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AN INEQUALITY FOR LINEAR POSITIVE FUNCTIONALS

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ABSTRACT. Using P_0-simple functionals, we generalise the result from Theorem 1.1 obtained by Professor F. Qi (F. QI, An algebraic inequality, *RGMIA Res. Rep. Coll.*, **2**(1) (1999), article 8).

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1. INTRODUCTION

In [4] Professor Dr. F. Qi proved the following algebraic inequality

Theorem 1.1. *Let $b > a > 0$ and $\delta > 0$ be real numbers, then for any given positive $r \in \mathbb{R}$, we have*

$$\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} > \frac{b}{b+\delta}. \tag{1.1}$$

The lower bound in (1.1) *is the best possible.*

In this paper we will present a generalization of the inequality (1.1).

2. SOME LEMMAS

It is well-known that

$$C[a,b] = \{f : [a,b] \to \mathbb{R};\ f \text{ is continuous on } [a,b]\},$$

and let

$$\omega(f;t) = \sup\{|f(x+h)-f(x)|;\ 0 \le h \le t,\ x, x+h \in [a,b]\}.$$

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The least concave majorant of this modulus with respect to the variable t is given by

$$\widetilde{\omega}(f;t)=\begin{cases}\sup\limits_{0\le x\le t\le y}\dfrac{(t-x)\omega(f;y)+(y-t)\omega(f;t)}{y-x}, & \text{for } 0\le t\le b-a,\\ \omega(f;b-a), & \text{for } t>b-a.\end{cases}$$

Let $I=[a,b]$ be a compact interval of the real axis, S a subspace of $C(I)$ and A a linear functional defined on S. The following definition was given by T. Popoviciu in [3].

Definition 2.1 ([3])**.** A linear functional A defined on the subspace S which contains all polynomials is called P_n-simple for $n\ge -1$ if

(i) $A(e_{n+1})\ne 0$;
(ii) For every $f\in S$ there exist $n+2$ distinct points $t_1,t_2,\dots,t_{n+2}$ in $[a,b]$ such that

$$A(f)=A(e_{n+1})[t_1,t_2,\dots,t_{n+2};f],$$

where $[t_1,t_2,\dots,t_{n+2};f]$ is the divided difference of the function f on the points $t_1,t_2,\dots,t_{n+2}$, and e_{n+1} denotes the monomial of degree $n+1$.

Lemma 2.1 ([2])**.** *Let A be a linear bounded functional, $A:C(I)\to\mathbb{R}$. If A is P_0-simple, then for all $f\in C(I)$ we have*

$$|A(f)|\le\frac{\|A\|}{2}\widetilde{\omega}\left(f;\frac{2A(e_1)}{\|A\|}\right). \tag{2.1}$$

Lemma 2.2 ([2])**.** *Let A be a linear bounded functional, $A:C(I)\to\mathbb{R}$. If $A(e_1)\ne 0$ and the inequality* (2.1) *holds for all $f\in C(I)$, then A is P_0-simple.*

A function $f\in C^{(k)}[a,b]$ is called P_n-nonconcave if the inequality

$$[t_1,t_2,\dots,t_{n+2};f]\ge 0$$

holds for any given $n+2$ points $t_1,t_2,\dots,t_{n+2}\in[a,b]$.

The following result was proved by I. Raşa in [5]:

Lemma 2.3 ([5])**.** *Let k be a natural number such that $0\le k\le n$ and $A:C^{(k)}[a,b]\to\mathbb{R}$ a linear bounded functional, $A\ne 0$, $A(e_i)=0$ for $i=0,1,\dots,n$ such that $A(f)\ge 0$ for every f which belongs to $C^{(k)}[a,b]$ and is P_0-nonconcave. Then A is P_0-simple.*

In [1], S. G. Gal gave the exact formula for the usual modulus of continuity of the nonconcave continuous functions on $[a,b]$. He proved the following result:

Lemma 2.4 ([1])**.** *Let $f\in C[a,b]$ be nonconcave and monotone on $[a,b]$. For any given $t\in(0,b-a)$ we have*

(i) *$\omega(f;t)=f(b)-f(b-t)$ if f is nondecreasing on $[a,b]$;*
(ii) *$\omega(f;t)=f(a)-f(a+t)$ if f is nonincreasing on $[a,b]$.*

