

# SUPERQUADRACITY OF FUNCTIONS AND REARRANGEMENTS OF SETS

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*Abstract:* In this paper we establish upper bounds of

$$\sum_{i=1}^n \left( f \left( \frac{x_i + x_{i+1}}{2} \right) + f \left( \frac{|x_i - x_{i+1}|}{2} \right) \right), \quad x_{n+1} = x_1$$

when the function  $f$  is superquadratic and the set  $(\mathbf{x}) = (x_1, \dots, x_n)$  is given except its arrangement.



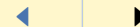
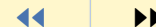
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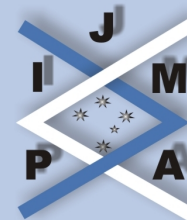
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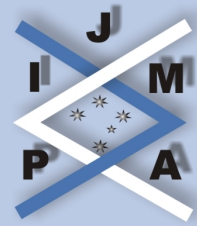
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## 1. Introduction

We start with the definitions and results of [1] and [5] which we use in this paper.

**Definition 1.1.** The sets  $(\mathbf{y}^-) = (y_1^-, \dots, y_n^-)$  and  $(^-\mathbf{y}) = (^-y_1, \dots, ^-y_n)$  are symmetrically decreasing rearrangements of an ordered set  $(\mathbf{y}) = (y_1, \dots, y_n)$  of  $n$  real numbers, if

$$(1.1) \quad y_1^- \leq y_n^- \leq y_2^- \leq \dots \leq y_{\lfloor \frac{n+2}{2} \rfloor}^-$$

and

$$(1.2) \quad ^-y_n \leq ^-y_1 \leq ^-y_{n-1} \leq \dots \leq ^-y_{\lfloor \frac{n+1}{2} \rfloor}.$$

A circular rearrangement of an ordered set  $(\mathbf{y}) = (y_1, \dots, y_n)$  is a cyclic rearrangement of  $(\mathbf{y})$  or a cyclic rearrangement followed by inversion.

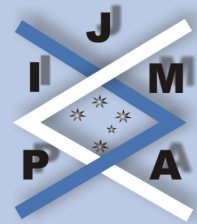
**Definition 1.2.** An ordered set  $(\mathbf{y}) = (y_1, \dots, y_n)$  of  $n$  real numbers is arranged in circular symmetric order if one of its circular rearrangements is symmetrically decreasing.

**Theorem A ([1]).** Let  $F(u, v)$  be a symmetric function defined for  $\alpha \leq u, v \leq \beta$  for which  $\frac{\partial^2 F(u, v)}{\partial u \partial v} \geq 0$ .

Let the set  $(\mathbf{y}) = (y_1, \dots, y_n)$ ,  $\alpha \leq y_i \leq \beta$ ,  $i = 1, \dots, n$  be given except its arrangement. Then

$$\sum_{i=1}^n F(y_i, y_{i+1}), \quad (y_{n+1} = y_1)$$

is maximal if  $(\mathbf{y})$  is arranged in circular symmetrical order.



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**Definition 1.3 ([5]).** A function  $f$ , defined on an interval  $I = [0, L]$  or  $[0, \infty)$  is superquadratic, if for each  $x$  in  $I$ , there exists a real number  $C(x)$  such that

$$f(y) - f(x) \geq C(x)(y - x) + f(|y - x|)$$

for all  $y \in I$ .

A function is subquadratic if  $-f$  is superquadratic.

**Lemma A ([5]).** Let  $f$  be a superquadratic function with  $C(x)$  as in Definition 1.3.

(i) Then  $f(0) \leq 0$ .

(ii) If  $f(0) = f'(0) = 0$ , then  $C(x) = f'(x)$  whenever  $f$  is differentiable.

(iii) If  $f \geq 0$ , then  $f$  is convex and  $f(0) = f'(0) = 0$ .

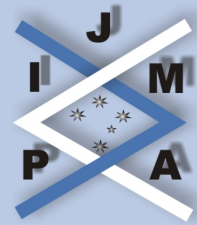
The following lemma presents a Jensen's type inequality for superquadratic functions.

**Lemma B ([6, Lemma 2.3]).** Suppose that  $f$  is superquadratic. Let  $x_r \geq 0$ ,  $1 \leq r \leq n$  and let  $\bar{x} = \sum_{r=1}^n \lambda_r x_r$ , where  $\lambda_r \geq 0$ , and  $\sum_{r=1}^n \lambda_r = 1$ . Then

$$\sum_{r=1}^n \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^n \lambda_r f(|x_r - \bar{x}|).$$

If  $f(x)$  is subquadratic, the reverse inequality holds.

From Lemma B we get an immediate result which we state in the following lemma.



