journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 4, Article 91, 11 pp.

TURÁN INEQUALITIES AND SUBTRACTION-FREE EXPRESSIONS

DAVID A. CARDON AND ADAM RICH
DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
PROVO, UT 84602, USA
cardon@math.byu.edu
URL: http://math.byu.edu/cardon

Received 01 January, 2008; accepted 29 June, 2008
Communicated by Q.I. Rahman

ABSTRACT. By using subtraction-free expressions, we are able to provide a new proof of the Turán inequalities for the Taylor coefficients of a real entire function when the zeros belong to a specified sector.

Key words and phrases: Turán inequalities, Subtraction-free expressions, Riemann zeta function, Zero-free region.

2000 *Mathematics Subject Classification.* 11M26, 30C15, 26D10.

1. INTRODUCTION

In the study of entire functions it is natural to ask whether simple conditions on the Taylor coefficients of a function can be used to determine the location of its zeros. For example, let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) > 1$ be the Riemann zeta function. The meromorphic continuation of $\zeta(s)$ to $\mathbb{C}$ has a simple pole at $s = 1$ and has simple zeros at the negative even integers. The Riemann ξ-function is defined by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Note that $\Gamma(s/2)$ has simple poles at the non-positive even integers. It is relatively straightforward to show that $\xi(s)$ is an entire function satisfying $\xi(s) = \xi(1-s)$ for all complex s and that the zeros of $\xi(s)$ satisfy $0 \leq \mathrm{Re}(s) \leq 1$. The prime number theorem is equivalent to the fact that the zeros of $\xi(s)$ satisfy the strict inequality $0 < \mathrm{Re}(s) < 1$, and the Riemann hypothesis is the conjecture that all of the zeros of $\xi(s)$ are on the line $\mathrm{Re}(s) = 1/2$. The ξ-function has a Taylor series representation

$$\xi(1/2 + iz) = \sum_{k=0}^{\infty} (-1)^k a_k \frac{z^{2k}}{(2k)!},$$

We wish to thank Craven and Csordas for their inspiring papers that helped us to become interested in this topic. Furthermore, we wish to thank the participants in a workshop on Pólya-Schur-Lax problems held at the American Institute of Mathematics in May 2007 for several very helpful conversations regarding this paper. Finally, the reviewer made a number of suggestions for improving the paper.
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where $a_k > 0$ for all k, and it is possible to state inequality conditions on the coefficients a_k in this representation of $\xi(s)$ that are equivalent to the Riemann hypothesis (see for example [5], [6], [7]). However, to date, the verification of such strong conditions has been intractable. Instead, it is reasonable to consider weaker conditions on the a_k that would be necessary should the Riemann hypothesis be true. It is known that a necessary condition for the zeros of $\xi(s)$ to satisfy $\mathrm{Re}(s) = 1/2$ is

$$D_k \equiv (2k+1)a_k^2 - (2k-1)a_{k-1}a_{k+1} \geq 0, \qquad k \geq 1. \tag{1.1}$$

The set of inequalities in (1.1) is an example of a class of inequalities called Turán-type inequalities which we will explain in more detail in §3 and §4.

The Turán inequalities for $\xi(s)$ have been studied by several authors. Matiyasevitch [10] outlined a proof of the positivity of D_k. In [3], Csordas, Norfolk, and Varga gave a complete proof that $D_k > 0$ for all $k \geq 1$. Csordas and Varga improved their earlier proof in [4]. Conrey and Ghosh [1] studied Turán inequalities for certain families of cusp forms. The argument of Csordas and Varga in [4] is based on an integral representation of D_k as

$$D_k = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^{2k}v^{2k}\Phi(u)\Phi(v)\left\{(v^2-u^2)\int_u^v\left(-\frac{\Phi'(t)}{t\,\Phi(t)}\right)' dt\right\} du\,dv, \tag{1.2}$$

where

$$\Phi(u) = \sum_{n=1}^{\infty}\left(4n^4\pi^2 e^{9u/2} - 6n^2\pi e^{5u/2}\right)e^{-n^2\pi e^{2u}}. \tag{1.3}$$

A long, detailed argument shows that the integrand of the innermost integral in (1.2) is positive, proving the Turán inequalities for $\xi(s)$. This important result relies heavily on the representation of $\Phi(u)$ in (1.3), making the generalization to other ξ-functions from number theory difficult.

In this paper, we study the Turán inequalities from a different point of view. Our main result is to represent the Turán inequalities in terms of *subtraction-free* expressions. This allows us to derive, as corollaries, several previously known results. Our method of proof is more combinatorial and algebraic in nature than the previously used analytic method which relied on the Gauss-Lucas theorem about the location of the zeros of the derivative of a polynomial.