3. MAIN RESULTS

Let a,b,d be real numbers such that $a<b<d$. Consider the functions u_b and u_b^* defined on $[a,d]$ by

$$u_b(t)=\begin{cases}1, & t\in[a,b];\\ 0, & t\in(b,d],\end{cases}$$

and

$$u_b^*(t)=\begin{cases}0, & t\in[a,b];\\ 1, & t\in(b,d].\end{cases}$$

It is clear that

$$u_b(t) + u_b^*(t) = 1, \quad t \in [a, d]. \tag{3.1}$$

Let A be a linear positive functional defined on the subspace S containing the functions u_b and u_b^*, which satisfies

(1) $0 < A(u_b) \leq A(e_0)$, $0 < A(u_b^*) \leq A(e_0)$;
(2) The functionals A_1 and A_2 defined by $A_1(f) = A(u_b f)$ and $A_2(f) = A(u_b^* f)$ are well defined for every $f \in C[a, b]$;
(3) $A(e_1)A(u_b) - A(e_0)A(u_b e_1) \neq 0$.

Theorem 3.1. *Let A be a linear positive functional which satisfies conditions 1, 2 and 3 above. Then the functional $B : C[a, d] \to \mathbb{R}$ defined by*

$$B(f) = \frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)} \tag{3.2}$$

is P_0-simple, and

$$\left| \frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)} \right| \leq \frac{A(u_b^*)}{A(e_0)} \widetilde{\omega}(f; t_b), \tag{3.3}$$

where

$$t_b = \frac{A(e_1 u_b^*)}{A(u_b^*)} - \frac{A(e_1 u_b)}{A(u_b)}.$$

Proof. In order to prove that the functional B is P_0-simple, from Lemma 2.3, it is sufficient to verify $B(f) \geq 0$ for every nondecreasing function f on $[a, d]$.

It is easy to see that

$$\begin{aligned} B(f) =& \frac{(A(fu_b) + A(fu_b^*))A(u_b) - A(fu_b)(A(u_b) + A(u_b^*))}{A(e_0)A(u_b)} \\ =& \frac{A(u_b)A(fu_b^*) - A(fu_b)A(u_b^*)}{A(e_0)A(u_b)}. \end{aligned} \tag{3.4}$$

From the definitions of functions u_b and u_b^* and f being nodecreasing, we have

$$\begin{aligned} fu_b^* \geq& f(b)u_b^* \\ -fu_b \geq& -f(b)u_b. \end{aligned} \tag{3.5}$$

Substitution of inequality (3.5) into (3.4) yields $B(f) \geq 0$ for every nondecreasing function $f \in C[a, d]$.

From the equality (3.4) we get

$$\|B\| = \frac{2A(u_b^*)}{A(e_0)} \tag{3.6}$$

and

$$B(e_1) = \frac{A(u_b)A(e_1 u_b^*) - A(e_1 u_b)A(u_b^*)}{A(e_0)A(u_b)}. \tag{3.7}$$

Since the functional B is P_0-simple, from Lemma 2.1, the inequality (3.3) follows. □

Corollary 3.1. *Let $f \in C[a, b]$ be nonconcave and monotone on $[a, b]$ and A a functional defined as in Theorem 3.1, then*

$$\frac{A(f)}{A(e_0)} - \frac{A(u_b f)}{A(u_b)} \leq \frac{A(u_b^*)}{A(e_0)}(f(d) - f(d - t_b)) \tag{3.8}$$

if f is nondecreasing on $[a,d]$, and

$$-\frac{A(f)}{A(e_0)}+\frac{A(u_b f)}{A(u_b)}\leq\frac{A(u_b^*)}{A(e_0)}(f(a)-f(a+t_b)) \tag{3.9}$$

if f is nonincreasing on $[a,d]$.

Proof. From Lemma 2.3 we have

$$\omega(f;t)=f(d)-f(d-t) \tag{3.10}$$

if f is nondecreasing on $[a,d]$, and

$$\omega(f;t)=f(a)-f(a+t) \tag{3.11}$$

if the function f is nonincreasing on $[a,d]$.

The functions $f(d)-f(d-\cdot)$ and $f(a)-f(a+\cdot)$ are concave on $[0,d-a)$ if the function f is a convex function. Since $\widetilde{\omega}(f;\cdot)$ is the least concave majorant of the function ω under above conditions, then we get $\widetilde{\omega}(f;\cdot)=\omega(f;\cdot)$.

