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BEST GENERALIZATION OF A HILBERT TYPE INEQUALITY

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ABSTRACT. By introducing a parameter λ , we have given generalization of Hilbert's type integral inequality with a best possible constant factor. Also its equivalent form is considered, and the generalized formula corresponding to the double series inequalities are built.

Key words and phrases: Hilbert's type integral inequality; Weight coefficient; Weight function; Hölder's inequality.

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1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$, satisfy $0 < \int_0^\infty f^p(t)dt < \infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left[\int_0^\infty f^p(t)dt\right]^{\frac{1}{p}} \left[\int_0^\infty g^q(t)dt\right]^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\pi/(\sin \pi/p)$ is the best possible. Inequality (1.1) is called Hardy-Hilbert's inequality (see [1]) and is important in analysis and applications (cf. Mitrinović et al. [2]). Recently, Yang gave an extension of (1.1) as (see [4, 5]):

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \\ < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left[\int_0^\infty t^{1-\lambda} f^p(t)dt\right]^{\frac{1}{p}} \left[\int_0^\infty t^{1-\lambda} g^q(t)dt\right]^{\frac{1}{q}}, \end{aligned} \tag{1.2}$$

where the constant factor $B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ $(\lambda > 2 - \min\{p, q\})$ is the best possible, and $B(u, v)$ is the β function. Hardy et al. [1] gave an inequality similar to (1.1) as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left[\int_0^\infty f^p(t)dt\right]^{\frac{1}{p}} \left[\int_0^\infty g^q(t)dt\right]^{\frac{1}{q}}, \tag{1.3}$$

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where the constant factor pq is the best possible. The double series inequality is:

(1.4) $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} b_m^q\right)^{\frac{1}{q}},$$

where the constant factor pq is the best possible. In particular, if $p = q = 2$, one has the Hilbert type integral inequality:

(1.5) $$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x,y\}}dxdy < 4\left[\int_0^{\infty} f^2(t)dt\int_0^{\infty} g^2(t)dt\right]^{\frac{1}{2}}.$$

Recently, Kuang gave an extension of (1.4) as (see [3]):

(1.6) $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left(\sum_{n=1}^{\infty}[pq - G(p,n)]a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}[pq - G(q,n)]b_m^p\right)^{\frac{1}{q}},$$

where $G(r,n) = \frac{r+1/3r-4/3}{(2n+1)^{1/r}} > 0$ $(r = p,q)$. Yang and Debnath have also considered other Hilbert type integral inequalities in [6].

The main objective of this paper is to build a new inequality with a best constant factor, related to the double integral $\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x^{\lambda},y^{\lambda}\}}dxdy$, which improves inequality (1.5). The equivalent form and the corresponding double series form are considered.

2. MAIN RESULTS

Theorem 2.1. *If* $\lambda > 0$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f,\ g \geq 0$ *such that* $0 < \int_0^{\infty} t^{p-1-\lambda} f^p(t)dt < \infty$, $0 < \int_0^{\infty} t^{q-1-\lambda} g^q(t)dt < \infty$, *then one has*

(2.1) $$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x^{\lambda},y^{\lambda}\}}dxdy < \frac{pq}{\lambda}\left[\int_0^{\infty} t^{p-1-\lambda}f^p(t)dt\right]^{\frac{1}{p}}\left[\int_0^{\infty} t^{q-1-\lambda}g^q(t)dt\right]^{\frac{1}{q}}$$

and

(2.2) $$\int_0^{\infty} y^{\lambda(p-1)-1}\left(\int_0^{\infty}\frac{f(x)}{\max\{x^{\lambda},y^{\lambda}\}}dx\right)^p < \left(\frac{pq}{\lambda}\right)^p\int_0^{\infty} x^{p-1-\lambda}f^p(x)dx,$$

where the constant factors $\frac{pq}{\lambda}$, $\left(\frac{pq}{\lambda}\right)^p$ *are the best possible. Inequality (2.1) is equivalent to (2.2). In particular, for* $\lambda = 1$, *(2.1) and (2.2) respectively reduce to the following two equivalent inequalities:*

(2.3) $$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq\left[\int_0^{\infty} t^{p-2}f^p(t)dt\right]^{\frac{1}{p}}\left[\int_0^{\infty} t^{q-2}f^q(t)dt\right]^{\frac{1}{q}}$$

and

(2.4) $$\int_0^{\infty} y^{p-2}\left(\int_0^{\infty}\frac{f(x)}{\max\{x,y\}}dx\right)^p dy < (pq)^p\int_0^{\infty} x^{p-2}f^p(x)dx.$$