2. STATEMENT OF MAIN RESULTS

Let $G(z)$ be a real entire function of genus 0 of the form

$$G(z) = \prod_k (1+\rho_k z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!},$$

where the numbers ρ_k are the negative reciprocal roots of $G(z)$. It is notationally simpler to work with the negative reciprocal roots rather than with the roots themselves. Denote the set of these negative reciprocal roots (with repetitions allowed) as R. The set R may be either infinite or finite, and we are interested in both cases. Since $G(z)$ is a real entire function, if $\rho \in R$, either ρ is real or the complex conjugate $\bar{\rho}$ is also in R. If $0 \leq |\mathrm{Im}(\rho_k)| < \mathrm{Re}(\rho_k)$ for all k, we will show in Theorem 2.2 that the strict Turán inequalities hold for $G(z)$, i.e.,

$$a_n^2 - a_{n-1}a_{n+1} > 0$$

for $1 \leq n \leq |R|$.

The Taylor coefficient a_n, expressed in terms of the negative reciprocal roots, is

$$a_n = n! s_R(n)$$

where $s_R(n)$ is the nth elementary symmetric function formed from the elements of R. That is,

$$s_R(n) = \sum_{i_1 < \cdots < i_n} \rho_{i_1} \cdots \rho_{i_n}$$

where the summation is over all possible strictly increasing lists of indices of length n. The expression $a_n^2 - a_{n-1}a_{n+1}$ becomes

$$a_n^2 - a_{n-1}a_{n+1} = n!(n-1)! \left[n s_R(n)^2 - (n+1) s_R(n-1) s_R(n+1) \right]$$

which we wish to be positive. It will be convenient to define a symmetric function $s_R(n,k)$ related to the elementary symmetric functions $s_R(n)$ that naturally arises when forming products of elementary symmetric functions. Let

$$s_R(n,k) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ k \text{ repetitions}}} \rho_{i_1} \cdots \rho_{i_n} \tag{2.1}$$

where the summation is taken over all lists of indices of the form

$$i_1 \leq i_2 \leq \cdots \leq i_n \tag{2.2}$$

such that k of the values are repeated exactly twice. In other words, k of the relations in (2.2) are equal signs, the remaining $n-1-k$ relations are strict inequalities, and no two consecutive relations are equal signs. Note that

$$s_R(n) = s_R(n,0).$$

We follow the convention that $s_R(m,k) = 0$ whenever its defining summation (2.1) is empty.

Example 2.1. If $A = \{\rho_1, \rho_2, \rho_3\}$, then

$$s_A(3,1) = \rho_1^2\rho_2 + \rho_1^2\rho_3 + \rho_1\rho_2^2 + \rho_2^2\rho_3 + \rho_1\rho_3^2 + \rho_2\rho_3^2$$

since the list of all possible ways to write ascending lists of the indices $\{1,2,3\}$ with exactly one repetition is

$$1 = 1 < 2, \quad 1 = 1 < 3, \quad 1 < 2 = 2, \quad 2 = 2 < 3, \quad 1 < 3 = 3, \quad 2 < 3 = 3.$$

The following theorem represents the Turán expression $a_n^2 - a_{n-1}a_{n+1}$ as a linear combination of the symmetric functions $s_R(n,k)$ in which all the coefficients are nonnegative. We refer to such a sum as a *subtraction-free* expression.

Theorem 2.1 (Subtraction-Free Expressions)**.** *The Turán expression $a_n^2 - a_{n-1}a_{n+1}$ may be written in terms of the symmetric functions $s_R(m,k)$ as*

$$a_n^2 - a_{n-1}a_{n+1} = n!(n-1)! \sum_{k=1}^{n} \frac{k}{n+1-k} \binom{2n-2k}{n-k} s_R(2n,k). \tag{2.3}$$

As a consequence of Theorem 2.1, we are able to obtain a new proof of the following previously known result without appealing to the Gauss-Lucas theorem on the location of the roots of the derivative of a polynomial.

Theorem 2.2. *Let $G(z)$ be a real entire function with product and series representations*

$$G(z) = \prod_k (1 + \rho_k z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

and suppose that $0 \leq |\operatorname{Im}(\rho_k)| < \operatorname{Re}(\rho_k)$ for all k. Then, the Turán inequalities

$$a_n^2 - a_{n-1}a_{n+1} > 0$$

hold for all $n \geq 1$ if $G(z)$ has infinitely many roots and for $1 \leq n \leq d$ if $G(z)$ is a polynomial of degree d.

