Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 2, Article 34, 2003

RATE OF CONVERGENCE OF SUMMATION-INTEGRAL TYPE OPERATORS WITH DERIVATIVES OF BOUNDED VARIATION

VIJAY GUPTA, VIPIN VASISHTHA, AND M.K. GUPTA

SCHOOL OF APPLIED SCIENCES,
NETAJI SUBHAS INSTITUTE OF TECHNOLOGY,
SECTOR 3 DWARKA, NEW DELHI-110045,
INDIA.
vijay@nsit.ac.in

DEPARTMENT OF MATHEMATICS,
HINDU COLLEGE,
MORADABAD -244001, INDIA

DEPARTMENT OF MATHEMATICS,
CH. CHARAN SINGH UNIVERSITY,
MEERUT-255004,
INDIA
mkgupta2002@hotmail.com

Received 28 January, 2003; accepted 29 March, 2003
Communicated by A. Fiorenza

ABSTRACT. In the present paper, we estimate the rate of convergence of the recently introduced generalized sequence of linear positive operators $G_{n,c}\left(f,x\right)$ with derivatives of bounded variation.

Key words and phrases: Linear positive operators, Bounded variation, Total variation, Rate of convergence.

2000 *Mathematics Subject Classification.* 41A25, 41A30.

1. INTRODUCTION

Let $DB_{\gamma}\left(0,\infty\right)$, $\left(\gamma\geq 0\right)$ be the class of all locally integrable functions defined on $\left(0,\infty\right)$, satisfying the growth condition $\left|f\left(t\right)\right|\leq Mt^{\gamma}$, $M>0$ and $f'\in BV$ on every finite subinterval of $\left[0,\infty\right)$. Then for a function $f\in DB_{\gamma}\left(0,\infty\right)$ we consider the generalized family of linear positive operators which includes some well known operators as special cases. The generalized

ISSN (electronic): 1443-5756

013-03

sequence of operators is defined by

$$G_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)\,f(t)\,dt + p_{n,0}(x;c)\,f(0), \quad x\in[0,\infty) \tag{1.1}$$

where $p_{n,k}(x;c) = (-1)^k \frac{x^k}{k!}\phi_{n,c}^{(k)}(x)$,

(i) $\phi_{n,c}(x) = e^{-nx}$ for $c=0$,
(ii) $\phi_{n,c}(x) = (1+cx)^{-n/c}$ for $c\in\mathbb{N}$,

and $\{\phi_{n,c}\}_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval $[0,b]$, $b>0$ having the following properties for every $n\in\mathbb{N}$, $k\in N_0$:

(i) $\phi_{n,c}\in C^{\infty}([a,b])$;
(ii) $\phi_{n,c}(0)=1$;
(iii) $\phi_{n,c}$ is completely monotone $(-1)^k\phi_{n,c}^{(k)}(x)\ge 0$;
(iv) There exists an integer c such that $\phi_{n,c}^{(k+1)} = -n\phi_{n+c,c}^{(k)}$, $n>\max\{0,-c\}$.

Remark 1.1. We may remark here that the functions $\phi_{n,c}$ have various applications in different fields, like potential theory, probability theory, physics and numerical analysis. A collection of most interesting properties of such functions can be found in [10, Ch. 4].

It is easily verified that the operators (1.1) are linear positive operators. Also $G_{n,c}(1,x)=1$. The generalized new sequence $G_{n,c}$ was recently introduced by Srivastava and Gupta [9].

For $c=0$ and $\phi_{n,c}(x)=e^{-nx}$ the operators $G_{n,c}$ reduce to the Phillips operators (see e.g. [7], [8]), which are defined by

$$G_{n,0}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x;0)\int_0^{\infty} p_{n,k-1}(t;0)\,f(t)\,dt + e^{-nx}f(0), \quad x\in[0,\infty), \tag{1.2}$$

where $p_{n,k}(x;0) = \frac{e^{-nx}}{k!}(nx)^k$.

