journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 2, Article 33, 4 pp.

ON A BRUNN-MINKOWSKI THEOREM FOR A GEOMETRIC DOMAIN FUNCTIONAL CONSIDERED BY AVHADIEV

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Received 28 December, 2006; accepted 26 April, 2007
Communicated by D. Hinton

ABSTRACT. Suppose two bounded subsets of $\mathbb{R}^n$ are given. Parametrise the Minkowski combination of these sets by t. The Classical Brunn-Minkowski Theorem asserts that the $1/n$-th power of the volume of the convex combination is a concave function of t. A Brunn-Minkowski-style theorem is established for another geometric domain functional.

Key words and phrases: Brunn-Minkowski, Prekopa-Leindler.

2000 *Mathematics Subject Classification.* 26D15, 52A40.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$. Define

$$I(k,\partial\Omega) = \int_\Omega \operatorname{dist}(z,\partial\Omega)^k d\mu_z \qquad \text{for } k>0. \tag{1.1}$$

Here $\operatorname{dist}(z,\Omega)$ denotes the distance of the point $z\in\Omega$ to the boundary $\partial\Omega$ of Ω. The integration uses the ordinary measure in $\mathbb{R}^n$ and is over all $z\in\Omega$. When $n=2$ and $k=1$ this functional was introduced, in [1], in bounds of the torsional rigidity $P(\Omega)$ of plane domains Ω. See also [10] where the inequalities

$$\frac{I(2,\partial\Omega)}{I(2,\partial B_1)} \le \frac{P(\Omega)}{P(B_1)} \le \frac{128}{3}\,\frac{I(2,\partial\Omega)}{I(2,\partial B_1)} \tag{1.2}$$

are presented. Here B_1 is the unit disk and

$$I(2,\partial B_1) = \frac{\pi}{6} = \frac{|B_1|^2}{6\pi}.$$

This inequality is one of many relating domain functionals such as these: see [9, 2, 7]. As an example, proved in [9], we instance

$$(\dot r(\Omega))^4 \le \frac{P(\Omega)}{P(B_1)} \le \left(\frac{|\Omega|}{|B_1|}\right)^2 \le \left(\frac{|\partial\Omega|}{|\partial B_1|}\right)^4 \tag{1.3}$$

giving bounds for the torsional rigidity in terms of the inner-mapping radius $\dot{r}$, the area $|\Omega|$ and the perimeter $|\partial\Omega|$.

We next define the Minkowski sum of domains by

$$\Omega_0 + \Omega_1 := \{z_0 + z_1 |\ z_0 \in \Omega_0,\ z_1 \in \Omega_1\}\ ,$$

and

$$\Omega(t) := \{(1-t)z_0 + tz_1 |\ z_0 \in \Omega_0,\ z_1 \in \Omega_1\}, \quad 0 \le t \le 1.$$

The classical Brunn-Minkowski Theorem in the plane is that $\sqrt{|\Omega(t)|}$ is a concave function of t for $0 \le t \le 1$, and it also happens that $|\partial\Omega(t)|$ is, for convex Ω, a linear, hence concave, function of t. Given a nonnegative quasiconcave function $f(t)$ for which, with $\alpha > 0$, $f(t)^\alpha$ is concave, we say that f is α-concave. In [3] it was shown that, for convex domains Ω, the torsional rigidity satisfies a Brunn-Minkowski style theorem: specifically $P(\Omega(t))$ is 1/4-concave. Thus inequalities (1.3) show that the 1/4-concave function $P(\Omega(t))$ is sandwiched between the 1/4-concave functions $|\Omega(t)|^2$ and $|\partial\Omega(t)|^4$. In [6] it is shown that the polar moment of inertia $I_c(\Omega(t))$ about the centroid of Ω, for which

$$\left(\frac{|\Omega|}{|B_1|}\right)^2 \le \frac{I_c(\Omega)}{I_c(B_1)} \le \left(\frac{|\partial\Omega|}{|\partial B_1|}\right)^4\ , \tag{1.4}$$

holds, is also 1/4-concave. (The 1/4-concavity of $\dot{r}(\Omega(t))^4$ has also been established by Borell.) In this paper we show that the same 1/4-concavity of the domain functions holds for the quantities in inequalities (1.2). Our main result will be the following.

Theorem 1.1. *Let $\mathcal{K}$ denote the set of convex domains in $\mathbb{R}^n$. For $\Omega_0,\ \Omega_1 \in \mathcal{K}$, $I(k, \partial\Omega(t))$ is $1/(n+k)$-concave in t.*

Our proof is an application of the Prekopa-Leindler inequality, Theorem 2.2 below.

2. PROOFS

The proof will use two little lemmas, Theorems 2.1 and 2.3, and one major theorem, the Prekopa-Leindler Theorem 2.2. None of these three results is new: the new item in this paper is their use.

