Journal of Inequalities in Pure and Applied Mathematics

ON HADAMARD'S INEQUALITY FOR THE CONVEX MAPPINGS DEFINED ON A CONVEX DOMAIN IN THE SPACE

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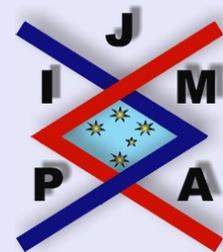
Abstract

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Abstract

In this paper we obtain some Hadamard type inequalities for triple integrals. The results generalize those obtained in (S.S. DRAGOMIR, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, *RGMIA* (preprint), 1999).

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a convex mapping defined on the interval $[a, b]$. The following double inequality

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality for convex mappings.

In [1] S.S. Dragomir considered the following mapping naturally connected to Hadamard's inequality

$$H : [0, 1] \to \mathbb{R}, \quad H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\, dx$$

and proved the following properties of this function

*(i)* H is convex and monotonic nondecreasing.

(ii) H has the bounds

$$\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

and

$$\inf_{t \in [0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$
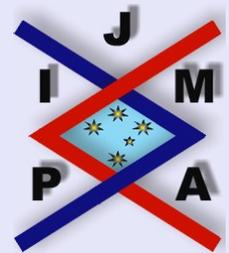
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In the recent paper [2], S.S. Dragomir gave some inequalities of Hadamard's type for convex functions defined on the ball $\overline{B}(C, R)$, where

$$C = (a, b, c) \in \mathbb{R}^3, \quad R > 0$$

and

$$\overline{B}(C, R) := \{(x, y, z) \in \mathbb{R}^3 |\ (x - a)r + (y - b)^2 + (z - c)^2 \leq R^2\}$$

More precisely he proved the following theorem.

Theorem 1.1. *Let* $f : \overline{B}(C, R) \to \mathbb{R}$ *be a convex mapping on the ball* $\overline{B}(C, R)$. *Then we have the inequality*

$$
\begin{aligned}
(1.2) \qquad f(a, b, c) &\leq \frac{1}{v(\overline{B}(C, R))} \iiint_{\overline{B}(C,R)} f(x, y, z)\, dxdydz \\
&\leq \frac{1}{\sigma(\overline{B}(C, R))} \iint_{S(C,R)} f(x, y, z)\, d\sigma
\end{aligned}
$$

where

$$S(C, R) := \{(x, y, z) \in \mathbb{R}^2 |\ (x - a)^2 + (y - b)^2 + (z - c)^2 = R^2\}$$

and

$$v(\overline{B}(C, R)) = \frac{4\pi R^3}{3}, \qquad \sigma(\overline{B}(C, R)) = 4\pi R^2.$$

In [2] S.S. Dragomir considers, for a convex mapping f defined on the ball $\overline{B}(C, R)$, the mapping $H : [0, 1] \to \mathbb{R}$ given by

$$H(t) = \frac{1}{v(\overline{B}(C, R))} \iiint_{\overline{B}(C,R)} f(t(x, y, z) + (1 - t)C)\, dxdydz.$$

The main properties of this mapping are contained in the following theorem.
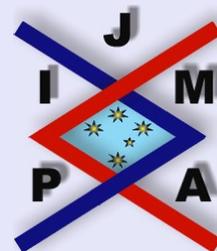
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Theorem 1.2. *With the above assumption, we have*

(i) *The mapping H is convex on $[0,1]$.*

(ii) *H has the bounds*

$$(1.3) \qquad \inf_{t\in[0,1]} H(t) = H(0) = f(C)$$

and

$$(1.4) \quad \sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{v(\overline{B}(C,R))} \iiint_{\overline{B}(C,R)} f(x,y,z)\, dx dy dz.$$

(iii) *The mapping H is monotonic nondecreasing on $[0,1]$.*

In this paper we shall give a generalization of the Theorem 1.2 for a positive linear functional defined on $C(D)$, where $D \subset \mathbb{R}^m$ ($m \in \mathbb{N}^*$) is a convex domain. We shall give also a generalization of the Theorem 1.1.
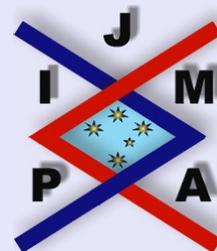
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2.   Results