Notice that the hypothesis of Theorem 2.2 requires G to have a genus 0 Weierstrass product which is equivalent to saying that $\sum_k |\rho_k|$ converges. Since the coefficients a_n are real, the non-real zeros of G occur in complex conjugate pairs. The condition $0 \leq |\operatorname{Im}(\rho_k)| < \operatorname{Re}(\rho_k)$ on the negative reciprocal roots of $G(z)$ is the same as saying that all of the zeros of G belong to the wedge shaped region

$$\{z \in \mathbb{C} \,|\, z \neq 0 \text{ and } 3\pi/4 < \arg(z) < 5\pi/4\}.$$

Our main interest is to apply Theorem 2.2 to the entire function $\xi_K(s)$ associated with the Dedekind zeta function $\zeta_K(s)$, where K is a number field. It is known that the function $\xi_K(s)$ is entire, has all zeros in the critical strip $0 \leq \operatorname{Re}(s) \leq 1$, and satisfies the functional equation $\xi_K(s) = \xi_K(1-s)$. For the general theory of the Dedekind ζ-functions and ξ-functions, see [8, Ch.13] or [11, Ch.7]. As a consequence of Theorem 2.2 we are able to deduce the following result about $\xi_K(s)$:

Corollary 2.3. *Let $s = 1/2 + iz$ and write*

$$\xi_K(s) = \xi_K(1/2 + iz) = \sum_{k=0}^{\infty} (-1)^k a_k \frac{z^{2k}}{(2k)!}.$$

If $\xi_K(s)$ has no zeros in the closed triangular region determined by the three points

$$1/2, \quad 1, \quad 1 + \left(\tfrac{1+\sqrt{2}}{2}\right) i,$$

then the (strict) Turán inequalities

$$(2k+1)a_k^2 - (2k-1)a_{k-1}a_{k+1} > 0$$

hold for $k \geq 1$.

The organization of the remainder of this paper is as follows: In §3 we recall several relevant facts about the Turán inequalities. In §4 we discuss how these inequalities are applicable to even real entire functions and to the study of Dedekind zeta functions. Proofs of Theorems 2.1 and 2.2 and Corollary 2.3 are given in §5. For the interested reader, in §6, we outline the original proof of Theorem 2.2 based on the Gauss-Lucas theorem. Finally, in §7 we state several questions for further study.

3. THE LAGUERRE-PÓLYA CLASS AND TURÁN INEQUALITIES

In this section we will review a few facts about the Laguerre-Pólya class and the Turán inequalities.

In the study of real entire functions having only real zeros, it is natural to begin with the simplest case: real polynomials with only real zeros. The set of functions obtained as uniform limits on compact sets of such polynomials is called the Laguerre-Pólya class, denoted $\mathcal{LP}$. It is known (see [9, Ch.8,Thm.3]) that a real entire function $f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ is in $\mathcal{LP}$ if and only if it has a Weierstrass product representation of the form

$$f(z) = cz^n e^{\alpha z - \beta z^2} \prod_k \left(1 - \tfrac{z}{\alpha_k}\right) e^{z/\alpha_k}$$

where $c, \alpha, \beta, \alpha_k \in \mathbb{R}$, $n \in \mathbb{Z}$, $n \geq 0$, $\beta \geq 0$, and $\alpha_k \neq 0$. Note that, if $\beta = 0$, the genus of $f(z)$ is 0 or 1. The subset of $\mathcal{LP}$ such that all the Taylor coefficients satisfy $c_n \geq 0$ is denoted by $\mathcal{LP}^+$. The derivative of the logarithmic derivative of $f(z)$ is

$$\left(\frac{f'(z)}{f(z)}\right)' = \frac{f''(z)f(z) - f'^2}{f(z)^2} = -\frac{n}{z^2} - 2\beta - \sum_k \frac{1}{(z-\alpha_k)^2}.$$

Consequently, for real z,

$$f'^2 - f(z)f''(z) \geq 0.$$

Since the derivative of a function in $\mathcal{LP}$ is also in $\mathcal{LP}$,

$$f^{(k)}(z) - f^{(k-1)}(z)f^{(k+1)}(z) \geq 0 \tag{3.1}$$

for all real z and all $k = 1, 2, 3, \ldots$. The inequalities in (3.1) are sometimes called the Laguerre inequalities.

As a consequence of (3.1), if $f(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!}$ is a real entire function of genus 0 or 1, a *necessary* condition for $f(z)$ to belong to $\mathcal{LP}$ is that

$$a_k^2 - a_{k-1}a_{k+1} \geq 0 \qquad (k \geq 1). \tag{3.2}$$

Definition 3.1. The inequalities in (3.2) are called the *Turán inequalities*. We say that $f(z)$ satisfies the *strict Turán inequalities* if

$$a_k^2 - a_{k-1}a_{k+1} > 0$$

for all $k \geq 1$ when $f(z)$ is a transcendental function or for $1 \leq k \leq n$ if $f(z)$ is a polynomial of degree n.

