JGSP **27** (2012) 109–113

JOURNAL OF
Geometry and Symmetry
in Physics

BOOK REVIEW

Structure and Geometry of Lie Groups, by Joachim Hilgert and Karl-Hermann Neeb, Springer Monographs in Mathematics, Springer, New York 2012, x+744 pp. ISBN: 978-0-387-84793-1 (Print) 978-0-387-84794-8 (Online).

The theory of Lie groups and Lie algebras lies at the heart of modern-day mathematics, with applications too numerous to mention. Not surprisingly, the literature abounds with texts on the subject, ranging from the introductory to treatises beyond and well beyond an introduction.

Many beginner texts, geared towards students, offer shortcuts into subjects such as a Semi-Simple Representation Theory or Complex Geometry, thereby evading intricacies of necessity. Other, more specialised tracts focus by definition on particular species of Lie groups – nilpotent, solvable, or semi-simple – and on questions specific to that lineage. This state of the literature is perfectly efficient and satisfactory to the novice and the expert. However, in the middle ground connecting these boundaries of the community, a compendium was lacking that was comprehensive without resorting to generalities.

As the title suggests, the present monograph addresses the topic from such a general vantage point: combining a hands-on initiation *via* matrix groups with the thorough treatment of the general structure theory of Lie algebras and the machinery of Differential Geometry, it culminates in the definitive study of the most salient structure theoretical questions – decompositions, conjugacy of tori and maximal compacts, closedness of subgroups, linearity, complexifications, disconnectedness, and covering theory.

Thus, the aim is to provide both, on one hand, a serious introduction to Lie theory, placing the neophyte on solid footing, and on the other, a profound manual to the fine points of structure theory, which any practioner of Lie groups is sure to stumble over, and for which even the expert may be in want of a reference. Further developments and applications are intentionally left to the existing literature.

As the authors state in the *Preface*, the present work is based on [3]. The latter book followed a not entirely different idea: Namely, to give at once a no-nonsense introduction to Lie theory and a presentation of the full structure theory in all its

Byzantine beauty – thus furnishing a replacement for Hochschild's time-honoured classic [4], which at the time was long out of print. However, the present volume differs in an essential respect from the latter two references. Motivated by developments in Lie theory and Representation Theory in last two decades, it brings the Differential Geometry back into the picture, thus considerably broadening its scope (it is twice as long as [3]).

The book is divided into four parts – entitled Matrix Groups – Lie Algebras – Manifolds and Lie Groups – Structure Theory of Lie Groups – totalling in eighteen chapters, with the addition of four Appendices – entitled Basic Covering Theory – Some Multilinear Algebra – Some Functional Analysis – Hints to the Exercises. Moreover, there is a slim but carefully compiled Bibliography and an excellent Index of symbols and terminology.

Part I begins by introducing the General Linear Group and its closed subgroups as topological groups, in *Chapter 2*, using methods from Linear Algebra and Point-Set Topology. Notable features are the inclusion of examples such as the Galilei and Poincaré groups, and a discussion of quaternionic matrix groups. Next, in *Chapter 3*, the matrix exponential function is introduced, and an analytic proof of the Baker–Campbell–Hausdorff formula is given. Finally, in *Chapter 4*, using the exponential function, the passage form Lie groups to Lie algebras is explained.

Part II concerns the general theory of Lie algebras. Firstly, in *Chapter 5*, the basic classes of Lie algebras – nilpotent, solvable, and semi-simple – are presented, and basic theorems about them are proved: Engels's theorems, Cartan's semi-simplicity criterion, Weyl's theorem on complete reduciblity. The basic structure theoretical results (by Levi and Malcev) on general Lie algebras are proved, and reductive Lie algebras are introduced. The general theory of Cartan subalgebras is treated in *Chapter 6*, a subject otherwise only accessible through the Bourbaki tract [2]. It is applied to obtain the root decomposition of semi-simple Lie algebras. In *Chapter 7*, the universal enveloping algebra is defined and the Poincaré-Birkhoff-Witt theorem is derived. As applications, free Lie algebras are discussed and proofs are given of Serre's theorem, the theorem of the highest weight, and of Ado's theorem.

At this point, Lie algebra cohomology is introduced. By way of motivation, it is applied to Abelian extensions and to the question of unimodularity. The relation of Weyl's and Levi's theorems to the Whitehead lemmas is explained and Whitehead's vanishing theorem is proved. As a further application, the conjugacy of Cartan subalgebras of solvable Lie algebras is discussed. Finally, there is a nice treatment of general extensions of Lie algebras by these methods.

