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PROJECTION ITERATIVE SCHEMES FOR GENERAL VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we propose some modified projection methods for general variational inequalities. The convergence of these methods requires the monotonicity of the underlying mapping. Preliminary computational experience is also reported.

Key words and phrases: General variational inequalities, Projection method, Monotonicity.

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1. INTRODUCTION

Let K be a nonempty closed convex set in Euclidean space $\mathbb{R}^n$. For given nonlinear operators $T, g : \mathbb{R}^n \to \mathbb{R}^n$, consider the problem of finding vector $u^* \in \mathbb{R}^n$ such that $g(u^*) \in K$ and

$$\langle T(u^*), g(u) - g(u^*)\rangle \geq 0, \ \forall\, g(u) \in K. \tag{1.1}$$

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This problem is called general variational inequality (GVI) which was introduced by Noor in [10]. General variational inequalities have important applications in many fields including economics, operations research and nonlinear analysis, see, e.g., [5], [10] – [15] and the references therein.

If $g(u) \equiv u$, then the general variational inequality (1.1) reduces to finding vector $u^* \in K$ such that

$$\langle T(u^*), u - u^* \rangle \geq 0, \; \forall\, u \in K, \tag{1.2}$$

which is known as the classical variational inequality and was introduced and studied by Stampacchia [18] in 1964. For the recent state-of-the-art, see e.g., [1] – [22].

If $K^{**} = \{u \in \mathbb{R}^n \,|\, \langle u, v\rangle \geq 0, \; \forall\, v \in K\}$ is a polar cone of a convex cone K in $\mathbb{R}^n$, then problem (1.1) is equivalent to finding $u^* \in \mathbb{R}^n$ such that

$$g(u) \in K, \; T(u) \in K^{**}, \; \langle g(u), T(u)\rangle = 0, \tag{1.3}$$

which is known as the general complementarity problem. If $g(u) = u - m(u)$, where m is a point-to-set mapping, then problem (1.3) is called quasi (implicit) complementarity problem. For $g(u) = u$, problem (1.3) is known as the generalized complementarity problem.

For general variational inequality, Noor [10] gave a fixed point equation reformulation, Pang and Yao [15] established some sufficient conditions for the existence of the solutions and investigated their stability, and He [5] proposed an inexact implicit method. In this paper, we consider a projection method for solving GVI under the assumptions that the solution set is nonempty and the underlying mapping is monotone in a generalized sense.

2. PRELIMINARIES

For nonempty closed convex set $K \subset \mathbb{R}^n$ and any vector $u \in \mathbb{R}^n$, the orthogonal projection of u onto K, i.e., $\arg\min\{||v-u|| \mid v \in K\}$, is denoted by $P_K(u)$. In the following, we state some well known properties of the projection operator.

Lemma 2.1. [23]. *Let K be a closed convex subset of $\mathbb{R}^n$, for any $u \in \mathbb{R}^n$, $v \in K$, then*

$$\langle P_K(u) - u, v - P_K(u)\rangle \geq 0.$$

From Lemma 2.1, it follows that the projection operator P_K is nonexpansive.

Invoking Lemma 2.1, one can prove that the general variational inequality (1.1) is equivalent to the fixed-point problem For GVI, this result is due to Noor [10].

Lemma 2.2. [10]. *A vector $u^* \in \mathbb{R}^n$ with $g(u^*) \in K$ is a solution of GVI if and only if $g(u^*) = P_K(g(u^*) - \rho T(u^*))$ for some $\rho > 0$.*

Based on this fixed-point formulation, various projection type iterative methods for solving general variational inequalities have been suggested and analyzed, see [5], [10] – [15].

In this paper, we suggest another projection method which needs two projections at each iteration and its convergence requires the following assumptions.

Assumptions.

(i) The solution set of GVI, denoted by K^*, is nonempty.

(ii) Mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is g-monotone, i.e.,

$$\langle T(u) - T(v), g(u) - g(v)\rangle \geq 0, \; \forall\, u, \; v \in \mathbb{R}^n.$$

(iii) Mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ is nonsingular, i.e., there exists a positive constant μ such that

$$||g(u) - g(v)|| \geq \mu ||u - v||, \; \forall\, u, \; v \in \mathbb{R}^n.$$

Note that for $g \equiv I$, g-monotonicity of mapping T reduces to the usual definition of monotone. Furthermore, every solvable monotone variational inequality of form (1.2) satisfies the above assumptions.