Let $D \subset \mathbb{R}^m$ be a convex domain and $A : C(D) \to \mathbb{R}$ be a given positive linear functional such that $A(e_0) = 1$, where $e_0(x) = 1$, $x \in D$. Let $x = (x_1, \ldots, x_m)$ be a point from D we note by p_i, $i = 1, 2, \ldots, m$ the function defined on D by

$$p_i(x) = x_i, \qquad i = 1, 2, \ldots, m$$

and by a_i, $i = 1, 2, \ldots, m$ the value of the functional A in p_i, i.e.

$$A(p_i) = a_i, \qquad i = 1, 2, \ldots, m.$$

In addition, let f be a convex mapping on D. We consider the mapping $H : [0, 1] \to \mathbb{R}$ associated with the function f and given by

$$H(t) = A(f(tx + (1 - t)a))$$

where $a = (a_1, a_2, \ldots, a_m)$ and the functional A acts analagous to the variable x.

Theorem 2.1. *With above assumption, we have*

(i) *The mapping H is convex on $[0, 1]$.*

(ii) *The bounds of the function H are given by*

(2.1) $$\inf_{t \in [0,1]} H(t) = H(0) = f(a)$$

and

(2.2) $$\sup_{t \in [0,1]} H(t) = H(1) = A(f).$$
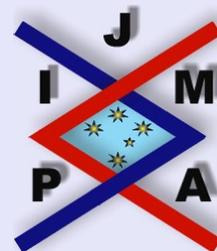
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(iii) *The mapping H is monotonic nondecreasing on $[0, 1]$.*

Proof. *(i)* Let $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$
\begin{aligned}
H(\alpha t_1 + \beta t_2) &= A[f((\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))a)] \\
&= A[f(\alpha(t_1 x + (1 - t_1)a) + \beta(t_2 x + (1 - t_2)a))] \\
&\leq \alpha A[f(t_1 x + (1 - t_1)a)] + BA[f(t_2 x + (1 - t_2)a)] \\
&= \alpha H(t_1) + \beta H(t_2)
\end{aligned}
$$

which proves the convexity of H on $[0, 1]$.

(ii) Let g be a convex function on D. Then there exist the real numbers $A_1, A_2, \ldots, A_m$ such that

$$(2.3) \qquad g(x) \geq g(a) + (x_1 - a_1)A_1 + (x_2 - a_2)A_2 + \cdots + (x_m - a_m)A_m$$

for any $x = (x_1, \ldots, x_m) \in D$.

   Using the fact that the functional A is linear and positive, from the inequality (2.3) we obtain the inequality

$$(2.4) \qquad\qquad\qquad A(g) \geq g(a).$$

   Now, for a fixed number t, $t \in [0, 1]$ the function $g : D \to \mathbb{R}$ defined by

$$g(x) = f(tx + (1 - t)a)$$

is a convex function. From the inequality (2.4) we obtain

$$A(f(tx + (1 - t)a)) \geq f(ta + (1 - t)a) = f(a)$$
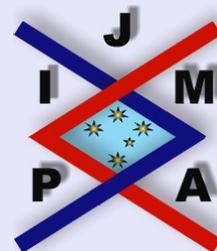
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or
$$H(t) \geq H(0)$$

for every $t \in [0, 1]$, which proves the equality (2.1).

Let $0 \leq t_1 < t_2 \leq 1$. By the convexity of the mapping H we have

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} \geq \frac{H(t_1) - H(0)}{t_1} \geq 0.$$

So the function H is a nondecreasing function and $H(t) \leq H(1)$. The theorem is proved. $\qquad\square$

Remark 2.1. For $m = 1$, $D = [a, b]$ and

$$A(f) = \frac{1}{b - a} \int_a^b f(x) \, dx$$

the function H is the function which was considered in the paper [1].

Remark 2.2. For $m = 3$ and $D = \overline{B}(C, R)$ and

$$A(f) = \frac{1}{v(\overline{B}(C, R))} \int \int \int_{\overline{B}(C,R)} f(x, y, z) \, dxdydz$$

a being C, the function H is the functional from the Theorem 1.2.
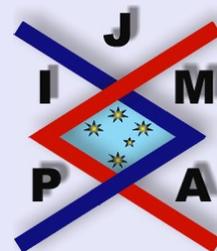
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Let D be a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. We define the notation

$$\sigma := \iint_S dS,$$

$$a_1 := \frac{1}{\sigma} \iint_S x\, dS,$$

$$a_2 := \frac{1}{\sigma} \iint_S y\, dS,$$

$$a_3 := \iint_S z\, dS,$$

$$v := \iiint_V f(x,y,z)\, dxdydz.$$

Let us assume that the surface S is oriented with the aid of the unit normal h directed to the exterior of D

$$h = (\cos\alpha, \cos\beta, \cos\gamma).$$

The following theorem is a generalization of the Theorem 1.1.