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**Lemma C.** Let  $f(x)$  be superquadratic on  $[0, L]$  and let  $x, y \in [0, L]$ ,  $0 \leq \lambda \leq 1$ , then

$$\begin{aligned} & \lambda f(x) + (1 - \lambda) f(y) \\ & \geq f(\lambda x + (1 - \lambda) y) + \lambda f((1 - \lambda)|y - x|) + (1 - \lambda) f(\lambda|y - x|) \\ & \geq f(\lambda x + (1 - \lambda) y) + \sum_{k=0}^{t-1} \left( f\left(2\lambda(1 - \lambda)|1 - 2\lambda|^k|x - y|\right) \right) \\ & \quad + \lambda f((1 - \lambda)|1 - 2\lambda|^t|x - y|) + (1 - \lambda) f(\lambda|1 - 2\lambda|^t|x - y|). \end{aligned}$$

If  $f$  is positive superquadratic we get that:

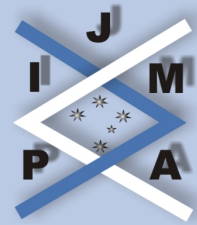
$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) + \sum_{k=0}^{t-1} \left( f\left(2\lambda(1 - \lambda)|1 - 2\lambda|^k|x - y|\right) \right)$$

More results related to superquadracity were discussed in [2] to [6].

In this paper we refine the results in [7] by showing that for positive superquadratic functions we get better bounds than in [7].

**Theorem B ([7, Thm. 1.2]).** If  $f$  is a convex function and  $x_1, x_2, \dots, x_n$  lie in its domain, then

$$\begin{aligned} & \sum_{i=1}^n f(x_i) - f\left(\frac{x_1 + \dots + x_n}{n}\right) \\ & \geq \frac{n-1}{n} \left[ f\left(\frac{x_1 + x_2}{2}\right) + \dots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right) \right]. \end{aligned}$$



**Theorem C ([7, Thm. 1.4]).** *If  $f$  is a convex function and  $a_1, \dots, a_n$  lie in its domain, then*

$$(n - 1) [f(b_1) + \dots + f(b_n)] \leq n [f(a_1) + \dots + f(a_n) - f(a)],$$

where  $a = \frac{a_1 + \dots + a_n}{n}$  and  $b_i = \frac{na - a_i}{n-1}$ ,  $i = 1, \dots, n$ .

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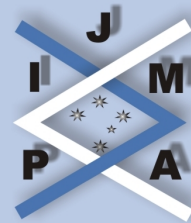
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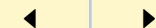
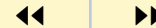
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## 2. The Main Results

**Theorem 2.1.** Let  $f(x)$  be a superquadratic function on  $[0, L]$ . Then for  $x_i \in [0, L]$ ,  $i = 1, \dots, n$ , where  $x_{n+1} = x_1$ ,

$$(2.1) \quad \frac{n-1}{n} \sum_{i=1}^n \left( f \left( \frac{x_i + x_{i+1}}{2} \right) + f \left( \frac{|x_i - x_{i+1}|}{2} \right) \right) \\ \leq \left( \sum_{i=1}^n f(x_i) \right) - f \left( \frac{\sum_{i=1}^n x_i}{n} \right) - \frac{1}{n} \sum_{i=1}^n f \left( \left| x_i - \frac{\sum_{j=1}^n x_j}{n} \right| \right)$$

holds. If  $f'''(x) \geq 0$  too, then

$$(2.2) \quad \frac{n-1}{n} \sum_{i=1}^n \left( f \left( \frac{x_i + x_{i+1}}{2} \right) + f \left( \frac{|x_i - x_{i+1}|}{2} \right) \right) \\ \leq \frac{n-1}{n} \sum_{i=1}^n \left( f \left( \frac{\hat{x}_i + \hat{x}_{i+1}}{2} \right) + f \left( \frac{|\hat{x}_i - \hat{x}_{i+1}|}{2} \right) \right) \\ \leq \left( \sum_{i=1}^n f(x_i) \right) - f \left( \frac{\sum_{i=1}^n x_i}{n} \right) - \frac{1}{n} \sum_{i=1}^n f \left( \left| x_i - \frac{\sum_{j=1}^n x_j}{n} \right| \right),$$

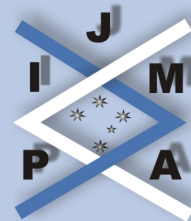
where  $(\hat{\mathbf{x}}) = (\hat{x}_1, \dots, \hat{x}_n)$  is a circular symmetrical rearrangement of  $(\mathbf{x}) = (x_1, \dots, x_n)$ .

*Example 2.1.* The functions

$$f(x) = x^n, \quad n \geq 2, \quad x \geq 0,$$

and the function

$$f(x) = \begin{cases} x^2 \log x, & x > 0, \\ 0, & x = 0 \end{cases}$$



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are superquadratic with an increasing second derivative and therefore (2.2) holds for these functions.