Throughout this paper, we define the residue vector $R_\rho(u)$ by the following relation

$$R_\rho(u) := g(u) - P_K(g(u) - \rho T(u)).$$

Invoking Lemma 2.2, one can easily conclude that vector u^* is a solution of GVI if and only if u^* is a root of the following equation:

$$R_\rho(u) = 0, \text{ for some } \rho \geq 0.$$

3. ALGORITHMS AND CONVERGENCE

The basic idea of our method is as follows. First, take an initial point $u^0 \in \mathbb{R}^n$ such that $g(u^0) \in K$ and compute the projection residue. If it is a zero vector, then stop; otherwise, take the negative projection residue as a direction and perform a line search along this direction to get a new point; after constructing a "descent direction" related to the current point and the new point, the next iterative point can be obtained by using a projection. Repeat this process until the projection residue is a zero vector. So the algorithm needs only two projections at each iteration.

Now, we formally describe our method for solving the GVI problem.

Algorithm 3.1.

Initial step: Choose $u^0 \in \mathbb{R}^n$ such that $g(u^0) \in K$, select any $\sigma, \gamma \in (0,1)$, $\rho \in (0, +\infty)$, let $k := 0$.

Iterative step: For $g\left(u^k\right) \in K$, take $w^k \in \mathbb{R}^n$ such that $g\left(w^k\right) := P_K\left(g\left(u^k\right) - \rho T\left(u^k\right)\right)$. If $\left\|R_\rho\left(u^k\right)\right\| = 0$, then stop. Otherwise, compute $v^k \in \mathbb{R}^n$ such that $g\left(v^k\right) := g\left(u^k\right) - \eta_k R_\rho\left(u^k\right)$, where $\eta_k = \gamma^{m_k}$ with m_k being the smallest nonnegative integer m satisfying

$$\rho\left\langle T\left(u^k\right) - T\left(v^k\right), R_\rho\left(u^k\right)\right\rangle \leq \sigma\left\|R_\rho\left(u^k\right)\right\|^2. \tag{3.1}$$

Compute u^{k+1} by solving the following equation $g\left(u^{k+1}\right) = P_K\left(g\left(u^k\right) + \alpha_k d_k\right)$, where $d_k = -\left(\eta_k R_\rho\left(u^k\right) + \eta_k T\left(u^k\right) + \rho T\left(v^k\right)\right)$, $\alpha_k = \frac{(1-\sigma)\eta_k\left\|R_\rho\left(u^k\right)\right\|^2}{\|d_k\|^2}$.

Remark 3.1. We analyze the step-size rule given in (3.1). If Algorithm 3.1 terminates with $R_\rho\left(u^k\right) = 0$, then u^k is a solution of GVI. Otherwise, by non-singularity of g and continuity of T and g, η_k satisfying (3.1) exists.

Remark 3.2. In Algorithm 3.1, several implicit equations of g must be solved at each iteration. If $g \equiv I$, then $v^k = (1-\eta_k)u^k + \eta_k w^k$.

Remark 3.3. We recall the searching directions appear in existing projection-type methods for solving VI of form (1.2). They are

(i) the direction $-T(\bar{u}^k)$ by Korpelevich [9], where $\bar{u}^k = P_K\left(u^k - \alpha_k T\left(u^k\right)\right)$;
(ii) the direction $-\left\{u^k - \bar{u}^k - \alpha_k\left[T\left(u^k\right) - T\left(\bar{u}^k\right)\right]\right\}$ by Solodov and Tseng [17], Tseng [20], Sun [19] and He [6].
(iii) the direction $-\left\{u^k - \bar{u}^k + T\left(\bar{u}^k\right)\right\}$ by Noor [13].
(iv) the direction $-T\left(v^k\right)$ by Iusem and Svaiter [7] and Solodov and Svaiter [16].
(v) the direction $-\left(\eta_k r\left(u^k\right) + T\left(v^k\right)\right)$ by Wang, Xiu and Wang [22].

