SUPERQUADRACITY OF FUNCTIONS AND REARRANGEMENTS OF SETS

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Abstract: In this paper we establish upper bounds of

$$\sum_{i=1}^{n}\left(f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{|x_i-x_{i+1}|}{2}\right)\right), \qquad x_{n+1}=x_1$$

when the function f is superquadratic and the set $(\mathbf{x})=(x_1,\dots,x_n)$ is given except its arrangement.

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1. Introduction

We start with the definitions and results of [1] and [5] which we use in this paper.

Definition 1.1. *The sets* $(\mathbf{y}^-) = (y_1^-, \dots, y_n^-)$ *and* $(^-\mathbf{y}) = (^-y_1, \dots, ^-y_n)$ *are symmetrically decreasing rearrangements of an ordered set* $(\mathbf{y}) = (y_1, \dots, y_n)$ *of* n *real numbers, if*

$$y_1^- \le y_n^- \le y_2^- \le \dots \le y^-_{\left[\frac{n+2}{2}\right]} \tag{1.1}$$

and

$$^-y_n \le {}^-y_1 \le {}^-y_{n-1} \le \dots \le {}^-y_{\left[\frac{n+1}{2}\right]}. \tag{1.2}$$

A circular rearrangement of an ordered set $(\mathbf{y}) = (y_1, \dots, y_n)$ *is a cyclic rearrangement of* $(\mathbf{y})$ *or a cyclic rearrangement followed by inversion.*

Definition 1.2. *An ordered set* $(\mathbf{y}) = (y_1, \dots, y_n)$ *of* n *real numbers is arranged in circular symmetric order if one of its circular rearrangements is symmetrically decreasing.*

Theorem A ([1]). *Let* $F(u, v)$ *be a symmetric function defined for* $\alpha \le u, v \le \beta$ *for which* $\frac{\partial^2 F(u,v)}{\partial u \partial v} \ge 0$.

Let the set $(\mathbf{y}) = (y_1, \dots, y_n)$, $\alpha \le y_i \le \beta$, $i = 1, \dots, n$ *be given except its arrangement. Then*

$$\sum_{i=1}^{n} F(y_i, y_{i+1}), \qquad (y_{n+1} = y_1)$$

is maximal if $(\mathbf{y})$ *is arranged in circular symmetrical order.*

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Definition 1.3 ([5]). *A function f, defined on an interval $I = [0, L]$ or $[0,\infty)$ is superquadratic, if for each x in I, there exists a real number $C(x)$ such that*

$$f(y) - f(x) \geq C(x)(y - x) + f(|y - x|)$$

for all $y \in I$.

A function is subquadratic if $-f$ is superquadratic.

Lemma A ([5]). *Let f be a superquadratic function with $C(x)$ as in Definition 1.3.*

(i) Then $f(0) \leq 0$.

(ii) If $f(0) = f'(0) = 0$, then $C(x) = f'(x)$ whenever f is differentiable.

(iii) If $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

The following lemma presents a Jensen's type inequality for superquadratic functions.

Lemma B ([6, Lemma 2.3]). *Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$ and let $\overline{x} = \sum_{r=1}^{n} \lambda_r x_r$, where $\lambda_r \geq 0$, and $\sum_{r=1}^{n} \lambda_r = 1$. Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\overline{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \overline{x}|).$$

If $f(x)$ is subquadratic, the reverse inequality holds.

From Lemma B we get an immediate result which we state in the following lemma.

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Lemma C. *Let $f(x)$ be superquadratic on $[0,L]$ and let $x,y\in[0,L]$, $0\le\lambda\le1$, then*

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) &\\ \geq f(\lambda x+(1-\lambda)y)&+\lambda f((1-\lambda)|y-x|)+(1-\lambda)f(\lambda|y-x|)\\ \geq f(\lambda x+(1-\lambda)y)&+\sum_{k=0}^{t-1}\left(f\left(2\lambda(1-\lambda)|1-2\lambda|^{k}|x-y|\right)\right)\\ &+\lambda f\left((1-\lambda)|1-2\lambda|^{t}|x-y|\right)+(1-\lambda)f\left(\lambda|1-2\lambda|^{t}|x-y|\right).\end{aligned}$$

If f is positive superquadratic we get that:

$$\lambda f(x)+(1-\lambda)f(y)\geq f(\lambda x+(1-\lambda)y)+\sum_{k=0}^{t-1}\left(f\left(2\lambda(1-\lambda)|1-2\lambda|^{k}|x-y|\right)\right)$$

More results related to superquadracity were discussed in [2] to [6].

In this paper we refine the results in [7] by showing that for positive superquadratic functions we get better bounds than in [7].

Theorem B ([7, Thm. 1.2]). *If f is a convex function and $x_1,x_2,\ldots,x_n$ lie in its domain, then*

$$\begin{aligned}\sum_{i=1}^{n}f(x_i)-f\left(\frac{x_1+\cdots+x_n}{n}\right)&\\ \geq\frac{n-1}{n}&\left[f\left(\frac{x_1+x_2}{2}\right)+\cdots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right].\end{aligned}$$

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Theorem C ([7, Thm. 1.4]). *If f is a convex function and $a_1, \ldots, a_n$ lie in its domain, then*

$$(n-1)\left[f(b_1)+\cdots+f(b_n)\right] \leq n\left[f(a_1)+\cdots+f(a_n)-f(a)\right],$$

where $a = \frac{a_1+\cdots+a_n}{n}$ and $b_i = \frac{na-a_i}{n-1}$, $i = 1, \ldots, n$.