For $c=1$ and $\phi_{n,c}(x) = (1+cx)^{-n/c}$ the operators $G_{n,c}$ reduce to the new sequence of summation integral type operators [6], which are defined by

$$G_{n,1}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x;1)\int_0^{\infty} p_{n+1,k-1}(t;1)\,f(t)\,dt + (1+x)^{-n}f(0), \quad x\in[0,\infty), \tag{1.3}$$

where

$$p_{n,k}(x;1) = \binom{n+k-1}{k} x^k(1+x)^{n-k}.$$

Remark 1.2. It may be noted that for $c=1$, we get the Baskakov basis functions $p_{n,k}(x;1)$ which are closely related to the well known Meyer-Konig and Zeller basis functions $m_{n,k}(t) = \binom{n+k-1}{k}t^k(1-t)^n$, $t\in[0,1]$ because by replacing the variable t with $\frac{x}{1+x}$ in the above MKZ basis functions we get the Baskakov basis functions. Zeng [11] obtained the exact bound for the Meyer Konig Zeller basis functions. Very recently Gupta et al. [6] used the bound of Zeng [11] and estimated the rate of convergence for the operators $G_{n,1}(f,x)$ on functions of bounded variation.

The operators (1.3) are slightly modified form of the operators introduced by Agarwal and Thamer [1], which are defined by

$$G_{n,1}^{*}(f,t) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x;1)\int_0^{\infty} p_{n,k-1}(t;1)\, f(t)\,dt + (1+x)^{-n} f(0), \qquad x\in[0,\infty), \tag{1.4}$$

where $p_{n,k}(x;1)$ is as defined by (1.3) above.

Recently Gupta [5] estimated the rate of approximation for the sequence (1.4) for bounded variation functions. Although the operators defined by (1.3) and (1.4) above are almost the same, but the main advantage to consider the operators in the form (1.3) rather than the form (1.4) is that some approximation properties become simpler in the analysis for the form (1.3) in comparison to the form (1.4). The rate of approximation with derivatives of bounded variation has been studied by several researchers. Bojanic and Cheng ([2], [3]) estimated the rate of convergence with derivatives of bounded variation for Bernstein and Hermite-Fejer polynomials by using different methods.

Alternatively we may rewrite the operators (1.1) as

$$G_{n,c}(f,x) = \int_0^{\infty} K_n(x,t;c)\, f(t)\,dt, \tag{1.5}$$

where

$$K_n(x,t;c) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\, p_{n+c,k-1}(t;c) + p_{n,0}(x;c)\, p_{n,0}(t;c)\,\delta(t),$$

$\delta(t)$ being the Dirac delta function. Also let

$$\beta_n(x,t;c) = \int_0^{t} K_n(x,s;c)\,ds \tag{1.6}$$

then

$$\beta_n(x,\infty;c) = \int_0^{\infty} K_n(x,s;c)\,ds = 1.$$

In the present paper we extend the results of [4] and [6] and study the rate of convergence by means of the decomposition technique of functions with derivatives of bounded variation. More precisely the functions having derivatives of bounded variation on every finite subinterval on the interval $[0,\infty)$ be defined as

$$f(x) = f(0) + \int_0^{x} \psi(t)\,dt, \qquad 0<a\le x\le b,$$

where ψ is a function of bounded variation on $[a,b]$ and c is a constant.

We denote the auxiliary function f_x, by

$$f_x(t) = \begin{cases} f(t)-f(x^-), & 0\le t<x; \\ 0, & t=x; \\ f(t)-f(x^+), & x<t<\infty. \end{cases}$$

2. AUXILIARY RESULTS

In this section we give certain results, which are necessary to prove the main result.

Lemma 2.1. [9]. *Let the function* $\mu_{n,m}(x)$, $m \in N^0$, *be defined as*

$$\mu_{n,m}(x;c) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)(t-x)^m\,dt + (-x)^m p_{n,0}(x;c).$$

Then

$$\mu_{n,0}(x;c) = 1,\;\; \mu_{n,1}(x;c) = \frac{cx}{(n-c)},$$
$$\mu_{n,2}(x;c) = \frac{x(1+cx)(2n-c)+(1+3cx)cx}{(n-c)(n-2c)},$$

and there holds the recurrence relation

$$\begin{aligned}&[n-c(m+1)]\mu_{n,m+1}(x;c)\\ &\qquad = x(1+cx)\left[\mu_{n,m}^{(1)}(x;c)+2m\mu_{n,m-1}(x;c)\right] + [m(1+2cx)+cx]\mu_{n,m}(x;c).\end{aligned}$$

Consequently for each $x \in [0,\infty)$, *we have from this recurrence relation that*

$$\mu_{n,m}(x;c) = O\left(n^{-[(m+1)/2]}\right).$$

Remark 2.2. In particular, given any number $\lambda > 2$ and $x > 0$ from Lemma 2.1, we have for $c \in N^0$ and n sufficiently large

$$G_{n,c}\left((t-x)^2, x\right) \equiv \mu_{n,2}(x;c) \le \frac{\lambda x(1+cx)}{n}. \tag{2.1}$$