Theorem 2.2. *Let f be a convex function on D. With the above assumption we have the following inequalities*

(2.5)
$$v \iint_S f\, ds - \sigma \iint_S [(a_1-x)\cos\alpha + (a_2-y)\cos\beta + (a_3-z)\cos\gamma] f(x,y,z)\, dS$$
$$\geq 4\sigma \iiint_D f(x,y,z)\, dxdydz$$
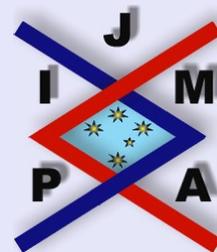
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and

$$(2.6) \qquad \iiint_D f(x,y,z)\,dxdydz \geq f(x_\sigma, y_\sigma, z_\sigma)v,$$

where

$$x_\sigma = \frac{1}{v}\iiint_D x\,dxdydz, \quad y_\sigma = \frac{1}{v}\iiint_D y\,dxdydz, \quad z_\sigma = \frac{1}{v}\iiint_D z\,dxdydz.$$

Proof. We can suppose that the function f has the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ and these are continuous on D.

For every point $(u,v,w) \in S$ and $(x,y,z) \in D$ the following inequality holds:

$$(2.7)$$
$$f(u,v,w) \geq f(x,y,z) + \frac{\partial f}{\partial x}(x,y,z)(u-x) + \frac{\partial f}{\partial y}(x,y,z)(v-y) + \frac{\partial f}{\partial z}(x,y,z)(w-z).$$

From the inequality (2.7) we have

$$(2.8) \quad \iint_S f(x,y,z)\,dS \geq f(x,y,z)\sigma + \frac{\partial f}{\partial x}(x,y,z)(a_1 - x)\sigma$$
$$+ \frac{\partial f}{\partial y}(x,y,z)(a_2 - y)\sigma + \frac{\partial f}{\partial z}(x,y,z)(a_3 - z)\sigma.$$

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The above inequality leads us to the inequality

$$
\text{(2.9)} \quad v \iint_S f(x,y,z)\,dS \geq \sigma \iiint_D f(x,y,z)\,dxdydz
$$

$$
+ \sigma \iiint_D \left[\frac{\partial}{\partial x}((a_1 - x)f(x,y,z)) + \frac{\partial}{\partial y}((a_2 - y)f(x,y,z)) \right.
$$

$$
\left. + \frac{\partial}{\partial z}((a_3 - z)f(x,y,z)) \right] dxdydz
$$

$$
+ 3\sigma \iiint_D f(x,y,z)\,dxdydz.
$$

Using the Gauss-Ostrogradsky' theorem we obtain the equality

$$
\text{(2.10)} \quad \iiint_D \left[\frac{\partial}{\partial x}((a_1 - x)f(x,y,z) + \frac{\partial}{\partial y}((a_2 - y)f(x,y,z)) \right.
$$

$$
\left. + \frac{\partial}{\partial z}((a_3 - z)f(z,y,z)) \right] dxdydz
$$

$$
= \iint_S [(a_1 - x)\cos\alpha + (a_2 - y)\cos\beta + (a_3 - z)\cos\gamma]f(x,y,z)\,dS.
$$

From the relations (2.9) and (2.10) we obtain the inequality (2.4). The in-
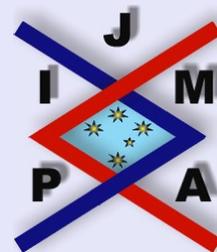
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equality (2.6) is the inequality (2.4) for the functional

$$A(f) = \frac{\iiint_D f(x, y, z)\, dxdydz}{\iiint_D dxdydz}.$$

□

Remark 2.3. For $D = \overline{B}(C, R)$ we have

$$(a_1, a_2, a_3) = C$$

and

$$\cos\alpha = \frac{x - a_1}{R}, \quad \cos\beta = \frac{y - a_2}{R}, \quad \cos\gamma = \frac{z - a_3}{R}.$$

In this case the inequality (2.4) becomes

$$\sigma \iiint_{\overline{B}(C,R)} f(x, y, z)\, dxdydz \le v \iint_{S(C,R)} f(x, y, z)\, d\sigma.$$
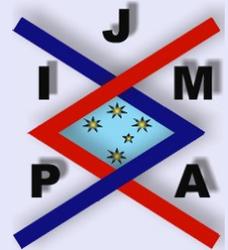
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