4. TURÁN INEQUALITIES FOR EVEN REAL ENTIRE FUNCTIONS

Consider a real entire function of genus 0 or 1 of the form

$$F(z) = \sum_{k=0}^{\infty} (-1)^k a_k \frac{z^{2k}}{(2k)!}$$

where $a_k > 0$ for $k \geq 0$. The Turán inequalities (3.2) are trivially true for $F(z)$. We wish to find a nontrivial application of the Turán inequalities to the function $F(z)$. We define a companion function $G(w)$ by making the substitution $-z^2 \mapsto w$.

$$F(z) = \sum_{k=0}^{\infty} a_k \frac{(-z^2)^k}{(2k)!} \quad \longleftrightarrow \quad G(w) = \sum_{k=0}^{\infty} \underbrace{\frac{k! a_k}{(2k)!}}_{b_k} \frac{w^k}{k!} \tag{4.1}$$

The series $F(z)$ in powers of z^2 has alternating coefficients while the associated series $G(w)$ in powers of w has positive coefficients. Observe that $F(z)$ has only real zeros if and only if $G(w)$ has only negative real zeros. Thus we consider the Turán inequalities for the companion function $G(w)$:

$$b_k^2 - b_{k-1}b_{k+1} \geq 0 \qquad (k \geq 1)$$

which hold if and only if

$$(2k+1)a_k^2 - (2k-1)a_{k-1}a_{k+1} \geq 0 \qquad (k \geq 1). \tag{4.2}$$

This explains condition (1.1) as a necessary condition for the Riemann hypothesis since $\xi(1/2 + iz)$ is an even entire function of genus 1 with alternating coefficients.

Definition 4.1. For an even entire function $F(z)$ with alternating coefficients, as in (4.1), we will refer to the inequalities (4.2) as the *Turán inequalities* for $F(z)$.

The following fundamental example helped us to discover our proof of Theorem 2.2.

Example 4.1. Let $F(z)$ be the monic polynomial with roots $\pm\alpha \pm \beta i$ where $\alpha, \beta \geq 0$. Then

$$F(z) = \underbrace{(\alpha^2+\beta^2)^2}_{a_0} - \underbrace{4(\alpha^2-\beta^2)}_{a_1}\frac{z^2}{2!} + \underbrace{24}_{a_2}\frac{z^4}{4!}.$$

The coefficients alternate signs provided that $\alpha > \beta$, which we assume to be the case. What additional hypothesis ensures that $F(z)$ satisfies the Turán inequalities? The only interesting inequality would be $(2k+1)a_k^2 - (2k-1)a_{k-1}a_{k+1} \geq 0$ with $k = 1$ (if this is possible). A short computation gives

$$3a_1^2 - a_0a_2 = 24\left[\left(\sqrt{2}+1\right)\alpha^2 - \left(\sqrt{2}-1\right)\beta^2\right]\left[\left(\sqrt{2}-1\right)\alpha^2 - \left(\sqrt{2}+1\right)\beta^2\right].$$

Since $\alpha > \beta$, the quantity $\left(\sqrt{2}+1\right)\alpha^2 - \left(\sqrt{2}-1\right)\beta^2$ is strictly positive. Then

$$\left(\sqrt{2}-1\right)\alpha^2 - \left(\sqrt{2}+1\right)\beta^2 > 0 \qquad \Leftrightarrow \qquad \alpha > \left(1+\sqrt{2}\right)\beta.$$

Thus, the strict Turán inequalities hold for $F(z)$ if and only if $\alpha > (1+\sqrt{2})\beta$.

Since $\tan(\pi/8) = -1+\sqrt{2} = (1+\sqrt{2})^{-1}$, the strict Turán inequalities hold for $F(z)$ if and only if the four roots of $F(z)$ lie in the region

$$\{z \in \mathbb{C} \mid z \neq 0 \text{ and } -\pi/8 < \arg(z) < \pi/8 \text{ or } 7\pi/8 < \arg(z) < 9\pi/8\}$$

if and only if the two roots of the companion polynomial $G(w)$, defined in (4.1), lie in the region

$$\{w \in \mathbb{C} \mid w \neq 0 \text{ and } 3\pi/4 < \arg(w) < 5\pi/4\}.$$

5. Proofs of Theorems 2.1 and 2.2 and Corollary 2.3

In this section we will prove Theorems 2.1 and 2.2 and Corollary 2.3. Let $G(z)$ be a real entire function of genus 0 of the form

$$G(z) = \prod_k (1+\rho_k z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!},$$

where the numbers ρ_k are the negative reciprocal roots of $G(z)$. The set of these negative reciprocal roots (with repetitions allowed) is denoted as R. The Taylor coefficient a_n, expressed in terms of the negative reciprocal roots, is

$$a_n = n! s_R(n)$$

where $s_R(n)$ is the nth elementary symmetric function formed from the elements of R. Recall from equation (2.1) that we define the symmetric function $s_R(n,k)$ as

$$s_R(n,k) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ k \text{ repetitions}}} \rho_{i_1} \cdots \rho_{i_n}$$

where the summation is taken over all lists of indices of the form

$$i_1 \leq i_2 \leq \cdots \leq i_n \tag{5.1}$$

such that k of the values are repeated exactly twice. In (5.1), k of the relations are equal signs, the remaining $n-1-k$ relations are strict inequalities, and no two consecutive relations are equal signs. We consider $s_R(m,k) = 0$ whenever its defining summation (2.1) is empty. Several of these trivial cases are listed in Lemma 5.1.