Remark 2.3. It is also noted from (2.1), that

$$G_{n,c}(|t-x|, x) \le \left(G_{n,c}\left((t-x)^2, x\right)\right)^{\frac{1}{2}} \le \frac{\sqrt{\lambda x(1+cx)}}{\sqrt{n}}. \tag{2.2}$$

Lemma 2.4. *Let* $x \in (0,\infty)$ *and* $K_n(x,t)$ *be defined by (1.5). Then for* $\lambda > 2$ *and for* n *sufficiently large, we have*

(i) $\beta_n(x,y;c) = \int_0^y K_n(x,t;c)\,dt \le \frac{\lambda x(1+cx)}{n(x-y)^2}$, $0 \le y < x$,

(ii) $1-\beta_n(x,z;c) = \int_z^{\infty} K_n(x,t;c)\,dt \le \frac{\lambda x(1+cx)}{n(z-x)^2}$, $x < z < \infty$.

Proof. First, we prove (i). In view of (2.1), we have

$$\begin{aligned}\int_0^y K_n(x,t;c)\,dt &\le \int_0^y \frac{(x-t)^2}{(x-y)^2}K_n(x,t;c)\,dt \le (x-y)^{-2}\mu_{n,2}(x;c)\\ &\le \frac{\lambda x(1+cx)}{n(x-y)^2}.\end{aligned}$$

The proof of (ii) is similar. □

3. MAIN RESULT

In this section we prove the following main theorem.

Theorem 3.1. *Let $f \in DB_{\gamma}(0,\infty)$, $\gamma > 0$, and $x \in (0,\infty)$. Then for $\lambda > 2$ and for n sufficiently large, we have*

$$\begin{aligned}
|G_{n,c}(f,x) - f(x)| \le{} & \frac{\lambda(1+cx)}{n}\left(\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} ((f')_x)\right) \\
&+ \frac{\lambda(1+cx)}{n}\left(\left|f(2x) - f(x) - xf'\left(x^+\right)\right| + |f(x)|\right) \\
&+ \frac{\sqrt{\lambda x(1+cx)}}{\sqrt{n}}\left(M2^{\gamma}O\left(n^{-\gamma/2}\right) + \left|f'\left(x^+\right)\right|\right) \\
&+ \frac{1}{2}\frac{\sqrt{\lambda x(1+cx)}}{\sqrt{n}}\left|f'\left(x^+\right) - f'\left(x^-\right)\right| \\
&+ \frac{cx}{2(n-c)}\left|f'\left(x^+\right) + f'\left(x^-\right)\right|,
\end{aligned}$$

where $\bigvee_a^b (f_x)$ denotes the total variation of f_x on $[a,b]$.

Proof. We have

$$\begin{aligned}
G_{n,c}(f,x) - f(x) &= \int_0^{\infty} K_n(x,t;c)(f(t) - f(x))\,dt \\
&= \int_0^{\infty}\left(\int_x^t K_n(x,t;c) f'(u)\,du\right)dt.
\end{aligned}$$

Using the identity

$$\begin{aligned}
f'(u) = \frac{1}{2}\left[f'\left(x^+\right) + f'\left(x^-\right)\right] + (f')_x(u) + \frac{1}{2}\left[f'\left(x^+\right) - f'\left(x^-\right)\right]\operatorname{sgn}(u-x) \\
+ \left[f'(x) - \frac{1}{2}\left[f'\left(x^+\right) + f'\left(x^-\right)\right]\right]\chi_x(u),
\end{aligned}$$

it is easily verified that

$$\int_0^{\infty}\left(\int_x^t f'(x) - \frac{1}{2}\left[f'\left(x^+\right) + f'\left(x^-\right)\right]\chi_x(u)\,du\right)K(x,t;c)\,dt = 0.$$

Also

$$\begin{aligned}
\int_0^{\infty}\left(\int_x^t \frac{1}{2}\left[f'\left(x^+\right) - f'\left(x^-\right)\right]\operatorname{sgn}(u-x)\,du\right)K_n(x,t;c)\,dt \\
= \frac{1}{2}\left[f'\left(x^+\right) - f'\left(x^-\right)\right]G_{n,c}(|t-x|,x)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\infty}\left(\int_x^t \frac{1}{2}\left[f'\left(x^+\right) + f'\left(x^-\right)\right]du\right)K(x,t;c)\,dt \\
= \frac{1}{2}\left[f'\left(x^+\right) + f'\left(x^-\right)\right]G_{n,c}((t-x),x).
\end{aligned}$$