Up to this point, although extended and updated in its presentation, the book follows a similar route as [3]. The main differences begin in *Part III*, in which manifolds and Lie groups are first considered. This part of the book begins with a condensed but nonetheless complete introduction to smooth manifolds on forty pages. The emphasis is on vector fields and local flows. This is immediately applied to introduce Lie groups, define the exponential function, and prove the Baker–Campbell–Hausdorff formula is this setting. The closed subgroup theorem is proved and, following a discussion of local Lie group data, so is the existence of integral subgroups and Lie's third theorem. Covering theory of Lie groups is treated, leading to first examples of non-linear Lie groups. Finally, Yamabe's theorem on integral subgroups (stating that any path-connected subgroup of a Lie group is integral) is proved, leading to a proof of the fact that any subgroup of a Lie group carries an initial Lie group structure.

In the final *Chapter 10* of Part III, actions of Lie groups are studied. Homogeneous spaces and frame bundles are introduced. Integration on manifolds is treated for both differential forms and densities, leading to a discussion of invariant integration on Lie groups. As applications, the non-vanishing of $H^n_{\mathrm{dR}}(G,\mathbb{R})$ for G compact and Palais' theorem are proved.

At this point, the basic theory of Lie groups has been established, and the authors turn to more specialised questions in *Part IV*, beginning with a discussion, in *Chapter 11*, of connectivity questions: fundamental groups of quotients (circumventing the use of the Long Exact Sequence in homotopy), smooth splitting of extensions, structure of nilpotent groups, and the Lie group structure on the automorphism group of a Lie group with finitely generated component group. Compact Lie groups receive, in *Chapter 12*, a careful treatment containing a study of Lie groups with compact Lie algebras, the conjugacy of maximal tori, the linearity of compact Lie groups (a version of Gelfand–Raikov), and a discussion of their topology.

In the short *Chapter 13*, basics of semi-simple Lie groups are explained: the Cartan decomposition, the existence of compact real forms, and the Iwasawa decomposition in the semi-simple case. Here, the authors do not go into excessive detail, since this part of the theory is well-covered in the literature, for instance in [5–7]. Following the discussion of various subclasses of Lie groups in previous chapters, the general structure theory is addressed in *Chapter 14*. Here, the existence and conjugacy of maximal compact subgroups of almost connected Lie groups is proved and the structure of the center of a connected is described. Subsequently, various splitting theorems for almost connected Lie groups are derived, consummate with an exposition of the general Iwasawa decomposition. The only other sources for the latter (save Iwasawa's original papers) known to the reviewer are [4] and Borel's lecture notes [1]. The exponential function of solvable Lie groups is studied, ending with a proof of Dixmier's theorem characterizing the simply connected Lie groups whose exponential function is a diffeomorphism. The chapter is rounded off with an Appendix on finitely generated Abelian groups.

The following *Chapter 15* concerns complex Lie groups. In particular, the universal complexification and complex Abelian Lie groups are studied, and the question when an almost connected complex Lie groups is linearly complex reductive (*i.e.,* is the universal complexification of a compact connected Lie group) is given a description which is likely the most comprehensive available in print. In*Chapter 16*, the case of linearly real reductive Lie groups is considered, and this property is characterized for semi-simple groups. Subsequently, the linearity of general Lie groups is discussed, and then specialized to the case of complex Lie groups.

In *Chapter 17*, the authors reap the fruits of their travails, applying the general theory to several classical groups. Notably, quaternionic groups, indefinite pin and spin groups, and conformal groups are considered. In the final *Chapter 18*, disconnected Lie groups are studied *via* extensions by discrete groups and the use of disconnected covering theory.

An impressive feat and quite a slab of knowledge to hold in hand, the sheer volume of this monograph may appear daunting. However – and this bespeaks the authors' wealth of experience not only as experts in the subject, but equally as teachers and expositors of their field of expertise – the text is never cluttered or ambulatory, but always crystal clear, terse, and to the point. This book is exceptionally well-structured. Moreover, the consistent cross-referencing and the meticulously composed indices of notation and terminology make its use as a reference manual a pure pleasure.

But that is not all. Most of the material presented in the book has been used for lecture courses and thoroughly tested in the classroom. Every section ends with well sought-out exercises, and there is an Appendix containing hints to their solution (but, thankfully, no more than that). A tiny but utterly remarkable gem is the half-page teaching guide in the Introduction which suggests no less than seven paths to be taken through the book, yielding one- to two-semester long courses.

In summary, this monograph is a highly welcome addition and complement to the existant Lie group literature. It will be an unfailing companion for serious students and practioners, and an invaluable one-step reference manual for the seasoned expert. In short, if you have spare space on your bookshelf, this is a must!

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