Lemma 5.1. *$s_R(m,k) = 0$ in all of the following cases:*

(i) *if $m < 1$ or $k < 0$,*

(ii) *if $k > m/2$ since there can be at most $m/2$ repeated values in an ascending list of length m,*
(iii) *if $m - k > |R|$ since the length of an ascending list can be at most $|R|$.*

Thus, necessary conditions for $s_R(m, k)$ to be nonzero are

$$m \geq 1 \quad \textit{and} \quad 0 \leq 2k \leq m \leq |R| + k.$$

The next lemma shows how to express the product of two elementary symmetric functions in terms of the functions $s_R(m, k)$.

Lemma 5.2. *Let $0 \leq m \leq n$. Then*

$$s_R(m)s_R(n) = \sum_{k=0}^{m} \binom{m+n-2k}{m-k} s_R(m+n, k).$$

Proof. Each term in the product of the sums $s_R(m)$ and $s_R(n)$ is a term in the sum $s_R(m+n, k)$ for some k with $0 \leq k \leq m$. Conversely, each term in the sum $s_R(m+n, k)$ with $0 \leq k \leq m$ is obtainable as a product of terms from the sums $s_R(m)$ and $s_R(n)$. We need to count how often this happens. A given term $\rho_{\ell_1} \cdots \rho_{\ell_{m+n}}$ containing exactly k repeated indices can be obtained as the product of $\rho_{i_1} \cdots \rho_{i_m}$ and $\rho_{j_1} \cdots \rho_{j_n}$ each of which shares k indices. The terms in the product $\rho_{i_1} \cdots \rho_{i_m}$ contain the k repeated terms as well as $m - k$ terms chosen from among the $m+n-2k$ non-repeated terms of $\rho_{\ell_1} \cdots \rho_{\ell_{m+n}}$. The choice of $\rho_{i_1} \cdots \rho_{i_m}$ determines the choice of $\rho_{j_1} \cdots \rho_{j_n}$. So, there are $\binom{m+n-2k}{m-k}$ ways to obtain the product $\rho_{\ell_1} \cdots \rho_{\ell_{m+n}}$. $\square$

We will now prove Theorem 2.1 by representing the Turán expression $a_n^2 - a_{n-1}a_{n+1}$ as a linear combination of the symmetric functions $s_R(m, k)$ having nonnegative coefficients. In other words, $a_n^2 - a_{n-1}a_{n+1}$ can be written as a subtraction-free expression.

Proof of Theorem 2.1. Since $a_m = m! s_R(m)$,

$$a_n^2 - a_{n-1}a_{n+1} = n!(n-1)! \left[n s_R(n)^2 - (n+1)s_R(n-1)s_R(n+1) \right].$$

Applying Lemma 5.2 to the expression on the right gives

$$\begin{aligned}
& n s_R(n)^2 - (n+1)s_R(n-1)s_R(n+1) \\
&= n \sum_{k=0}^{n} \binom{2n-2k}{n-k} s_R(2n, k) - (n+1) \sum_{k=0}^{n-1} \binom{2n-2k}{n+1-k} s_R(2n, k) \\
&= n s_R(2n, n) + \sum_{k=0}^{n-1} \left[n \binom{2n-2k}{n-k} - (n+1) \binom{2n-2k}{n+1-k} \right] s_R(2n, k) \\
&= \sum_{k=1}^{n} \frac{k}{n+1-k} \binom{2n-2k}{n-k} s_R(2n, k).
\end{aligned}$$

$\square$

Lemma 5.3, below, will provide conditions under which $s_R(n, k)$ is positive when its defining sum is not empty as in Lemma 5.1. Then the subtraction-free expression in Theorem 2.1 is also positive.

Lemma 5.3. *Let A be a nonempty set (finite or countable) of nonzero complex numbers (with repetitions allowed) such that*

(1) *if $\rho \in A$, then $\bar{\rho} \in A$ with the same multiplicity,*
(2) *if $\rho \in A$, then $0 \leq |\operatorname{Im}(\rho)| < \operatorname{Re}(\rho)$, and*
(3) $\sum_{\rho \in A} |\rho| < \infty$.

Then, $s_A(m,k) > 0$ *whenever* $m > 0$ *and* $0 \le 2k \le m \le |A| + k$.

Note that the condition on the negative reciprocal roots in Lemma 5.3 coincides with the condition in the statement of Theorem 2.2. The third condition, $\sum_{\rho \in A} |\rho| < \infty$, guarantees convergence of the product $\prod_{\rho \in A}(1 + \rho z)$ and convergence of the sum $S_A(m,k)$ when A is an infinite set.

Proof. If A is a finite set, we will prove the lemma by induction on the cardinality of A. The case in which A is an infinite set will follow immediately from the finite case.

