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Complements, Approximations, Smoothings and Invariance Properties

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Proximal methods are used to determine the relationship between normal cones to a closed set in $\Re^n$ and those to the closure of its complement. The geometry of outer and inner set approximations is explored as well. In the context of differential inclusions, we study the extent to which approximations and associated smoothings inherit invariance and tangentiality properties posited for the original set. This analysis leads to the construction of a Lipschitz feedback law which achieves penetration of the interior of compact sets satisfying hypotheses which include a strict tangentiality condition. Analytic versions of some of the geometric results are provided, when the set of interest is the epigraph of a continuous function. In that setting, invariance properties correspond to types of monotonicity along trajectories, and we obtain smoothing results which are couched in terms of convolution operations and Hamilton-Jacobi inequalities, as well as a result on the existence of a universal Lipschitz feedback law which achieves monotonicty along trajectories in an approximate sense.

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1. Introduction

A closed set $S \subset \Re^n$ is said to be *epi-Lipschitz* at a boundary point x provided that $N_S^C(x)$, the Clarke normal cone to S at x, is pointed. This is a geometric attribute which plays an important role in nonsmooth analysis. Rockafellar [24] showed that the epi-Lipschitz property is equivalent to the existence of an isometry A (a unitary matrix) such that for some neighborhood U of x, the set $U \cap S = U \cap A[\mathrm{epi}(f)]$, where epi$(f)$ denotes the epigraph of a Lipschitz function f. Let us denote the closure of the complement of S by $\hat{S}$. Then, upon exploiting the connection between N_S^C and the C-subdifferential $\partial_C f$, as well as the C-calculus fact that $\partial_C f = -\partial_C(-f)$, one can show that

$$N_S^C(x) = -N_{\hat{S}}^C(x). \tag{1}$$

In fact, in this epi-Lipschitz setting setting, formula (1) can be regarded as folklore within the theory of generalized gradients.

In this article, we shall provide analogs of this "complementary normal formula" in terms of $N_S^P(x)$, the proximal normal cone to S at x, for S not necessarily epi-Lipschitz. In the most general of these results, Theorem 3.1 below, we require only the mild condition that x be a limit of interior points of S. The result is that for such x,

$$-N_S^P(x) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} [N_{\hat{S}}^P(y)]^\delta \right\} \quad \forall\, \varepsilon > 0, \quad \forall\, \delta > 0, \tag{2}$$

where "$\overline{\mathrm{co}}$" denotes the closure of the convex hull and $[N_{\hat{S}}^P(y)]^\delta$ is a certain "directional" approximator of $N_{\hat{S}}^P(y)$ (which agrees with $N_{\hat{S}}^P(y)$ when $\delta = 0$). A measure of the generality of Theorem 3.1 is that its proof does not depend upon properties of the proximal subdifferential; indeed, the assumptions on S do not imply that S is locally homeomorphic to an epigraph, and so reliance on P-calculus should not be expected. It is an open question as to whether formula (2) holds in general with $\delta = 0$, or as in (2) with $\delta > 0$ in infinite dimensions. On the other hand, we shall obtain, via an independent proof utilizing a mean value inequality of Clarke and Ledyaev [11], the following specialization of Theorem 3.1 with $\delta = 0$, valid in an infinite dimensional Hilbert space: When S is the epigraph of a function f continuous at x, then one has the complementary P-normal formula

$$-N_S^P(x, f(x)) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} N_{\hat{S}}^P(y, f(y)) \right\} \quad \forall\, \varepsilon > 0. \tag{3}$$

In fact, formula (2) above is valid with $\delta = 0$ when S is merely locally isometric to the epigraph of a continuous function. (In particular, this result holds when S is epi-Lipschitz; it is explained how this can also be obtained via a C-calculus proof, not depending on the mean value inequality.) Independently of the above complementary P-normal formulas, but also in an intrinsically geometric vein, we shall utilize properties of proximal normals to offer a new proof of the complementary C-normal formula (1).

