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COEFFICIENT INEQUALITIES FOR CERTAIN CLASSES OF MEROMORPHICALLY STARLIKE AND MEROMORPHICALLY CONVEX FUNCTIONS

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ABSTRACT. Let Σ_r be the class of meromorphic functions $f(z)$ in $\mathbb{D}_r$ with a simple pole at the origin. Two subclasses $T_r^*(\alpha)$ and $\mathcal{C}_r(\alpha)$ of Σ_r are considered. Some coefficient properties of functions $f(z)$ to be in the classes $T_r^*(\alpha)$ and $\mathcal{C}_r(\alpha)$ of Σ_r are discussed. Also, the starlikeness and the convexity of functions $f(z)$ in Σ_r are discussed.

Key words and phrases: Meromorphic functions, Univalent functions, Starlike functions, Convex functions.

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1. INTRODUCTION

Let Σ_r denote the class of functions $f(z)$ of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured disk $\mathbb{D}_r = \{z \in \mathbb{C} : 0 < |z| < r \leqq 1\}$. A function $f(z) \in \Sigma_r$ is said to be starlike of order α if it satisfies the inequality:

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{D}_r) \tag{1.2}$$

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for some $\alpha\,(0 \leqq \alpha < 1)$. We say that $f(z)$ is in the class $\mathcal{T}_r^*(\alpha)$ for such functions. A function $f(z) \in \Sigma_r$ is said to be convex of order α if it satisfies the inequality:

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \qquad (z \in \mathbb{D}_r) \tag{1.3}$$

for some $\alpha\,(0 \leqq \alpha < 1)$. We say that $f(z)$ is in the class $\mathcal{C}_r(\alpha)$ if it is convex of order α in $\mathbb{D}_r$. We note that $f(z) \in \mathcal{C}_r(\alpha)$ if and only if $-zf'(z) \in \mathcal{T}_r^*(\alpha)$. There are many papers discussing various properties of classes consisting of univalent, starlike, convex, multivalent, and meromorphic functions in the book by Srivastava and Owa [3].

Ozaki [2] has shown that the necessary and sufficient condition that $f(z) \in \Sigma_r$ with $a_n \geqq 0$ $(n = 1, 2, 3, \cdots)$ is meromorphic and univalent in $\mathbb{D}_r$ is that there should exist the relation:

$$\sum_{n=1}^{\infty} n a_n r^{n+1} \leqq 1$$

between its coefficients.

Our results in the present paper are an improvement and extension of the above theorem by Ozaki [2].

2. COEFFICIENT INEQUALITIES FOR FUNCTIONS

Our first result for the functions $f(z) \in \Sigma_r$ is contained in

Theorem 2.1. *If $f(z) \in \Sigma_r$ satisfies*

$$\sum_{n=0}^{\infty}(n+k+|2\alpha+n-k|)|a_n|r^{n+1} \leqq 2(1-\alpha) \tag{2.1}$$

for some α $(0 \leqq \alpha < 1)$ and k $(\alpha < k \leqq 1)$, then $f(z) \in \mathcal{T}_r^*(\alpha)$.

Proof. For $f(z) \in \Sigma_r$, we know that

$$\begin{aligned}
&|zf'(z)+kf(z)| - |zf'(z)+(2\alpha-k)f(z)| \\
&\qquad = \left|(k-1)\frac{1}{z}+\sum_{n=0}^{\infty}(n+k)a_n z^n\right| - \left|(2\alpha-k-1)\frac{1}{z}+\sum_{n=0}^{\infty}(2\alpha+n-k)a_n z^n\right|.
\end{aligned}$$

Therefore, applying the condition of the theorem, we have

$$\begin{aligned}
&r\,|zf'(z)+kf(z)| - r\,|zf'(z)+(2\alpha-k)f(z)| \\
&\quad \leqq (k-1)+\sum_{n=0}^{\infty}(n+k)|a_n|r^{n+1} - (k+1-2\alpha) + \sum_{n=0}^{\infty}|2\alpha+n-k||a_n|r^{n+1} \\
&\quad = 2(\alpha-1)+\sum_{n=0}^{\infty}(n+k+|2\alpha+n-k|)|a_n|r^{n+1} \\
&\quad \leqq 0,
\end{aligned}$$

which shows that

$$\sum_{n=0}^{\infty}(n+k+|2\alpha+n-k|)|a_n|r^{n+1} \leqq 2(1-\alpha).$$

It follows from the above that

$$\left|\frac{zf'(z)+kf(z)}{zf'(z)+(2\alpha-k)f(z)}\right| \leqq 1,$$

so that

$$\mathrm{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}_r).$$

□

Letting $k = 0$ in Theorem 2.1, we have

Corollary 2.2. *If $f(z) \in \Sigma_r$ satisfies*

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1} \leqq 1-\alpha \tag{2.2}$$

for some α ($\frac{1}{2} \leqq \alpha < 1$), then $f(z) \in \mathcal{T}_r^*(\alpha)$.