First, suppose $A = \{\rho\}$ consists of a single positive number. Since $|A| = 1$, the set of possible choices for (m,k) is $\{(1,0),(2,1)\}$. Then

$$\begin{aligned} s_A(1,0) &= \rho > 0, \\ s_A(2,1) &= \rho^2 > 0. \end{aligned}$$

The lemma holds in this case. Next, suppose $A = \{\rho, \bar{\rho}\}$ and $0 \le |\operatorname{Im}(\rho)| < \operatorname{Re}(\rho)$. Since $|A| = 2$, the set of all possible choices for (m,k) is

$$\{(1,0),(2,0),(2,1),(3,1),(4,2)\}.$$

Then

$$\begin{aligned} s_A(1,0) &= \rho + \bar{\rho} = 2\operatorname{Re}(\rho) > 0, \\ s_A(2,0) &= \rho\bar{\rho} = |\rho|^2 > 0, \\ s_A(2,1) &= \rho^2 + \bar{\rho}^2 = 2[\operatorname{Re}(\rho)^2 - \operatorname{Im}(\rho)^2] > 0, \\ s_A(3,1) &= \rho^2\bar{\rho} + \rho\bar{\rho}^2 = 2|\rho|^2 \operatorname{Re}(\rho) > 0, \\ s_A(4,2) &= \rho^2\bar{\rho}^2 = |\rho|^4 > 0. \end{aligned}$$

The lemma also holds in this case.

Let A be a finite set (with repetitions allowed) as in the statement of the lemma. Assume, by way of induction, that the lemma holds for the set A. Thus, $s_A(n,\ell) > 0$ whenever $n > 1$ and $0 \le 2\ell \le n \le |A| + \ell$. Let ρ be a positive number and let $B = A \cup \{\rho\}$. From the definition of $s_B(m,k)$, it follows that

$$s_B(m,k) = s_A(m,k) + \rho\, s_A(m-1,k) + \rho^2\, s_A(m-2,k-1). \tag{5.2}$$

Choose the pair (m,k) so that $m \ge 1$ and $0 \le 2k \le m \le |B| + k$. By the induction hypothesis, each term on the right hand side of equation (5.2) is either positive or zero. Potentially, some of the terms on the right hand side of (5.2) could be zero by Lemma 5.1. It will suffice to show that at least one term is positive. Let

$$L_A = \left\{(m,k) \,|\, 0 < m \quad \text{and} \quad 0 \le 2k \le m \le |A| + k\right\},$$

and let L_B be similarly defined. Since $|A| < |B| = |A| + 1$, $L_A \subset L_B$. If $(m,k) \in L_A$, then $s_A(m,k) > 0$ which implies that $s_B(m,k) > 0$. Now, assume $(m,k) \in L_B$ but $(m,k) \notin L_A$. In this case, $m = |A| + 1 + k$ where $0 \le k \le |A| + 1$. If $m = |A| + 1 + k$ and $0 \le k \le |A|$, the pair $(m-1,k) = (|A| + k, k)$ is in L_A. Then $s_A(m-1,k) > 0$ which implies, by (5.2), that $s_B(m,k) > 0$. If $m = |A| + 1 + k$ and $k = |A| + 1$, the pair $(m-2,k-1)$ is in L_A. Then $s_A(m-2,k-1) > 0$ so that $s_B(m,k) > 0$. This proves that the lemma holds when the set A is enlarged by adjoining a positive real number.

Next we will enlarge A by adjoining a pair $\{\rho, \bar{\rho}\}$. Let $C = A \cup \{\rho, \bar{\rho}\}$ where $0 \le |\operatorname{Im}(\rho)| < \operatorname{Re}(\rho)$. From the definition of $s_C(m,k)$ it follows that

$$\begin{aligned}(5.3)\quad s_C(m,k) &= s_A(m,k) + (\rho+\bar{\rho})\, s_A(m-1,k) \\ &\quad + \rho\bar{\rho}\, s_A(m-2,k) + (\rho^2+\bar{\rho}^2)\, s_A(m-2,k-1) \\ &\quad + \rho\bar{\rho}(\rho+\bar{\rho})\, s_A(m-3,k-1) + \rho^2\bar{\rho}^2\, s_A(m-4,k-2).\end{aligned}$$