Theorem 2.1 (Knothe)**.** *Let $0 < t < 1$ and $\Omega_0,\ \Omega_1 \in \mathcal{K}$. With*

$$z_t = (1-t)z_0 + tz_1,$$

we have

$$\operatorname{dist}(z_t, \partial\Omega(t)) \ge (1-t)\operatorname{dist}(z_0, \partial\Omega_0) + t\operatorname{dist}(z_1, \partial\Omega_1). \tag{2.1}$$

Proof. Let $z_t \in \Omega(t)$ be as above. Denote the usual Euclidean norm with $|\cdot|$. Let $w_t \in \partial\Omega(t)$ be a point such that

$$|z_t - w_t| = \operatorname{dist}(z_t, \partial\Omega(t)).$$

Define the direction u by

$$u = \frac{z_t - w_t}{|z_t - w_t|}.$$

Define $v_0 \in \Omega_0$, and $v_1 \in \Omega_1$ as the points on these boundaries which are on the rays, in direction u, from z_0 and z_1 respectively. Thus

$$v_0 = z_0 + |z_0 - v_0|u, \qquad v_1 = z_1 + |z_1 - v_1|u.$$

Now let p be any unit vector perpendicular to u. The preceding definitions give that

$$\langle w_t - ((1-t)v_0 + tv_1), p\rangle = 0,$$

from which, on defining

$$v_t = (1-t)v_0 + tv_1 \text{ we have } w_t = v_t + \eta u.$$

for some number η. Now, we do not know (or care) if v_t is on the boundary of $\Omega(t)$, but we do know that v_t is in the closed set $\overline{\Omega(t)}$. Using the convexity of $D(t)$ we have that v_t is on the ray joining z_t with w_t, and between z_t and w_t. From this,

$$\begin{aligned}\operatorname{dist}(z_t, \partial\Omega(t)) &= |z_t - w_t| \geq |z_t - v_t|, \\ &= (1-t)|z_0 - v_0| + t|z_1 - v_1|, \\ &\geq (1-t)\operatorname{dist}(z_0, \partial\Omega_0) + t\operatorname{dist}(z_1, \partial\Omega_1),\end{aligned}$$

as required. □

Theorem 2.2 (Prekopa-Leindler). *Let $0 < t < 1$ and let f_0, f_1, and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying*

$$h\left((1-t)x + ty\right) \geq f_0(x)^{1-t} f_1(y)^t, \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,dx \geq \left(\int_{\mathbb{R}^n} f_0(x)\,dx\right)^{1-t} \left(\int_{\mathbb{R}^n} f_1(x)\,dx\right)^t. \tag{2.3}$$

For references to proofs, see [5].

Theorem 2.3 (Homogeneity Lemma). *If F is positive and homogeneous of degree 1,*

$$F(s\Omega) = sF(\Omega) \quad \forall s > 0, \Omega\ ,$$

and quasiconcave

$$F(\Omega(t)) \geq \min(F(\Omega(0)), F(\Omega(1))) \quad \forall 0 \leq t \leq 1,\ \forall \Omega_0, \Omega_1 \in \mathcal{K}, \tag{2.4}$$

then it is concave:

$$F(\Omega(t)) \geq (1-t)F(\Omega(0)) + tF(\Omega(1)) \quad \forall 0 \leq t \leq 1\ .$$

Proof. See [5]. Replace Ω_0 by $\Omega_0/F(\Omega_0)$, Ω_1 by $\Omega_1/F(\Omega_1)$. Using the homogeneity of degree 1, and applying (2.4), we have

$$F\left((1-t)\frac{\Omega_0}{F(\Omega_0)} + t\frac{\Omega_1}{F(\Omega_1)}\right) \geq 1\ .$$

With

$$t = \frac{F(\Omega_1)}{F(\Omega_0) + F(\Omega_1)}\ , \qquad \text{so } (1-t) = \frac{F(\Omega_0)}{F(\Omega_0) + F(\Omega_1)}\ ,$$

the last inequality on F becomes

$$F\left(\frac{\Omega_0 + \Omega_1}{F(\Omega_0) + F(\Omega_1)}\right) \geq 1\ .$$

Finally, using the homogeneity we have

$$F(\Omega_0 + \Omega_1) \geq F(\Omega_0) + F(\Omega_1)\ ,$$

and using homogeneity again,

$$F((1-t)\Omega_0 + t\Omega_1) \geq (1-t)F(\Omega_0) + tF(\Omega_1)\ ,$$

as required. □

Proof of the Main Theorem 1.1. Knothe's Lemma 2.1 and the AGM inequality give

$$\operatorname{dist}(z_t, \partial\Omega(t)) \geq \operatorname{dist}(z_0, \partial\Omega_0)^{(1-t)} \operatorname{dist}(z_1, \partial\Omega_1)^t, \tag{2.5}$$

and similarly for any positive k-th power of the distance. Denote the characteristic function of Ω by χ_Ω. A standard argument, as given in [5] for example, establishes that

$$\chi_{\Omega(t)}((1-t)z_0 + tz_1) \geq \chi_{\Omega_0}(z_0)^{1-t}\chi_{\Omega_1}(z_1)^t.$$

So, with

$$h(z) = \operatorname{dist}(z, \partial\Omega(t))\chi_{\Omega(t)}(z),$$
$$f_0(z) = \operatorname{dist}(z, \partial\Omega_0)\chi_{\Omega_0}(z),$$
$$f_1(z) = \operatorname{dist}(z, \partial\Omega_1)\chi_{\Omega_1}(z),$$

the conditions of the Prekopa-Leindler Theorem are satisfied. This gives that $I(k, \partial\Omega(t))$ is log-concave in t. Now define $F(\Omega(t)) := I(k, \partial\Omega(t))^{1/(n+k)}$. The function F is quasiconcave in t (as it inherits the stronger property of logconcavity in t from $I(k, \partial\Omega(t))$). Since $I(k, \partial\Omega(t))$ is homogeneous of degree $n + k$, F is homogeneous of degree 1. The Homogeneity Lemma applied to F yields that $I(k, \partial\Omega(t))$ is $1/(n + k)$-concave. □

ACKNOWLEDGEMENTS

I thank Richard Gardner for calling to my attention that I had omitted Borell's paper [3] from my survey paper [7].

I thank Sever Dragomir for his hospitality during a visit in 2006 to Victoria University, and for continuing his interest, begun in [4], in the elastic torsion problem in convex domains.

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