Thus we have

$$
\begin{aligned}
(3.1)\quad & \left|G_{n,c}(f,x)-f(x)\right| \\
& \le \left|\int_x^\infty \left(\int_x^t (f')_x(u)\,du\right) K_n(x,t;c)\,dt - \int_0^x \left(\int_x^t (f')_x(u)\,du\right) K_n(x,t;c)\,dt\right| \\
& \quad + \frac{1}{2}\left|f'\left(x^+\right)-f'\left(x^-\right)\right| G_{n,c}\left(\left|t-x\right|,x\right) \\
& \quad + \frac{1}{2}\left|f'\left(x^+\right)+f'\left(x^-\right)\right| G_{n,c}\left(\left(t-x\right),x\right) \\
& = \left|A_n(f,x;c)+B_n(f,x;c)\right| + \frac{1}{2}\left|f'\left(x^+\right)-f'\left(x^-\right)\right| G_{n,c}\left(\left|t-x\right|,x\right) \\
& \quad + \frac{1}{2}\left|f'\left(x^+\right)+f'\left(x^-\right)\right| G_{n,c}\left(\left(t-x\right),x\right).
\end{aligned}
$$

To complete the proof of the theorem it is sufficient to estimate the terms $A_n(f,x;c)$ and $B_n(f,x;c)$. Applying integration by parts, using Lemma 2.4 and taking $y = x - x/\sqrt{n}$, we have

$$
\left|B_n(f,x;c)\right| = \left|\int_0^x \left(\int_x^t (f')_x(u)\,du\right) dt\left(\beta_n(x,t;c)\right)\right|,
$$

$$
\begin{aligned}
\int_0^x \beta_n(x,t;c)(f')_x(t)\,dt & \le \left(\int_0^y + \int_y^x\right) \left|(f')_x(t)\right| \left|\beta_n(x,t;c)\right| dt \\
& \le \frac{\lambda x(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \int_y^x \bigvee_t^x ((f')_x)\,dt \\
& \le \frac{\lambda x(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).
\end{aligned}
$$

Let $u = x/(x-t)$. Then we have

$$
\begin{aligned}
\frac{\lambda x(1+cx)}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(x-t)^2} dt & = \frac{\lambda x(1+cx)}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x ((f')_x)\,du \\
& \le \frac{\lambda(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^x ((f')_x).
\end{aligned}
$$

Thus

$$
(3.2)\qquad \left|B_n(f,x;c)\right| \le \frac{\lambda(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^x ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x).
$$

On the other hand, we have

$$
\begin{aligned}
(3.3)\quad & |A_n(f,x;c)| \\
&= \left| \int_x^\infty \left(\int_x^t (f')_x(u)\,du \right) K_n(x,t;c)\,dt \right| \\
&= \left| \int_{2x}^\infty \left(\int_x^t (f')_x(u)\,du \right) K_n(x,t;c)\,dt \right. \\
&\qquad \left. + \int_x^{2x} \left(\int_x^t (f')_x(u)\,du \right) dt\,(1-\beta_n(x,t;c)) \right| \\
&\le \left| \int_{2x}^\infty (f(t)-f(x))\,K_n(x,t;c)\,dt \right| + \left| f'\left(x^+\right) \right| \left| \int_{2x}^\infty (t-x)\,K_n(x,t;c)\,dt \right| \\
&\quad + \left| \int_x^{2x} (f')_x(u)\,du \right| |1-\beta_n(x,2x;c)| + \int_x^{2x} |(f')_x(t)|\,|1-\beta_n(x,t;c)|\,dt \\
&\le \frac{M}{x} \int_{2x}^\infty K_n(x,t;c)\,t^\gamma\,|t-x|\,dt + \frac{|f(x)|}{x^2} \int_{2x}^\infty K_n(x,t;c)\,(t-x)^2\,dt \\
&\quad + \left| f'\left(x^+\right) \right| \int_{2x}^\infty K_n(x,t;c)\,|t-x|\,dt + \frac{\lambda(1+cx)}{nx} \left| f(2x) - f(x) - xf'\left(x^+\right) \right| \\
&\quad + \frac{\lambda(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} ((f')_x).
\end{aligned}
$$