Choose the pair (m,k) so that $m \ge 1$ and $0 \le 2k \le m \le |C| + k$. By the induction hypothesis, each term on the right hand side of equation (5.3) is nonnegative. It will suffice to show that at least one term is positive. If $(m,k) \in L_A$, then $s_A(m,k) > 0$ so that $s_C(m,k) > 0$. If $(m,k) \in L_C$, but $(m,k) \notin L_A$, then $m = |A|+1+k$ where $0 \le k \le |A|+1$ or $m = |A|+2+k$ where $0 \le k \le |A|+2$. The case $m = |A|+1+k$ is exactly the same as in the previous paragraph. If $m = |A|+2+k$ and $0 \le k \le |A|$, then $(m-2,k)$ is in L_A so that $s_A(m-2,k) > 0$ and $s_B(m,k) > 0$. If $m = |A|+2+k$ and $k = |A|+1$, then $(m-2,k-1)$ is in L_A so that $s_A(m-2,k-1) > 0$ and $s_C(m,k) > 0$. If $m = |A|+2+k$ and $k = |A|+2$, then $(m-4,k-2)$ is in L_A so that $s_A(m-4,k-2) > 0$ and $s_C(m,k) > 0$. This proves that the lemma holds when the set A is enlarged by adjoining a pair $\{\rho, \bar{\rho}\}$. Thus, the lemma holds if A is a finite set.

Suppose now that A is an infinite set (with repetitions allowed) as in the statement of the lemma and suppose $0 \le 2k \le m$. Let

$$B_1 \subset B_2 \subset B_3 \subset \cdots$$

be a sequence of finite subsets of A satisfying the hypotheses in the lemma such that $A = \cup_{n=1}^{\infty} B_n$. Then

$$\lim_{n\to\infty} s_{B_n}(m,k) = s_A(m,k).$$

The nonnegativity of each term on the right hand sides of equations (5.2) and (5.3) implies that

$$s_{B_1}(m,k) \le s_{B_2}(m,k) \le s_{B_3}(m,k) \le \cdots \le s_A(m,k).$$

Since $s_{B_n}(m,k) > 0$ as soon as $|B_n|$ is sufficiently large, it follows that $s_A(m,k) > 0$. Therefore, the lemma also holds when A is an infinite set. □

Combining Theorem 2.1 and Lemma 5.3 immediately gives Theorem 2.2.

For Corollary 2.3, we recall from analytic number theory that the Dedekind ξ-function for a finite extension K of $\mathbb{Q}$ has a Taylor series representation of the form

$$\xi_K(1/2+iz) = \sum_{k=0}^{\infty} (-1)^k a_k \frac{z^{2k}}{(2k)!},$$

where $a_k > 0$ for all k. Then $\xi_K(1/2+iz)$ is a real entire function with alternating coefficients to which Theorem 2.2 applies. By standard facts from analytic and algebraic number theory, $\xi_K(s)$ has no zeros outside the closed strip $0 \le \operatorname{Re}(s) \le 1$, and the prime number theorem, generalized to number fields, is equivalent to the fact there are no zeros outside the open strip $0 < \operatorname{Re}(s) < 1$. Combining this with Theorem 2.2 shows that the strict Turán inequalities hold for $\xi_K(1/2+iz)$ if there are no roots of $\xi_K(s)$ in the closed triangular region determined by the three points $1/2$, 1, and $1 + \left(\frac{1+\sqrt{2}}{2}\right) i$, which completes the proof.

6. PROOF OF THEOREM 2.2 USING THE GAUSS-LUCAS THEOREM

We will now briefly recall the proof of Theorem 2.2 that relies on the Gauss-Lucas theorem. This argument would have been known to researchers such as Jensen, Laguerre, Pólya, and Turán. (See, for example, Theorem 2.4.2 and Lemma 5.4.4 in [12]).

Let $f(z)$ be a real monic polynomial whose negative reciprocal roots lie in the sector $0 \leq |\operatorname{Im}(z)| < \operatorname{Re}(z)$ as in Theorem 2.2. If the real roots are $r_1, \ldots, r_m$ and the complex roots are $\alpha_1 \pm i\beta_1, \ldots, \alpha_n \pm i\beta_n$, then

$$f(z) = \sum_{k=0}^{m+2n} a_k \frac{z^k}{k!} = \prod_{j=1}^{m}(z - r_j) \prod_{k=1}^{n}\left((z-\alpha_k)^2 + \beta_k^2\right).$$

Taking the derivative of the logarithmic derivative of $f(z)$ results in

$$\frac{[f'^2 - f(z)]f''(z)}{f(z)^2} = \sum_{j=1}^{m} \frac{1}{(z-r_j)^2} + 2\sum_{k=1}^{n} \frac{(z-\alpha_k)^2 - \beta_k^2}{[(z-\alpha_k)^2 + \beta_k^2]^2}. \tag{6.1}$$

The hypothesis causes the right hand side of (6.1) to be positive for $z = 0$ giving

$$a_1^2 - a_0 a_2 > 0.$$

The Gauss-Lucas theorem (see Theorem 2.1.1 in [12]) says that every convex set containing the zeros of $f(z)$ also contains the zeros of $f'(z)$. Since the negative reciprocal roots of $f(z)$ belong to the sector $0 \leq |\operatorname{Im}(z)| < \operatorname{Re}(z)$, which is a convex region, the negative reciprocal roots of $f'(z)$ also belong to that sector. By the previous argument applied to $f'(z)$,

$$a_2^2 - a_1 a_3 > 0.$$

Proceeding in this manner for the remaining derivatives of $f(z)$ proves the theorem.