In our algorithm, when $g \equiv I$, the searching direction reduces to

$$-\left(\eta_k r\left(u^k\right) + \eta_k T\left(u^k\right) + \rho T\left(v^k\right)\right).$$

It is a combination of the projection residue and T, and differs from the above five types of directions.

Now, we discuss the convergence of Algorithm 3.1. From the iterative procedure, we know that $g\left(u^k\right), g\left(v^k\right), g\left(w^k\right) \in K$ for all k. For any $g\left(u^*\right) \in K^*$, by Assumption (ii), we have

$$\left\langle \rho T\left(u^k\right), g\left(u^k\right) - g\left(u^*\right)\right\rangle \geq 0. \tag{3.2}$$

From Lemma 2.1, we know that

$$\left\langle g\left(u^k\right) - \rho T\left(u^k\right) - g\left(w^k\right), g\left(w^k\right) - g\left(u^*\right)\right\rangle \geq 0,$$

which can be written as

$$\begin{aligned}&\left\langle g\left(u^k\right) - \rho T\left(u^k\right) - g\left(w^k\right), g\left(w^k\right) - g\left(u^k\right)\right\rangle \\ &\qquad + \left\langle g\left(u^k\right) - g\left(w^k\right) - \rho T\left(u^k\right), g\left(u^k\right) - g\left(u^*\right)\right\rangle \geq 0.\end{aligned}$$

Combining with inequality (3.2), we obtain

$$\left\langle R_\rho\left(u^k\right), g\left(u^k\right) - g\left(u^*\right)\right\rangle \geq \left\|R_\rho\left(u^k\right)\right\|^2 - \rho\left\langle T\left(u^k\right), R_\rho\left(u^k\right)\right\rangle. \tag{3.3}$$

So

$$\begin{aligned}&\left\langle g\left(u^k\right) - g\left(u^*\right), -d_k\right\rangle \\ &\quad = \left\langle g\left(u^k\right) - g\left(u^*\right), \eta_k R_\rho\left(u^k\right) + \eta_k T\left(u^k\right) + \rho T\left(v^k\right)\right\rangle \\ &\quad = \left\langle g\left(u^k\right) - g\left(u^*\right), \eta_k R_\rho\left(u^k\right)\right\rangle + \left\langle g\left(u^k\right) - g\left(u^*\right), \eta_k T\left(u^k\right)\right\rangle \\ &\qquad + \left\langle g\left(u^k\right) - g\left(u^*\right), \rho T\left(v^k\right)\right\rangle \\ &\quad \geq \eta_k\left\|R_\rho\left(u^k\right)\right\|^2 - \rho\eta_k\left\langle T\left(u^k\right), R_\rho\left(u^k\right)\right\rangle + \left\langle g\left(u^k\right) - g\left(v^k\right), \rho T\left(v^k\right)\right\rangle \\ &\quad = \eta_k\left\|R_\rho\left(u^k\right)\right\|^2 - \rho\eta_k\left\langle T\left(u^k\right), R_\rho\left(u^k\right)\right\rangle + \eta_k\left\langle R_\rho\left(u^k\right), \rho T\left(v^k\right)\right\rangle \\ &\quad = \eta_k\left\|R_\rho\left(u^k\right)\right\|^2 - \rho\eta_k\left\langle T\left(u^k\right) - T\left(v^k\right), R_\rho\left(u^k\right)\right\rangle \\ &\quad \geq \eta_k\left\|R_\rho\left(u^k\right)\right\|^2 - \sigma\eta_k\left\|R_\rho\left(u^k\right)\right\|^2 \\ &\quad = (1-\sigma)\eta_k\left\|R_\rho\left(u^k\right)\right\|^2,\end{aligned}$$

where the first inequality uses (3.3) and the g-monotonicity of T, the second inequality follows from inequality (3.1).