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2. The Main Results

Theorem 2.1. *Let* $f(x)$ *be a superquadratic function on* $[0,L]$. *Then for* $x_i \in [0,L]$, $i=1,\dots,n$, *where* $x_{n+1}=x_1$,

$$\frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\sum_{i=1}^{n}\frac{x_i+x_{i+1}}{2}\right)+f\left(\sum_{i=1}^{n}\frac{|x_i-x_{i+1}|}{2}\right)\right) \leq \left(\sum_{i=1}^{n}f(x_i)\right)-f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)-\frac{1}{n}\sum_{i=1}^{n}f\left(\left|x_i-\sum_{j=1}^{n}\frac{x_j}{n}\right|\right) \tag{2.1}$$

holds. If $f'''(x)\geq 0$ *too, then*

$$\begin{aligned}\frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\sum_{i=1}^{n}\frac{x_i+x_{i+1}}{2}\right)+f\left(\sum_{i=1}^{n}\frac{|x_i-x_{i+1}|}{2}\right)\right) &\leq \frac{n-1}{n}\sum_{i=1}^{n}\left(f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)+f\left(\frac{|\widehat{x}_i-\widehat{x}_{i+1}|}{2}\right)\right)\\ &\leq \left(\sum_{i=1}^{n}f(x_i)\right)-f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)-\frac{1}{n}\sum_{i=1}^{n}f\left(\left|x_i-\sum_{j=1}^{n}\frac{x_j}{n}\right|\right),\end{aligned} \tag{2.2}$$

where $(\widehat{\mathbf{x}})=(\widehat{x}_1,\dots,\widehat{x}_n)$ *is a circular symmetrical rearrangement of* $(\mathbf{x})=(x_1,\dots,x_n)$.

Example 2.1. *The functions*

$$f(x)=x^n, \qquad n\geq 2, \qquad x\geq 0,$$

and the function

$$f(x)=\begin{cases} x^2\log x, & x>0,\\ 0, & x=0\end{cases}$$

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are superquadratic with an increasing second derivative and therefore (2.2) holds for these functions.

Proof. Let f be a superquadratic function on $[0, L]$. Then by Lemma B we get for $0 \leq \alpha \leq 1, \quad 1 \leq k \leq n$ and $x_i \in [0, L]$, $x_{n+1} = x_1$,

$$\begin{aligned} \sum_{i=1}^{n} f(x_i) &= \frac{n-k}{n} \sum_{i=1}^{n} f(x_i) + \frac{k}{n} \sum_{i=1}^{n} f(x_i) \\ &= \frac{n-k}{n} \sum_{i=1}^{n} (\alpha f(x_i) + (1-\alpha) f(x_{i+1})) + \frac{k}{n} \sum_{i=1}^{n} f(x_i) \\ &\geq \frac{n-k}{n} \sum_{i=1}^{n} f(\alpha x_i + (1-\alpha) x_{i+1}) \\ &\quad + \frac{n-k}{n} \sum_{i=1}^{n} (\alpha f((1-\alpha)|x_{i+1} - x_i|) + (1-\alpha) f(\alpha |x_{i+1} - x_i|)) \\ &\quad + k \left(f\left(\frac{\sum_{i=1}^{n} x_i}{n} \right) + \sum_{i=1}^{n} \frac{1}{n} f\left(\left| x_i - \frac{\sum_{i=1}^{n} x_i}{n} \right| \right) \right). \end{aligned} \tag{2.3}$$

For $k = 1$ and $\alpha = \frac{1}{2}$ we get that (2.1) holds.

If $f'''(x) \geq 0$, then $\frac{\partial^2 F(u,v)}{\partial u \partial v} \geq 0$, where

$$F(u, v) = f(u + v) + f(|u - v|), \qquad u, v \in [0, L].$$

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Therefore according to Theorem A, the sum

$$\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{|x_i+x_{i+1}|}{2}\right), \qquad x_{n+1}=x_1,$$

is maximal for $(\widehat{\mathbf{x}})=(\widehat{x}_1,\dots,\widehat{x}_n)$, which is the circular symmetric rearrangement of $(\mathbf{x})$. Therefore in this case (2.2) holds as well. □

Remark 1. For a positive superquadratic function f, which according to Lemma A is also a convex function, (2.1) is a refinement of Theorem B.

If $f'''(x)\geq 0$, (2.2) is a refinement of Theorem B as well.

Remark 2. Theorem B is refined by

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)-f\left(\frac{\sum_{i=1}^{n}x_i}{n}\right) &\geq \frac{n-1}{n}\left(\sum_{i=1}^{n} f\left(\frac{\widehat{x}_i+\widehat{x}_{i+1}}{2}\right)\right)\\ &\geq \frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right),\end{aligned}$$

because a convex function f satisfies the conditions of Theorem A for $F(u,v)=f(u+v)$.

The following inequality is a refinement of Theorem C for a positive superquadratic function f, which is therefore also convex. The inequality results easily from Lemma B and the identity

$$\sum_{i=1}^{n} f(a_i)=\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j=1}^{n} f(a_j)(1-\delta_{ij})\right)$$

(where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$), therefore the proof is omitted.

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Theorem 2.2. *Let f be a superquadratic function on $[0, L]$, and let $x_i \in [0, L]$, $i = 1, \dots, n$. Then*

$$\frac{n}{n-1}\left(\left(\sum_{i=1}^{n} f(x_i)\right) - f(\overline{x})\right) - \sum_{i=1}^{n} f(y_i) \geq \frac{1}{n-1}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} f(|y_i - x_j|)(1-\delta_{ij})\right) + \frac{1}{n-1}\sum_{i=1}^{n} f(|\overline{x} - x_i|),$$

where $\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n}$, $y_i = \left(\frac{n\overline{x} - x_i}{n-1}\right)$, $i = 1, \dots, n$.

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