Beyond investigating complementary normality issues, a related goal of this article is to illuminate the geometry of the *r-outer approximation*

$$S^r := \{z : d_S(z) \leq r\}$$

and the *r-inner approximation*

$$S_r := \{z : d_{\hat{S}}(z) \geq r\}$$

of S, where $r > 0$ and d_S denotes the Euclidean distance to S. Observe that one always has

$$\text{bdry}(S_r) = \{z : d_{\hat{S}}(z) = r\},$$

but in general, only the containment

$$\text{bdry}(S^r) \subseteq \{z : d_S(z) = r\}$$

holds for a given r.

An idea which features in our analysis is that when S is compact and epi-Lipschitz, the inner approximations S_r are *proximally smooth* for small r. This means that double approximations of the form $(S_r)^\delta$ are smooth for correspondingly small δ; we shall make use of characterizations of proximal smoothness derived recently in Clarke, Stern and Wolenski [14]. The conclusions reached relate to results of Benoist [8], who first studied the smoothing properties of double approximations of this type, and may be related to forthcoming work of Cornet and Czarnecki [19]. We remark that epi-Lipschitz and other properties of sets and their approximations play an important role in the existence of equilibrium points in nonconvex sets [18].

We shall apply our geometric results to certain issues which arise in the context of differential inclusions of the form

$$\dot{x}(t) \in F(x(t)) \quad \text{a.e.,} \quad t \geq 0. \tag{4}$$

A question of considerable interest is the determination of the manner in which complements, approximations and smoothings of S inherit invariance and corresponding tangentiality properties posited for S itself. The types of invariance that we have in mind are *strong invariance* (every trajectory of (4) starting in S remains in S) and *weak invariance* (for every initial point in S, some trajectory remains in S); this is also called "viability" by Aubin et alli [4] [5] [3]. Under the topological assumption that $S = \text{cl}[(\text{int}(S)]$, and suitable hypotheses on the multifunction F, we will show that strong invariance of (S, F) and $(\hat{S}, -F)$ are equivalent. While this conclusion seems quite natural (and of course, it is), parallel considerations for "complementary" weak invariance are more delicate, and further regularity assumptions on S will need to be imposed. As for the inheritance of weak invariance by approximations, we shall see that when S is compact and (S, F) is weakly invariant, then for given $\varepsilon > 0$, $(S^r, F + \varepsilon\overline{B})$ is weakly invariant (where $\overline{B}$ denotes the closure of B, the open unit ball). However, in order to obtain a similar result involving inner approximations S_r, it transpires that extra conditions on S need to be imposed here too. A by-product of our analysis of these issues is the construction of a Lipschitz feedback law which achieves penetration of the interior of sets satisfying certain hypotheses, including a kind of strict tangentiality (or "inwardness") with respect to F. This relates to work on set attainability and penetration in Clarke and Wolenski [16].

Given a continuous function $f : \Re^n \to \Re$, a property which generalizes "monotonicity along solutions" from the classical Liapounov theory is the following: (f, F) is *weakly decreasing* if for any initial point x_0 there exists a trajectory $x(\cdot)$ of (4) such that the function $t \to f(x(t))$ is nonincreasing. This property is rephrasable as weak invariance of the epigraph of f with respect to a certain augmentation of F, and versions of some of our geometric results can be brought to bear (with particular emphasis on the case of Lipschitz f). In this context, approximations and smoothings of $S = \text{epi}(f)$ are expressible in terms of the epigraphs of certain single and double convolutions involving f. Here we make contact with work of Lasry and Lions [22], Attouch and Azé [2], Seeger [26] and Ioffe [21]. The results obtained lead to interesting conclusions regarding solvability (in an approximate sense) of Hamilton-Jacobi inequalities by smooth functions. In addition, we obtain a result on the existence of a universal Lipschitz feedback law yielding monotonicity along trajectories in an approximate sense.