Theorem 2.1 gives us the following results.

Corollary 2.3. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n = |a_n|e^{-\frac{n+1}{2\pi}i}$, then $f(z) \in \mathcal{T}_r^*(\alpha)$ if and only if*

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1} \leqq 1-\alpha \tag{2.3}$$

for some α ($\frac{1}{2} \leqq \alpha < 1$).

Proof. In view of Theorem 2.1, we see that if the coefficient inequality (2.3) holds true for some α ($\frac{1}{2} \leqq \alpha < 1$), then $f(z) \in \mathcal{T}_r^*(\alpha)$.

Conversely, let $f(z)$ be in the class $\mathcal{T}_r^*(\alpha)$, then

$$\mathrm{Re}\left(-\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1-\sum_{n=0}^{\infty} na_n z^{n+1}}{1+\sum_{n=0}^{\infty} a_n z^{n+1}}\right) > \alpha$$

for all $z \in \mathbb{D}_r$. Letting $z = re^{\frac{1}{2\pi}i}$, we have that $a_n z^{n+1} = |a_n|r^{n+1}$. This implies that

$$1 - \sum_{n=0}^{\infty} n|a_n|r^{n+1} \geqq \alpha\left(1+\sum_{n=0}^{\infty}|a_n|r^{n+1}\right),$$

which is equivalent to (2.3). □

Example 2.1. The function $f(z)$ given by

$$f(z) = \frac{1}{z} + a_0 + \left(\frac{1-\alpha-\alpha|a_0|}{n+\alpha}\right)e^{i\theta}z^n$$

belongs to the class $\mathcal{T}_r^*(\alpha)$ for some real θ with $\frac{1}{2} \leqq \alpha \leqq \frac{1}{1+|a_0|} < 1$.

Remark 2.4. If $f(z) \in \Sigma_r$ with $a_0 = 0$, then Corollary 2.3 holds true for some α $(0 \leqq \alpha < 1)$.

Corollary 2.5. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n \geqq 0$, then $f(z) \in \mathcal{T}_r^*(\alpha)$ if and only if*

$$\sum_{n=0}^{\infty}(n+\alpha)a_n r^{n+1} \leqq 1-\alpha$$

for some α ($\frac{1}{2} \leqq \alpha < 1$).

Remark 2.6. If $f(z) \in \Sigma_r$ with $a_0 = 0$, then Corollary 2.5 holds true for $0 \leqq \alpha < 1$.

Remark 2.7. Juneja and Reddy [1] have given that $f(z) \in \Sigma_1$ with $a_0 = 0$ and $a_n \geqq 0$ belongs to the class $\mathcal{T}_1^*(\alpha)$ if and only if

$$\sum_{n=1}^{\infty}(n+\alpha)a_n \leqq 1-\alpha.$$

Theorem 2.8. *If $f(z) \in \Sigma_r$ satisfies*

$$\sum_{n=1}^{\infty} n(n+\alpha)|a_n|r^{n+1} \leqq 1-\alpha \tag{2.4}$$

for some $\alpha\,(0 \leqq \alpha < 1)$, then $f(z)$ belongs to the class $\mathcal{C}_r(\alpha)$.

Proof. Noting that $f(z) \in \mathcal{C}_r(\alpha)$ if and only if $-zf'(z) \in \mathcal{T}_r^*(\alpha)$, and that

$$-zf'(z) = \frac{1}{z} - \sum_{n=1}^{\infty} n a_n z^n,$$

we complete the proof of the theorem with the aid of Theorem 2.1. □

Corollary 2.9. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n = |a_n|e^{-\frac{n+1}{2\pi}i}$, then $f(z) \in \mathcal{C}_r(\alpha)$ if and only if the inequality (2.4) holds true for some $\alpha\,(0 \leqq \alpha < 1)$.*

Example 2.2. The function $f(z)$ given by

$$f(z) = \frac{1}{z} + a_0 + \left(\frac{1-\alpha}{n(n+\alpha)}\right) e^{i\theta} z^n$$

belongs to the class $\mathcal{C}_r(\alpha)$ for some real θ with $0 \leqq \alpha < 1$.