For any $\alpha > 0$, one has

$$\begin{aligned}&\left\|P_K\left(g\left(u^k\right) + \alpha d_k\right) - g\left(u^*\right)\right\|^2 \\ &\qquad \leq \left\|g\left(u^k\right) - g\left(u^*\right) + \alpha d_k\right\|^2 \\ &\qquad = \left\|g\left(u^k\right) - g\left(u^*\right)\right\|^2 + \alpha^2\left\|d_k\right\|^2 + 2\alpha\left\langle d_k, g\left(u^k\right) - g\left(u^*\right)\right\rangle \\ &\qquad \leq \left\|g\left(u^k\right) - g\left(u^*\right)\right\|^2 + \alpha^2\|d_k\|^2 - 2\alpha(1-\sigma)\eta_k\left\|R_\rho\left(u^k\right)\right\|^2,\end{aligned}$$

where the first inequality uses non-expansiveness of projection operator.

Based on the above analysis, we show that Algorithm 3.1 converges under Assumptions (i) – (iii).

Theorem 3.4. *Under Assumptions (i) – (iii), if Algorithm 3.1 generates an infinite sequence $\{u^k\}$, then $\{u^k\}$ globally converges to a solution u^* of GVI.*

Proof. Let $\alpha := \alpha_k = \frac{(1-\sigma)\eta_k||R_\rho(u^k)||^2}{\|d_k\|^2}$ in the aforementioned inequalities, we obtain

$$\left\|g(u^{k+1}) - g\left(u^*\right)\right\| \leq \left\|g\left(u^k\right) - g\left(u^*\right)\right\|^2 - \frac{(1-\sigma)^2\eta_k^2\left\|R_\rho\left(u^k\right)\right\|^4}{||d_k||^2}.$$

So $\left\{\left\|g\left(u^k\right) - g\left(u^*\right)\right\|\right\}$ is a non-increasing sequence, and $\left\{g\left(u^k\right)\right\}$ is a bounded sequence. Since g is nonsingular, we conclude that $\{u^k\}$ is a bounded sequence. Short discussion leads to that $\{d_k\}$ is bounded. So, there exists an infinite subset N_1 such that

$$\lim_{k\in N_1, k\to\infty} \left\|R_\rho\left(u^k\right)\right\| = 0$$

or an infinite subset N_2 such that

$$\lim_{k\in N_2, k\to\infty} \eta_k = 0.$$

If $\lim\limits_{k\in N_1,k\to\infty} \left\|R_\rho\left(u^k\right)\right\| = 0$, we know that any cluster $\tilde{u}$ of $\{u^k : k \in N_1\}$ is a solution of GVI. Since $\left\{\left\|g\left(u^k\right) - g\left(u^*\right)\right\|\right\}$ is non-increasing, if we take $u^* = \tilde{u}$, then we know that $\{g\left(u^k\right)\}$ globally converges to $g(\tilde{u})$ and thus $\{u^k\}$ globally converges to $\tilde{u}$ from Assumption (iii).

If $\lim\limits_{k\in N_2,k\to\infty} \eta_k = 0$, let $\bar{v}^k \in R^n$ such that $g\left(v^k\right) = g\left(u^k\right) - \frac{\eta_k}{\gamma R_\rho\left(u^k\right)}$. From the linear searching procedure of η_k, we have

$$\rho\left\langle T\left(u^k\right) - T(\bar{v}^k), R_\rho\left(u^k\right)\right\rangle > \sigma\left\|R_\rho\left(u^k\right)\right\|^2, \text{ for sufficiently large } k \in N_2.$$

Therefore,

$$\rho\left\|T\left(u^k\right) - T(\bar{v}^k)\right\| > \sigma\left\|R_\rho\left(u^k\right)\right\|, \text{ for sufficiently large } k \in N_2.$$

This, plus $\lim\limits_{k\in N_2,k\to\infty} \frac{\eta_k}{\gamma} = 0$, yields $\lim\limits_{k\in N_2,k\to\infty} \left\|R_\rho\left(u^k\right)\right\| = 0$. Similar discussion leads to that any cluster of $\left\{u^k : k \in N_2\right\}$ is a solution to GVI. Replacing u^* by this cluster point yields the desired result. □

If we replace ρ with ρ_k in Algorithm 3.1, then we obtain the following improved algorithm to GVI.

Algorithm 3.2.

Initial step: Choose $u^0 \in \mathbb{R}^n$ such that $g(u^0) \in K$, select any $\sigma, \gamma \in (0,1)$, $\eta_{-1} = 1, \theta > 0$. Let $k = 0$.