Proof. By Hölder's inequality, one has

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy = \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(\max\{x^\lambda, y^\lambda\})^{\frac{1}{p}}} \left(\frac{y}{x}\right)^{\frac{\lambda}{pq}-\frac{1}{p}} x^{\frac{1}{q}-\frac{1}{p}} \right.$$
$$\left. \times \frac{g(y)}{(\max\{x^\lambda, y^\lambda\})^{\frac{1}{q}}} \left(\frac{x}{y}\right)^{\frac{\lambda}{pq}-\frac{1}{q}} y^{\frac{1}{p}-\frac{1}{q}} \right] dxdy$$
$$\leq \left\{ \int_0^\infty \int_0^\infty \frac{f^p(x)}{\max\{x^\lambda, y^\lambda\}} \left(\frac{y}{x}\right)^{\frac{\lambda}{q}-1} x^{\frac{p}{q}-1} dxdy \right\}^{\frac{1}{p}}$$
$$\times \left\{ \int_0^\infty \int_0^\infty \frac{g^q(y)}{\max\{x^\lambda, y^\lambda\}} \left(\frac{x}{y}\right)^{\frac{\lambda}{p}-1} y^{\frac{q}{p}-1} dxdy \right\}^{\frac{1}{q}}. \tag{2.5}$$

Equality holds in (2.5) if there are two constants A, B, such that $A^2 + B^2 \neq 0$ and

$$A \frac{f^p(x)}{\max\{x^\lambda, y^\lambda\}} \left(\frac{y}{x}\right)^{\frac{\lambda}{q}-1} x^{\frac{p}{q}-1} = B \frac{g^q(y)}{\max\{x^\lambda, y^\lambda\}} \left(\frac{x}{y}\right)^{\frac{\lambda}{p}-1} y^{\frac{q}{p}-1}$$

a.e. in $(0, \infty) \times (0, \infty)$, or $Ax^{p-\lambda} f^p(x) = By^{q-\lambda} g^q(y) =$constant a.e. in $(0, \infty) \times (0, \infty)$, this contradicts the fact that $0 < \int_0^\infty t^{p-1-\lambda} f^p(t)dt < \infty$. Thus the inequality (2.5) is strict.

Define the weight function $w_\lambda(r, t)$ as:

$$w_\lambda(r, t) = t^{\lambda-1} \int_0^\infty \frac{1}{\max\{t^\lambda, u^\lambda\}} \left(\frac{u}{t}\right)^{\frac{\lambda}{r}-1} du, \qquad r = p, q; t \in (0, \infty).$$

By computing, one has:

$$w_\lambda(p, t) = \frac{pq}{\lambda} = w_\lambda(q, t), \tag{2.6}$$

and we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy$$
$$< \left[\int_0^\infty w_\lambda(q, t) t^{p-1-\lambda} f^p(t) dt \right]^{\frac{1}{p}} \left[\int_0^\infty w_\lambda(p, t) t^{q-1-\lambda} f^q(t) dt \right]^{\frac{1}{q}}$$
$$= \frac{pq}{\lambda} \left[\int_0^\infty t^{p-1-\lambda} f^p(t) dt \right]^{\frac{1}{p}} \left[\int_0^\infty t^{q-1-\lambda} f^q(t) dt \right]^{\frac{1}{q}}.$$

For $0 < \varepsilon < \lambda$, setting $\widetilde{f}(t), \widetilde{g}(t)$ as: $t \in (0, 1)$, $\widetilde{f}(t) = \widetilde{g}(t) = 0$; $t \in [1, \infty)$, $\widetilde{f}(t) = t^{\frac{\lambda-p-\varepsilon}{p}}$, $\widetilde{g}(t) = t^{\frac{\lambda-q-\varepsilon}{q}}$

$$\int_0^\infty \int_0^\infty \frac{\widetilde{f}(x)\widetilde{g}(y)}{\max\{x^\lambda, y^\lambda\}} dxdy \tag{2.7}$$
$$= \int_1^\infty y^{\frac{\lambda-q-\varepsilon}{q}} \left[\int_1^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} x^{\frac{\lambda-p-\varepsilon}{p}} dx \right] dy$$
$$= \int_1^\infty y^{\frac{\lambda-q-\varepsilon}{q}} \left[\int_1^y \frac{1}{y^\lambda} x^{\frac{\lambda-p-\varepsilon}{p}} dx \right] dy + \int_1^\infty y^{\frac{\lambda-q-\varepsilon}{q}} \left[\int_y^\infty \frac{1}{x^\lambda} x^{\frac{\lambda-p-\varepsilon}{p}} dx \right] dy$$

$$= \frac{p}{\lambda - \varepsilon}\left(\frac{1}{\varepsilon} + \frac{q}{\lambda - \lambda q - \varepsilon}\right) + \frac{1}{\varepsilon} \cdot \frac{p}{\lambda p - \lambda + \varepsilon}$$
$$= \frac{1}{\varepsilon}\left(\frac{p}{\lambda - \varepsilon} + \frac{p}{\lambda p - \lambda + \varepsilon}\right) - \frac{pq}{(\lambda - \varepsilon)(\lambda q - \lambda + \varepsilon)}.$$