After a section on preliminaries in nonsmooth analysis, the plan of the article follows the order of the preceding discussion: In section 3 and section 4 we shall provide intrinsically geometric proofs of complementary P-normal and complementary C-normal formulas, respectively, and some other results in this general vein. Geometric results are given in section 5, regarding approximations and smoothings. These facts are then applied in section 6, in the context of differential inclusions, in order to study the inheritance of invariance properties of a given set by complements, approximations and smoothings, as well as the construction of penetrative Lipschitz feedback laws. In section 7, analytic versions of some of the results are given.

2. Preliminaries in nonsmooth analysis

In this section we provide a whirlwind review of required material from nonsmooth analysis and differential inclusions. General references and literature guides to these subjects are provided by Clarke [10], [9], Loewen [23] and Clarke, Ledyaev, Stern and Wolenski [17], [12], Aubin and Cellina [4], and Aubin [3]. Throughout this article, *S will always denote a nonempty closed subset of $\Re^n$*, with extra hypotheses added as required. The Euclidean distance of a point u to S is given by

$$d_S(u) := \min\{\|u - x\| : x \in S\};$$

this function is globally Lipschitz of rank 1 for any S. The (possibly empty) set of closest points to u in S is denoted

$$proj_S(u) := \{x \in S : \|u - x\| = d_S(u)\}.$$

If $u \notin S$ and $x \in proj_S(u)$, then we say that the vector $u - x$ is a *perpendicular* to S at x. The set of all nonnegative multiples of such perpendiculars is denoted $N_S^P(x)$, and is referred to as the *proximal normal cone* (or P-normal cone) to S at x. One can show that $\zeta \in N_S^P(x)$ if and only if there exists $M > 0$ (generally depending upon x) such that the following *proximal normal inequality* holds:

$$M\|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall\, y \in S.$$

If $x \in int(S)$ or no perpendiculars to S exist at x, then by convention, we set $N_S^P(x) = \{0\}$. The P–normal cone is convex (but may not be closed), and is nonzero on a dense subset of bdry(S).

Let $f : \Re^n \to (-\infty, \infty]$ be a lower semicontinuous extended real valued function, and denote its epigraph by

$$\text{epi}(f) := \{(x, y) : x \in \text{dom}(f),\ y \geq f(x)\},$$

where dom(f) is the set of points where f is finite. The lower semicontinuity assumption is equivalent to epi(f) being closed. A vector $\zeta \in \Re^n$ is said to be a *proximal subgradient* (or P-subgradient) of f at x provided that

$$(\zeta, -1) \in N^P_{\text{epi}(f)}(x, f(x)).$$

The set of all such vectors is called the P-*subdifferential* of f at x, and is denoted $\partial_P f(x)$. This subdifferential is empty for points not in the domain of f, and is nonempty for a dense subset of the domain. Furthermore, one can prove that $\zeta \in \partial_P f(x)$ if and only if there exist positive numbers σ and γ such that the following *proximal subgradient inequality* is satisfied:

$$f(y) - f(x) + \sigma \|y - x\|^2 \geq \langle \zeta, y - x \rangle \quad \forall\, y \in x + \gamma B.$$

We shall require the following fact due to Rockafellar [25] concerning the approximation of horizontal P-normals to an epigraph by non-horizontal ones: Suppose that f is finite at x, and let $(\zeta, 0) \in N^P_{\text{epi} f}(x, \alpha)$, where $\zeta \neq 0$. Then

$$(\zeta, 0) \in N^P_{\text{epi} f}(x, f(x)),$$

and for any given $\varepsilon > 0$, there exists $\tilde{x} \in \text{dom}(f)$ such that

$$\|(\tilde{x}, f(\tilde{x})) - (x, f(x))\| \leq \varepsilon$$

and such that there exists $(\tilde{\zeta}, -\gamma) \in N^P_{\text{epi} f}(\tilde{x}, f(\tilde{x}))$, $\gamma > 0$, for which

$$\|(\zeta, 0) - (\tilde{\zeta}, -\gamma)\| \leq \varepsilon.$$

The P-*superdifferential* of an upper semicontinuous function f at x is defined as