Iterative step: For $g\left(u^k\right) \in K$, define $\rho_k = \min\{\theta\eta_{k-1}, 1\}$, and take $w^k \in R^n$ such that $g\left(w^k\right) = P_K(g\left(u^k\right) - \rho_k T\left(u^k\right))$.
If $R_{\rho_k}\left(u^k\right) = 0$, then stop. Otherwise, take $v^k \in R^n$ in the following way:
$g\left(v^k\right) = (1-\eta_k)g\left(u^k\right) + \eta_k g\left(w^k\right)$,
where $\eta_k = \gamma^{m_k}$, with m_k being the smallest nonnegative integer m satisfying
$\rho_k\langle T\left(u^k\right) - T\left(v^k\right), R_{\rho_k}\left(u^k\right)\rangle \leq \sigma\left\|R_{\rho_k}\left(u^k\right)\right\|^2$.
Compute u^{k+1} by solving the following equation:
$g(u^{k+1}) = P_K(g\left(u^k\right) + \alpha_k d_k)$
where $d_k = -\left(\eta_k R_{\rho_k}\left(u^k\right) + \eta_k T\left(u^k\right) + \rho_k T\left(v^k\right)\right)$,
$\alpha_k = \frac{(1-\sigma)\eta_k\left\|R_{\rho_k}\left(u^k\right)\right\|^2}{\|d_k\|^2}$.

The convergence of Algorithm 3.2 can be proved similarly.

Dimension	Alg. 3.1 ($\rho = 1$)	Alg. 3.2 ($\theta = 400$)
$n = 10$	73	56
$n = 20$	75	58
$n = 50$	78	58
$n = 80$	81	60
$n = 100$	84	60
$n = 200$	97	60

Table 4.1: Numbers of iterations for Example 4.1

4. PRELIMINARY COMPUTATIONAL EXPERIENCE

In the following, we present some numerical experiments for Algorithms 3.1 and 3.2. For these algorithms, we used $\left\|r\left(x^k, \rho_k\right)\right\| \leq 10^{-8}$ as stopping criteria.

Throughout the computational experiments, the parameters used were set as $\sigma = 0.5, \gamma = 0.8$. All computational results were undertaken on a PC-II by MATLAB.

Example 4.1. This example is a quadratic subproblem of the trust region approach for solving medium-size nonlinear programming problem:

$$\min\left\{\frac{1}{2}x^\top Hx + c^\top x \mid x \in C\right\}.$$

This problem is equivalent to VI(F, C) with $F(x) = Hx + c$. the data is chosen as: $H = VWV$, where $V = I - 2\frac{vv^\top}{||v||^2}$ is a Householder matrix and $W = \text{diag}(\sigma_i)$ with $\sigma_i = \cos\frac{i\pi}{n+1} + 1000$. The vectors v and c contain pseudo-random numbers:

$$v_1 = 13846,\ v_i = (42108v_{i-1} + 13846)\text{mod } 46273,\ i = 2, \dots, n;$$

$$c_1 = 13846,\ c_i = (45287c_{i-1} + 13846)\text{mod } 46219,\ i = 2, \dots, n.$$

For this test problems, the domain set $C = \{x \in \mathbb{R}^n \mid ||x|| \leq 10^5\}$. Table 4.1 gives the numerical results for this example with starting point $x^0 = (0, 0, \dots, 0)^T$ for different dimensions n.

Example 4.2. This example is a general variational inequality with $g(x) = Ax + q$ and $F(x) = x$, where

$$A = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

For this test problems, the domain set $C = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ \text{for } i = 1, 2, \cdots n\}$. Table 4.2 gives the results for this example with starting point $x^0 = -A^{-1}q$ for different dimensions n.

From Table 4.1 and Table 4.2, one observes that Algorithms 3.1 and 3.2 work quite well for these examples, respectively, and there is not much difference to the choice of parameter ρ_k in the second algorithm, especially for Example 4.2.

Dimension	Alg. 3.1 ($\rho = 1$)	Alg. 3.2 ($100 \leq \theta \leq 400$)
$n = 10$	492	492
$n = 20$	489	489
$n = 50$	484	484
$n = 80$	481	481
$n = 100$	480	480
$n = 200$	476	476

Table 4.2: Numbers of iterations for Example 4.2

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