Next applying Hölder's inequality, and Lemma 2.1, we proceed as follows for the estimation of the first two terms in the right hand side of (3.3):

$$
\begin{aligned}
(3.4)\qquad & \frac{M}{x} \int_{2x}^\infty K_n(x,t;c)\,t^\gamma\,|t-x|\,dt + \frac{|f(x)|}{x^2} \int_{2x}^\infty K_n(x,t;c)\,(t-x)^2\,dt \\
&\le \frac{M}{x} \left(\int_{2x}^\infty K_n(x,t;c)\,t^{2\gamma} dt \right)^{\frac{1}{2}} \left(\int_0^\infty K_n(x,t;c)\,(t-x)^2\,dt \right)^{\frac{1}{2}} \\
&\quad + \frac{|f(x)|}{x^2} \int_{2x}^\infty K_n(x,t;c)\,(t-x)^2\,dt \\
&\le M2^\gamma O\left(n^{-\gamma/2}\right) \frac{\sqrt{\lambda x(1+cx)}}{\sqrt{n}} + |f(x)| \frac{\lambda(1+cx)}{nx}.
\end{aligned}
$$

Also the third term of the right side of (3.3) is estimated as

$$
\begin{aligned}
& \left| f'\left(x^+\right) \right| \int_{2x}^\infty K_n(x,t;c)\,|t-x|\,dt \\
&\le \left| f'\left(x^+\right) \right| \int_0^\infty K_n(x,t;c)\,|t-x|\,dt \\
&\le \left| f'\left(x^+\right) \right| \left(\int_0^\infty K_n(x,t;c)\,(t-x)^2\,dt \right)^{\frac{1}{2}} \left(\int_0^\infty K_n(x,t;c)\,dt \right)^{\frac{1}{2}} \\
&= \left| f'\left(x^+\right) \right| \frac{\sqrt{\lambda x(1+cx)}}{\sqrt{n}}.
\end{aligned}
$$

Combining the estimates (3.1) – (3.4), we get the desired result.

This completes the proof of Theorem 3.1. □

Remark 3.2. For negative values of c, the operators $G_{n,c}$ may be defined in different ways. Here we consider one such example, when $c = -1$ then $\phi_{n,c}(x) = (1-x)^n$, the operator reduces to

$$G_{n,-1}(f,x) = n\sum_{k=1}^{n} p_{n,k}(x;-1)\int_0^1 p_{n-1,k-1}(t;-1)\, f(t)\, dt \\ + (1-x)^n f(0), \quad x \in [0,1],$$

where

$$p_{n,k}(x;-1) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The rate of convergence for the operators $G_{n-1}(f,x)$ is analogous so we omit the details.

REFERENCES

[1] P.N. AGARWAL AND K.J. THAMAR, Approximation of unbounded function by a new sequence of linear positive operators, *J. Math. Anal. Appl.,* **225** (1998), 660–672.

[2] R. BOJANIĆ AND F. CHENG, Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation, *J. Math. Anal Appl.*, **141** (1989), 136–151.

[3] R. BOJANIĆ AND F. CHENG, Rate of convergence of Hermite Fejer polynomials for functions with derivatives of bounded variation, *Acta Math. Hungar*., **59** (1992), 91–102.

[4] V. GUPTA AND G.S. SRIVASTAVA, On the rate of convergence of Phillips operators for functions of bounded variation, *Commentation Math*., **XXXVI** (1996), 123–130.

[5] V. GUPTA, Rate of approximation by new sequence of linear positive operators, *Comput. Math. Appl*., **45**(12) (2003), 1895–1904.

[6] V. GUPTA, M.K. GUPTA AND V. VASISHTHA, An estimate on the rate of convergence of Bezier type summation integral operators, *Kyungpook Math. J*., (to appear).

[7] C.P. MAY, On Phillips operators, *J. Approx. Theory,* **20** (1977), 315–322.

[8] R.S. PHILLIPS, An inversion formula for semi groups of linear operators, *Ann. Math*., **59** (1954), 352–356.

[9] H.M. SRIVASTAVA AND V. GUPTA, A certain family of summation integral type operators, *Math. Comput. Modelling*, **37** (2003), 1307–1315.

[10] D.V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

[11] X.M. ZENG, Bounds for Bernstein basis functions and Meyer Konig and Zeller basis functions, *J. Math. Anal. Appl*., **219** (1998), 364–376.