Combining (3.10) and (3.11) with Theorem 3.1 leads to inequalities (3.8) and (3.9). □

4. APPLICATIONS

Let a,b and d be positive numbers such that $0<a<b<d$. Consider the functional $A:C[a,d]\to\mathbb{R}$ defined by

$$A(f)=\int_a^d w(t)f(t)dt, \tag{4.1}$$

where $w:(a,d)\to\mathbb{R}$ is a positive weight function.

It is easy to verify that the functional A defined by (4.1) satisfies conditions in Theorem 3.1 and the functional B can be expressed as

$$B(f)=\frac{\int_a^d w(t)f(t)dt}{\int_a^d w(t)f(t)dt}-\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(t)f(t)dt}.$$

Then, from Theorem 3.1, we obtain

Theorem 4.1. *For every $f\in C[a,b]$,*

$$\left|\frac{\int_a^d w(t)f(t)dt}{\int_a^d w(t)f(t)dt}-\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(t)f(t)dt}\right|\leq\frac{\int_b^d w(t)dt}{\int_a^d w(t)dt}\widetilde{\omega}(f;t_b), \tag{4.2}$$

where

$$t_b=\frac{\int_b^d tw(t)dt}{\int_b^d w(t)dt}-\frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}.$$

Corollary 4.1. *Let a,b and c be positive numbers such that $0<a<b<d$. Then we have the following inequalities:*

$$0<\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt-\frac{da}{d-a}\int_a^d\frac{f(t)}{t^2}dt\leq\frac{d-b}{d-a}\cdot\frac{a}{b}(f(a)-f(a+t_b)) \tag{4.3}$$

for every convex and nonincreasing function f on $[a,d]$, where

$$t_b=\frac{bd\ln\frac{d}{b}}{d-b}-\frac{ab\ln\frac{b}{a}}{b-a}.$$

Proof. Taking $w(t)=\frac{1}{t^2}$, $t\in[a,d]$ in Theorem 4.1 produces inequality (4.3). □

Remark 1. Letting $f(t) = \frac{1}{t^r}$, $r > 0$ in inequality (4.3) gives us

$$\frac{b^{r+1} - a^{r+1}}{d^{r+1} - a^{r+1}} \cdot \frac{d-a}{b-a} > \frac{b^r}{d^r} \tag{4.4}$$

and

$$\frac{b^{r+1} - a^{r+1}}{d^{r+1} - a^{r+1}} \cdot \frac{d-a}{b-a} < \frac{b^r}{d^r} + (r+1)(d-b)\left(\frac{b}{a+t_b}\right)^r \frac{(a+t_b)^r - a^r}{d^{r+1} - a^{r+1}} \cdot \frac{a}{b}. \tag{4.5}$$

If we let $d = b+\delta$ in inequality (4.4), inequality (1.1) follows. Thus Theorem 1.1 by Professor Dr. F. Qi in [4] is generalized.

Remark 2. We can obtain some discrete inequalities if we select the functional A of the form

$$A(f) = \sum_{k=1}^{n+m} \lambda_k f(x_k),$$

where x_k, $k = 1, 2, \dots, n+m$, are $n+m$ distinct points such that

$$x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_{n+m},$$

and λ_k, $k = 1, 2, \dots, n+m$, are $n+m$ positive numbers.

Choose the point $b = x_n$, then from Theorem 3.1, we obtain the discrete analogue of Theorem 4.1:

$$\left| \frac{\sum_{k=1}^{n+m} \lambda_k f(x_k)}{\sum_{k=1}^{n+m} \lambda_k} - \frac{\sum_{k=n+1}^{n+m} \lambda_k f(x_k)}{\sum_{k=n+1}^{n+m} \lambda_k} \right| \le \frac{\sum_{k=n+1}^{n+m} \lambda_k}{\sum_{k=1}^{n+m} \lambda_k} \widetilde{\omega}(f; t_b),$$

where

$$t_b = \frac{\sum_{k=n+1}^{n+m} \lambda_k x_k}{\sum_{k=n+1}^{n+m} \lambda_k} - \frac{\sum_{k=1}^{n} \lambda_k x_k}{\sum_{k=n+1}^{n+m} \lambda_k}.$$

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