y)}z\| \leq \eta\|x - y\|$$

for all $x, y, z \in H$.

Remark 4.1. In many important applications, $K(u)$ has the following form:

$$K(u) = m(u) + K,$$

where $m : H \to H$ is a single-valued mapping and K is a nonempty closed convex subset of H. If m is Lipschitz continuous with constant λ, it is easy to see that $P_{K(x)}$ is Lipschitz continuous with the Lipschitz constant $\mu = 2\lambda$.

Theorem 4.1. *Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let mappings $T, g : H \to H$ be Lipschitz continuous with Lipschitz constants β and γ, respectively, and g be strongly monotone with constant δ. Let a mapping $N : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with Lipschitz constants τ and ξ, respectively. Let set-valued mappings F, G, S : $H \to CB(H)$ be H-Lipschitz continuous with H-Lipschitz constants η, σ, ϵ, respectively, and G be strongly monotone with respect to T with constant α. Suppose that $P_{K(x)}$ is Lipschitz continuous with the Lipschitz constant μ. If the following conditions hold:*

$$\begin{aligned} &|\rho - \tfrac{\alpha}{\beta^2\eta^2}| < \tfrac{\sqrt{\alpha^2 - \beta^2\eta^2 k(2-k)}}{\beta^2\eta^2}, \\ &\alpha > \beta\eta\sqrt{k(2-k)}, \\ &k = 2\sqrt{1 - 2\delta + \gamma^2} + \mu + \xi\gamma\epsilon + \tau\sigma < 1, \end{aligned} \tag{4.1}$$

then there exist $u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ which are a solution of the generalized strongly nonlinear implicit quasivariational inequality (2.1) and

$$u_n \to u, \quad x_n \to x, \quad y_n \to y, \quad z_n \to z \quad (n \to \infty),$$

where the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by Algorithm 3.1.

Proof. From Algorithm 3.1, Lemma 3.3 and the Lipschitz continuity of $P_{K(x)}$, we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \lambda\|u_n-u_{n-1}-[g(u_n)-g(u_{n-1})]\| \\
&\quad+(1-\lambda)\|u_n-u_{n-1}\|+\lambda\|P_{K(u_n)}Q(u_n)-P_{K(u_{n-1})}Q(u_{n-1})\| \\
&\le \lambda\|u_n-u_{n-1}-[g(u_n)-g(u_{n-1})]\| \\
&\quad+(1-\lambda)\|u_n-u_{n-1}\|+\lambda\|P_{K(u_n)}Q(u_n)-P_{K(u_n)}Q(u_{n-1})\| \\
&\quad+\lambda\|P_{K(u_n)}Q(u_{n-1})-P_{K(u_{n-1})}Q(u_{n-1})\| \\
&\le \lambda\|u_n-u_{n-1}-[g(u_n)-g(u_{n-1})]\| \\
&\quad+(1-\lambda)\|u_n-u_{n-1}\|+\lambda\|Q(u_n)-Q(u_{n-1})\|+\lambda\mu\|u_n-u_{n-1}\| \\
&\le 2\lambda\|u_n-u_{n-1}-[g(u_n)-g(u_{n-1})]\|+(1-\lambda)\|u_n-u_{n-1}\| \\
&\quad+\lambda\|\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\|+\lambda\mu\|u_n-u_{n-1}\|,
\end{aligned} \tag{4.2}$$

where $Q(u_n)=g(u_n)-u_n+\varphi(u_n,x_n,y_n,z_n)$. By (3.1), we have

$$\begin{aligned}
&\|\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\|^2 \\
&\quad=\langle\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1}),\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\rangle \\
&\quad\le|\langle u_n-u_{n-1}-\rho(T(x_n)-T(x_{n-1})),\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\rangle| \\
&\qquad+|\langle N(y_n,g(z_n))-N(y_{n-1},g(u_{n-1})),\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\rangle| \\
&\quad\le[\|u_n-u_{n-1}-\rho(T(x_n)-T(x_{n-1}))\| \\
&\qquad+\|N(y_n,g(z_n))-N(y_{n-1},g(z_{n-1}))\|]\cdot\|\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\|
\end{aligned}$$

and so

$$\begin{aligned}
&\|\varphi(u_n,x_n,y_n,z_n)-\varphi(u_{n-1},x_{n-1},y_{n-1},z_{n-1})\| \\
&\qquad\le\|u_n-u_{n-1}-\rho(T(x_n)-T(x_{n-1}))\|+\|N(y_n,g(z_n))-N(y_{n-1},g(z_{n-1}))\|.
\end{aligned} \tag{4.3}$$

Since G and S are H-Lipschitz continuous, g is Lipschitz continuous and N is Lipschitz continuous with respect to the first and second arguments, respectively, we get

$$\begin{aligned}
&\|N(y_n,g(z_n))-N(y_{n-1},g(z_{n-1}))\| \\
&\qquad\le\|N(y_n,g(z_n))-N(y_{n-1},g(z_n))\|+\|N(y_{n-1},g(z_n))-N(y_{n-1},g(z_{n-1}))\| \\
&\qquad\le\tau\|y_n-y_{n-1}\|+\xi\|g(z_n)-g(z_{n-1})\| \\
&\qquad\le\tau\sigma(1+\frac{1}{n})\|u_n-u_{n-1}\|+\xi\gamma\epsilon(1+\frac{1}{n})\|u_n-u_{n-1}\| \\
&\qquad\le(\tau\sigma+\xi\gamma\epsilon)(1+\frac{1}{n})\|u_n-u_{n-1}\|.
\end{aligned} \tag{4.4}$$