On the other hand,

$$\left[\int_0^\infty t^{p-1-\lambda} \widetilde{f}^p(t)dt\right]^{\frac{1}{p}} \left[\int_0^\infty t^{q-1-\lambda} \widetilde{g}^q(t)dt\right]^{\frac{1}{q}} = \frac{1}{\varepsilon}. \tag{2.8}$$

If the constant factor $\frac{pq}{\lambda}$ in (2.1) is not the best possible, then there exists a positive number k (with $k < \frac{pq}{\lambda}$), such that (2.1) is still valid if one replaces $\frac{pq}{\lambda}$ by k. By (2.7) and (2.8), one has:

$$\begin{aligned} &\frac{p}{\lambda - \varepsilon} + \frac{p}{\lambda p - \lambda + \varepsilon} - \frac{pq\varepsilon}{(\lambda - \varepsilon)(\lambda q - \lambda + \varepsilon)} \\ &= \varepsilon \int_0^\infty \int_0^\infty \frac{\widetilde{f}(x)\widetilde{f}(y)}{\max\{x^\lambda, y^\lambda\}} dxdy \\ &< \varepsilon k \left[\int_0^\infty t^{p-1-\lambda} \widetilde{f}^p(t)dt\right]^{\frac{1}{p}} \left[\int_0^\infty t^{q-1-\lambda} \widetilde{g}^q(t)dt\right]^{\frac{1}{q}} \\ &= k. \end{aligned} \tag{2.9}$$

Setting $\varepsilon \to 0^+$, then $\frac{p}{\lambda} + \frac{p}{(p-1)\lambda} \leq k$ or $\frac{pq}{\lambda} \leq k$. By this contradiction we can conclude that the constant factor $\frac{pq}{\lambda}$ in (2.1) is the best possible.

Define the weight function $g(y)$ as:

$$g(y) = y^{\lambda(p-1)-1} \left[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}} dx\right]^{p-1}, \qquad y \in (0, \infty).$$

By (2.1), one has:

$$\begin{aligned} 0 &< \left[\int_0^\infty y^{q-1-\lambda} g^q(y)dy\right]^p \\ &= \left\{\int_0^\infty y^{\lambda(p-1)-1} \left[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}}\right]^p dy\right\}^p \\ &= \left[\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy\right]^p \\ &\leq \left(\frac{pq}{\lambda}\right)^p \int_0^\infty x^{p-1-\lambda} f^p(x)dx \left[\int_0^\infty y^{q-1-\lambda} g^q(y)dy\right]^{p-1}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} 0 &< \int_0^\infty y^{q-1-\lambda} g^q(y)dy \\ &= \int_0^\infty y^{\lambda(p-1)-1} \left[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}} dx\right]^p dy \\ &\leq \left(\frac{pq}{\lambda}\right)^p \int_0^\infty x^{p-1-\lambda} f^p(x)dx < \infty. \end{aligned} \tag{2.11}$$

Hence by using (2.1), (2.10) takes the form of strict inequality; so does (2.11). One then has (2.2). On other hand, if (2.2) holds, by Hölder's inequality, one has

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dxdy &= \int_0^\infty \left[y^{\frac{\lambda+1-q}{q}} \int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}} dx \right] \left[y^{\frac{q-1-\lambda}{q}} g(y) \right] dy \\ &\le \left\{ \int_0^\infty y^{\lambda(p-1)-1} \left[\int_0^\infty \frac{f(x)}{\max\{x^\lambda, y^\lambda\}} \right]^p \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q-1-\lambda} g^q(y) dy \right\}^{\frac{1}{q}} \end{aligned} \tag{2.12}$$

By (2.2), we have (2.1).

If the constant factor in(2.2) is not the best possible, we may show that the constant factor in (2.1) is not the best possible by using (2.12). This is a contradiction. Hence the constant factor in (2.2) is the best possible. Inequality (2.1) is equivalent to (2.2). Thus the theorem is proved. □

Remark 2.2. For $p = q = 2$, (2.3) reduces to (1.5); (2.1), (2.3) are generalizations of (1.5), but (2.1) is not a generalization of (1.3).