*Proof.* Let  $f$  be a superquadratic function on  $[0, L]$ . Then by Lemma B we get for  $0 \leq \alpha \leq 1$ ,  $1 \leq k \leq n$  and  $x_i \in [0, L]$ ,  $x_{n+1} = x_1$ ,

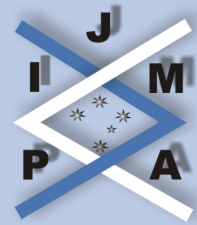
$$\begin{aligned} (2.3) \quad & \sum_{i=1}^n f(x_i) \\ &= \frac{n-k}{n} \sum_{i=1}^n f(x_i) + \frac{k}{n} \sum_{i=1}^n f(x_i) \\ &= \frac{n-k}{n} \sum_{i=1}^n (\alpha f(x_i) + (1-\alpha) f(x_{i+1})) + \frac{k}{n} \sum_{i=1}^n f(x_i) \\ &\geq \frac{n-k}{n} \sum_{i=1}^n f(\alpha x_i + (1-\alpha) x_{i+1}) \\ &\quad + \frac{n-k}{n} \sum_{i=1}^n (\alpha f((1-\alpha)|x_{i+1} - x_i|) + (1-\alpha) f(\alpha|x_{i+1} - x_i|)) \\ &\quad + k \left( f\left(\frac{\sum_{i=1}^n x_i}{n}\right) + \sum_{i=1}^n \frac{1}{n} f\left(\left|x_i - \frac{\sum_{i=1}^n x_i}{n}\right|\right) \right). \end{aligned}$$

For  $k = 1$  and  $\alpha = \frac{1}{2}$  we get that (2.1) holds.

If  $f'''(x) \geq 0$ , then  $\frac{\partial^2 F(u,v)}{\partial u \partial v} \geq 0$ , where

$$F(u, v) = f(u+v) + f(|u-v|), \quad u, v \in [0, L].$$





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Therefore according to Theorem A, the sum

$$\sum_{i=1}^n f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{|x_i + x_{i+1}|}{2}\right), \quad x_{n+1} = x_1,$$

is maximal for  $(\hat{\mathbf{x}}) = (\hat{x}_1, \dots, \hat{x}_n)$ , which is the circular symmetric rearrangement of  $(\mathbf{x})$ . Therefore in this case (2.2) holds as well.  $\square$

*Remark 1.* For a positive superquadratic function  $f$ , which according to Lemma A is also a convex function, (2.1) is a refinement of Theorem B.

If  $f'''(x) \geq 0$ , (2.2) is a refinement of Theorem B as well.

*Remark 2.* Theorem B is refined by

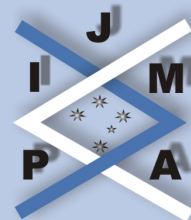
$$\begin{aligned} \sum_{i=1}^n f(x_i) - f\left(\frac{\sum_{i=1}^n x_i}{n}\right) &\geq \frac{n-1}{n} \left( \sum_{i=1}^n f\left(\frac{\hat{x}_i + \hat{x}_{i+1}}{2}\right) \right) \\ &\geq \frac{n-1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1}}{2}\right), \end{aligned}$$

because a convex function  $f$  satisfies the conditions of Theorem A for  $F(u, v) = f(u + v)$ .

The following inequality is a refinement of Theorem C for a positive superquadratic function  $f$ , which is therefore also convex. The inequality results easily from Lemma B and the identity

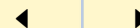
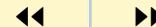
$$\sum_{i=1}^n f(a_i) = \sum_{i=1}^n \left( \frac{1}{n-1} \sum_{j=1}^n f(a_j) (1 - \delta_{ij}) \right)$$

(where  $\delta_{ij} = 1$  for  $i = j$  and  $\delta_{ij} = 0$  for  $i \neq j$ ), therefore the proof is omitted.



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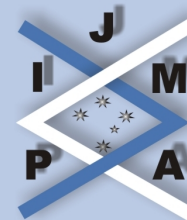
**Theorem 2.2.** Let  $f$  be a superquadratic function on  $[0, L]$ , and let  $x_i \in [0, L]$ ,  $i = 1, \dots, n$ . Then

$$\begin{aligned} & \frac{n}{n-1} \left( \left( \sum_{i=1}^n f(x_i) \right) - f(\bar{x}) \right) - \sum_{i=1}^n f(y_i) \\ & \geq \frac{1}{n-1} \left( \sum_{i=1}^n \sum_{j=1}^n f(|y_i - x_j|) (1 - \delta_{ij}) \right) + \frac{1}{n-1} \sum_{i=1}^n f(|\bar{x} - x_i|), \end{aligned}$$

where  $\bar{x} = \sum_{i=1}^n \frac{x_i}{n}$ ,  $y_i = \left( \frac{n\bar{x} - x_i}{n-1} \right)$ ,  $i = 1, \dots, n$ .

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