By the Lipschitz continuity and strong monotonicity of g, we obtain

$$\|u_n-u_{n-1}-(g(u_n)-g(u_{n-1}))\|^2\le(1-2\delta+\gamma^2)\|u_n-u_{n-1}\|^2. \tag{4.5}$$

Further, since T is Lipschitz continuous and F is H-Lipschitz continuous and strongly monotone with respect to T, we get

$$\|u_n-u_{n-1}-\rho(T(x_n)-T(x_{n-1}))\|^2\le(1-2\rho\alpha+\rho^2\beta^2\eta^2(1+\frac{1}{n})^2)\|u_n-u_{n-1}\|^2. \tag{4.6}$$

From (4.2) – (4.6), it follows that

$$\|u_{n+1}-u_n\|\le\theta_n\|u_n-u_{n-1}\|, \tag{4.7}$$

where

$$\theta_n = \lambda k_n + (1-\lambda) + \lambda\sqrt{1 - 2\rho\alpha + \rho^2\beta^2\eta^2(1+\frac{1}{n})^2},$$
$$k_n = 2\sqrt{1-2\delta+\gamma^2} + \mu + (\xi\gamma\epsilon + \tau\sigma)(1+\frac{1}{n}).$$

Letting

$$\theta = \lambda k + (1-\lambda) + \lambda\sqrt{1-2\rho\alpha+\rho^2\beta^2\eta^2},$$

we know that $\theta_n \searrow \theta$ as $n \to \infty$. It follows from (4.1) that $\theta < 1$. Hence $\theta_n < 1$ for n sufficiently large. Therefore, (4.7) implies that $\{u_n\}$ is a Cauchy sequence in H and we can assume that $u_n \to u \in H$.

Now we prove that $x_n \to x \in Fu$, $y_n \to y \in Gu$ and $z_n \to z \in Su$, respectively. In fact, it follows from Algorithm 3.1 that

$$\|x_n - x_{n-1}\| \le \left(1+\frac{1}{n}\right)\eta\|u_n - u_{n-1}\|,$$
$$\|y_n - y_{n-1}\| \le \left(1+\frac{1}{n}\right)\sigma\|u_n - u_{n-1}\|,$$
$$\|z_n - z_{n-1}\| \le \left(1+\frac{1}{n}\right)\epsilon\|u_n - u_{n-1}\|$$

and so $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all Cauchy sequences in H. Let $x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to \infty$. Further we have

$$\begin{aligned} d(x, Fu) &= \inf\{\|x - z\| : z \in Fu\} \\ &\le \|x - x_n\| + d(x_n, Fu) \\ &\le \|x - x_n\| + H(Fu_n, Fu) \\ &\le \|x - x_n\| + \eta\|u_n - u\| \to 0 \quad (n \to \infty). \end{aligned}$$

Hence, $x \in Fu$. Similarly, we have $y \in Gu$ and $z \in Su$. This completes the proof. □

From Theorem 4.1, we can get the following results:

Theorem 4.2. *Let $K : H \to 2^H$ be a set-valued mapping such that for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let mappings T, g, A, $B : H \to H$ be Lipschitz continuous with Lipschitz constants β, γ, ξ and τ, respectively, and g be strongly monotone with constant δ. Let set-valued mappings F, G, $S : H \to CB(H)$ be H-Lipschitz continuous with H-Lipschitz constants η, σ and ϵ, respectively, and G be strongly monotone with respect to T with constant α. Suppose that $P_{K(x)}$ is Lipschitz continuous with Lipschitz constant μ. If the condition (4.1) in Theorem 4.1 holds, then there exist $u \in H$, $x \in Fu$, $y \in Gu$ and $z \in Su$ which is a solution of the problem (2.11) and*

$$u_n \to u, \quad x_n \to x, \quad y_n \to y, \quad z_n \to z \quad (n \to \infty),$$

where the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are defined by Algorithm 3.2.

Theorem 4.3. *Let $K : H \to 2^H$ be a set-valued mapping such that, for each $u \in H$, $K(u)$ is a nonempty closed convex set of H. Let mappings $T, g : H \to H$ be Lipschitz continuous with Lipschitz constants β and γ, respectively, and g be strongly monotone with constant δ. Let mapping $N : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with Lipschitz constants τ and ξ, respectively. Let set-valued mappings F, $G : H \to CB(H)$ be H-Lipschitz continuous with H-Lipschitz constants η and σ, respectively, and G be* *strongly monotone with respect to T with constant α. Suppose that $P_{K(x)}$ is Lipschitz continuous with Lipschitz constant μ. If the condition (4.1) in Theorem 4.1 holds for*

$$k = 2\sqrt{1 - 2\delta + \gamma^2} + \mu + \xi\gamma + \tau\sigma < 1,$$

then there exist $u \in H$, $x \in Fu$ and $y \in Gu$ which are a solution of the problem (2.12) and

$$u_n \to u, \quad x_n \to x, \quad y_n \to y \quad (n \to \infty),$$

where the sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ are defined by Algorithm 3.3.

Remark 4.2. For a suitable choice of the mappings K, g, F, G, S, T and N, we can obtain several known results in [18], [24] and [27] as special cases of Theorem 4.1.

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