Theorem 2.3. *If* $p > 1$, $1/p + 1/q = 1$, $0 < \lambda \le \min\{p, q\}$, $a_n \ge 0$, $b_n \ge 0$ *such that* $0 < \sum_{n=1}^\infty n^{p-1-\lambda} a_n^p < \infty$, $0 < \sum_{n=1}^\infty n^{q-1-\lambda} a_n^q < \infty$, *one has:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \frac{pq}{\lambda} \left(\sum_{n=1}^\infty n^{p-1-\lambda} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{q-1-\lambda} b_n^q \right)^{\frac{1}{q}} \tag{2.13}$$

and

$$\sum_{n=1}^\infty n^{\lambda(p-1)-1} \left(\sum_{m=1}^\infty \frac{a_m}{\max\{m^\lambda, n^\lambda\}} \right)^p < \left(\frac{pq}{\lambda} \right)^p \sum_{n=1}^\infty n^{p-1-\lambda} a_n^p, \tag{2.14}$$

where the constant factors $\frac{pq}{\lambda}$, $\left(\frac{pq}{\lambda}\right)^p$ *are the best possible. Inequality* (2.13) *is equivalent to* (2.14).

In particular, for $\lambda = 1$, (2.13) *and* (2.14) *respectively reduce to the following two equivalent inequalities:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m, n\}} < pq \left(\sum_{n=1}^\infty n^{p-2} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{q-2} b_n^q \right)^{\frac{1}{q}} \tag{2.15}$$

and

$$\sum_{n=1}^\infty n^{p-2} \left(\sum_{m=1}^\infty \frac{a_m}{\max\{m, n\}} \right)^p < (pq)^p \sum_{n=1}^\infty n^{p-2} a_n^p. \tag{2.16}$$

Proof. Define the weight coefficient $\widetilde{w}_\lambda(\lambda, n)$ as:

$$\widetilde{w}_\lambda(r, n) = n^{\lambda-1} \sum_{m=1}^\infty \frac{1}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n} \right)^{\frac{\lambda}{r}-1}, \qquad r = p, q;\ n \in \mathbb{N}. \tag{2.17}$$

Since $0 < \lambda \le \min\{p,q\}$, we have:

$$\widetilde{w}_\lambda(r,n) < n^{\lambda-1}\sum_{m=1}^{\infty}\int_{m-1}^{m}\frac{1}{\max\{u^\lambda,n^\lambda\}}\left(\frac{u}{n}\right)^{\frac{\lambda}{r}-1}du \tag{2.18}$$
$$= n^{\lambda-1}\int_0^\infty \frac{1}{\max\{u^\lambda,n^\lambda\}}\left(\frac{u}{n}\right)^{\frac{\lambda}{r}-1}du$$
$$= w_\lambda(r,n) = \frac{pq}{\lambda}, \quad r = p,q$$

By Hölder's inequality and (2.17), following the method of proof in Theorem 2.1, one has:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} < \left(\sum_{n=1}^{\infty}\widetilde{w}_\lambda(q,n)n^{p-1-\lambda}a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\widetilde{w}_\lambda(p,n)n^{q-1-\lambda}a_n^p\right)^{\frac{1}{q}}$$

By (2.18) we have (2.13), for $0 < \varepsilon < \lambda$, setting $\widetilde{a}_n = n^{\frac{\lambda-p-\varepsilon}{p}}$, $\widetilde{b}_n = n^{\frac{\lambda-q-\varepsilon}{q}}$, $n \in \mathbb{N}$. Since $0 < \lambda \le \min\{p,q\}$, by (2.9), we get:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\max\{m^\lambda,n^\lambda\}} > \int_1^\infty\int_1^\infty\frac{\widetilde{f}(x)\widetilde{g}(y)}{\max\{x^\lambda,y^\lambda\}}dxdy$$
$$= \frac{1}{\varepsilon}\left(\frac{p}{\lambda-\varepsilon}+\frac{p}{\lambda p-\lambda+\varepsilon}\right) - \frac{pq}{(\lambda-\varepsilon)(\lambda q-\lambda+\varepsilon)}.$$

On other hand,

$$\left(\sum_{n=1}^{\infty}n^{p-1-\lambda}\widetilde{a}_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}n^{q-1-\lambda}\widetilde{b}_n^q\right)^{\frac{1}{q}} = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < 1+\frac{1}{\varepsilon}.$$

By using the above inequalities and the method of proof in Theorem 2.1, we may show that the constant factor in (2.13) is the best possible.

Setting b_n as:

$$b_n = n^{\lambda(p-1)-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right)^{p-1},$$

we obtain:

$$\sum_{n=1}^{\infty}n^{q-1-\lambda}b_n^q = \sum_{n=1}^{\infty}n^{\lambda(p-1)-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\right)^p$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}.$$

By (2.13) and using the same method of Theorem 2.1, we have (2.14). We may show that the constant factor in (2.14)is the best possible, and inequality (2.13) is equivalent to (2.14). □

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