Corollary 2.10. *Let the function $f(z) \in \Sigma_r$ be given by (1.1) with $a_n \geqq 0$, then $f(z) \in \mathcal{C}_r(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} n(n+\alpha)a_n r^{n+1} \leqq 1-\alpha \tag{2.5}$$

for some $\alpha\,(0 \leqq \alpha < 1)$.

3. STARLIKENESS AND CONVEXITY OF FUNCTIONS

We consider the radius problems for starlikeness and convexity of functions $f(z)$ belonging to the class Σ_r.

Theorem 3.1. *A function $f(z) \in \Sigma_r$ belongs to the class $\mathcal{T}_r^*(\alpha)$ for $0 \leqq r < r_0$, where r_0 is the smallest positive root of the equation*

$$\alpha|a_0|r^3 - (\delta+1-\alpha)r^2 - \alpha|a_0|r + 1 - \alpha = 0, \tag{3.1}$$

and

$$\delta = \sqrt{\sum_{n=1}^{\infty} n|a_n|^2} + \alpha\sqrt{\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^2}. \tag{3.2}$$

Proof. Using the Cauchy inequality, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1} \\ &= \alpha|a_0|r + \sum_{n=1}^{\infty}|a_n|r^{n+1} \\ &\leqq \alpha|a_0|r + \sqrt{\sum_{n=1}^{\infty} n|a_n|^2}\sqrt{\sum_{n=1}^{\infty} nr^{2n+2}} + \alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}\sqrt{\sum_{n=1}^{\infty} nr^{2n+2}} \\ &= \alpha|a_0|r + \sqrt{\frac{r^4}{(1-r^2)^2}}\left(\sqrt{\sum_{n=1}^{\infty} n|a_n|^2} + \alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}\right) \\ &= \alpha|a_0|r + \frac{r^2}{1-r^2}\delta < 1-\alpha,\end{aligned}$$

where δ is given by (3.2). Therefore, an application of Corollary 2.2 gives us that $f(z) \in \mathcal{T}_r^*(\alpha)$ for $0 \leqq r < r_0$. □

Letting $a_0 = 0$ in Theorem 3.1, we have

Corollary 3.2. *A function $f(z) \in \Sigma_r$ with $a_0 = 0$ belongs to the class $\mathcal{T}_r^*(\alpha)$ for $0 \leqq r < r_0$, where*

$$r_0 = \sqrt{1 - \frac{\delta}{\delta + 1 - \alpha}}$$

and δ is given by (3.2).

Example 3.1. If we consider the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}e^{i\theta_n}z^n \qquad (\theta_n \text{ is real}),$$

then $f(z) \in \mathcal{T}_r^*(\alpha)$ for $0 \leqq r < r_0$ with

$$\begin{aligned}\delta &= \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}} + \alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}} \\ &= \sqrt{\zeta(2)} + \alpha\sqrt{\zeta(4)} \\ &= \pi\left(\frac{1}{\sqrt{6}} + \frac{\pi\alpha}{3\sqrt{10}}\right).\end{aligned}$$

Further, letting $\alpha = 0$, we have that

$$\delta = \frac{\pi}{\sqrt{6}} \approxeq 1.282550$$

and

$$r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6}+\pi}} \approxeq 0.661896.$$

Finally, for convexity of functions $f(z)$, we derive

Theorem 3.3. *A function* $f(z) \in \Sigma_r$ *belongs to the class* $\mathcal{C}_r(\alpha)$ *for* $0 \leqq r < r_1$*, where*

$$r_1 = \sqrt{1 - \frac{\sigma}{\sigma + 1 - \alpha}}$$

and

$$\sigma = \sqrt{\sum_{n=1}^{\infty} n^3 |a_n|^2} + \alpha \sqrt{\sum_{n=1}^{\infty} n |a_n|^2}.$$

Example 3.2. Let us consider the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} e^{i\theta_n} z^n \qquad (\theta_n \text{ is real}).$$

We see that $f(z) \in \mathcal{C}_r(\alpha)$ for $0 \leqq r < r_0$ with

$$\delta = \pi \left(\frac{1}{\sqrt{6}} + \frac{\pi\alpha}{3\sqrt{10}} \right).$$

Taking $\alpha = 0$, we obtain

$$\delta = \frac{\pi}{\sqrt{6}} \approxeq 1.282550$$

and

$$r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6} + \pi}} \approxeq 0.661896.$$

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