Note that the original proof of Theorem 2.2 is not really shorter than our proof. Including the details of the proof of the Gauss-Lucas theorem and its extension to transcendental entire functions would make the argument as long and complicated as our new proof.

7. QUESTIONS FOR FURTHER STUDY

We conclude the paper by stating several problems suggested by our studies.

In proving Theorem 2.1, we actually proved the *stronger* result (Lemma 5.3) that if the negative reciprocal roots ρ_k of the real entire function

$$G(z) = \prod_k (1 + \rho_k z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

satisfy $0 \leq |\operatorname{Im}(\rho_k)| < \operatorname{Re}(\rho_k)$, then $s_R(m,k) > 0$ whenever $m > 0$ and $0 \leq 2k \leq m \leq |R| + k$ where R is the set of negative reciprocal roots (with repetitions allowed). In other words, we produced a stronger set of inequalities than the set of Turán inequalities since the Turán expressions were formed as subtraction-free expressions involving the symmetric functions $s_R(m,k)$.

Problem 7.1. Determine other interesting sets of inequalities related to the location of the zeros of $G(z)$ that naturally result from considering subtraction-free expressions.

To be more concrete, if $\phi(z)$ is a real entire function, set

$$\mathcal{T}_k^{(1)}\big(\phi(z)\big) := \big(\phi^{(k)}(z)\big)^2 - \phi^{(k-1)}(z)\phi^{(k+1)}(z) \quad \text{if } k \geq 1,$$

and for $n \geq 2$, set

$$\mathcal{T}_k^{(n)}\big(\phi(z)\big) := \big(\mathcal{T}_k^{(n-1)}\big(\phi(z)\big)\big)^2 - \mathcal{T}_{k-1}^{(n-1)}\big(\phi(z)\big)\mathcal{T}_{k+1}^{(n-1)}\big(\phi(z)\big) \quad \text{if } k \geq n \geq 2.$$

For $\phi(z) \in \mathcal{LP}^+$ (defined in §3) Craven and Csordas asked in [2] if it is true that

$$\mathcal{T}_k^{(n)}\big(\phi(z)\big) \geq 0 \tag{7.1}$$

for all $z \geq 0$ and $k \geq n$. They refer to the inequalities in (7.1) as *iterated Laguerre inequalities*. Our own studies have suggested that $\mathcal{T}_k^{(n)}\big(\phi(z)\big)\big|_{z=0}$ can be expressed in terms of subtraction-free expressions. Hence we have the problem:

Problem 7.2. Represent $\mathcal{T}_k^{(n)}\big(\phi(z)\big)\big|_{z=0}$ in terms of subtraction-free expressions and determine sectors in $\mathbb{C}$ such that if the negative reciprocal roots belong to the sectors, then $\mathcal{T}_k^{(n)}\big(\phi(z)\big)\big|_{z=0} \geq 0$ for certain values of n and k which depend on the sector.

REFERENCES

[1] J. B. CONREY AND A. GHOSH, Turán inequalities and zeros of Dirichlet series associated with certain cusp forms, *Trans. Amer. Math. Soc.*, **342**(1) (1994), 407–419.

[2] T. CRAVEN AND G. CSORDAS, Iterated Laguerre and Turán inequalities, *J. Inequal. Pure Appl. Math.*, **3**(3) (2002), Art. 39. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=191`].

[3] G. CSORDAS, T.S. NORFOLK, AND R.S. VARGA, The Riemann hypothesis and the Turán inequalities, *Trans. Amer. Math. Soc.*, **296**(2) (1986), 521–541.

[4] G. CSORDAS AND R.S. VARGA, Moment inequalities and the Riemann hypothesis, *Constr. Approx.*, **4**(2) (1988), 175–198.

[5] G. CSORDAS AND R.S. VARGA, Fourier transforms and the Hermite-Biehler theorem, *Proc. Amer. Math. Soc.*, **107**(3) (1989), 645–652.

[6] G. CSORDAS AND R.S. VARGA, Integral transforms and the Laguerre-Pólya class, *Complex Variables Theory Appl.*, **12**(1-4) (1989), 211–230.

[7] G. CSORDAS AND R.S. VARGA, Necessary and sufficient conditions and the Riemann hypothesis, *Adv. in Appl. Math.*, **11**(3) (1990), 328–357.

[8] S. LANG, *Algebraic Number Theory*, second ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994.

[9] B. Ja. LEVIN, Distribution of zeros of entire functions, revised ed., *Translations of Mathematical Monographs*, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman.

[10] Yu. V. MATIYASEVICH, Yet another machine experiment in support of Riemann's conjecture, *Cybernetics*, **18**(6) (1983), 705–707.

[11] J. NEUKIRCH, *Algebraic Number Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[12] Q.I. RAHMAN AND G. SCHMEISSER, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002.