$$\partial^P f(x) := -\partial_P(-f)(x), \tag{5}$$

with the members of this set referred to as P–*supergradients*. We will later require the fact that for an upper semicontinuous continuous function,

$$(\eta, -1) \in N^P_{\text{epi}(-f)}(x, -f(x)) \iff (\eta, 1) \in N^P_{\text{hyp}(f)}(x, f(x)), \tag{6}$$

where

$$\text{hyp}(f) := \{(x, y) : x \in \text{dom}(f),\ y \leq f(x)\}$$

is the hypograph of f.

We will make reference to the following specialization of the mean value inequality of [11]; see also [17]: For f as above, any points x, y, any number $r < f(y) - f(x)$, and any given $\varepsilon > 0$, there exist $z \in [x, y] + \varepsilon B$ and $\zeta \in \partial_P f(z)$ such that

$$r < \langle \zeta, y - x \rangle.$$

(Here $[x, y]$ denotes the line segment from x to y.)

An important fact regarding proximal subdifferentiability of the distance function is the following: Let $u \notin S$, and suppose that $\partial_P d_S(u) \neq \phi$. Then $\mathrm{proj}_S(u)$ is a singleton, say $\{x\}$, and $\partial_P d_S(u)$ is the singleton $\{\zeta\}$, where

$$\zeta = \frac{u - x}{\|u - x\|} \in N_S^P(x); \tag{7}$$

see Clarke, Ledyaev and Wolenski [13]. If $d_S(u) = r > 0$, then by a result in [14], one always has

$$\partial_P d_S(u) = N_{S^r}^P(u) \cap \Omega, \tag{8}$$

where emptiness is not precluded; here and later we denote the unit sphere by $\Omega := \mathrm{bdry}(B)$. Now let $u \in S$. Then it can be shown that

$$N_S^P(u) = \mathrm{cone}[\partial_P d_S(u)], \tag{9}$$

where this denotes the cone generated by the P-subdifferential of the distance function.

The *L-normal cone* (or limiting normal cone) to S at $x \in S$ is defined to be the set

$$N_S^L(x) := \{\zeta : \zeta_i \to \zeta,\ \zeta_i \in N_S^P(x_i),\ x_i \to x\}.$$

In particular, $N_S^P(x) \subseteq N_S^L(x)$. Also, it is not hard to show that N_S^L is zero on the interior of S, nonzero on the boundary of S, and that the multifunction N_S^L is closed on S. The *C-normal cone* (or Clarke, or convexified normal cone) to S at x is defined as

$$N_S^C(x) := \overline{\mathrm{co}}[N_S^L(x)].$$

We say that S is *epi-Lipschitz* at $x \in \mathrm{bdry}(S)$ provided that the cone $N_S^L(x)$ (or equivalently, $N_S^C(x)$) is pointed; if the property holds at each boundary point, then we say S is an epi-Lipschitz set. It is known that if S is epi-Lipschitz at x, then the multifunction N_S^C is closed at x.

The normal cones defined above lead to corresponding nonempty subdifferential sets for f:

$$\partial_L f(x) := \{\zeta : (\zeta, -1) \in N_{\mathrm{epi}(f)}^L(x, f(x)\}.$$

$$\partial_C f(x) := \{\zeta : (\zeta, -1) \in N_{\mathrm{epi}(f)}^C(x, f(x)\}.$$

These are the *L-subdifferential* and *C-subdifferential* of f at x, respectively. One has

$$\partial_L f(x) = \{\zeta : \zeta_i \to \zeta,\ \zeta_i \in \partial_P f(x_i),\ x_i \to x,\ f(x_i) \to x\}.$$

Let us now assume that f is Lipschitz of rank K on an open set U. By a result in [15], or by appealing to the mean value inequality, this is equivalent to $\|\zeta\| \leq K$ for every $\zeta \in \partial_P f(x)$ for each $x \in U$ (bearing in mind that the P-subdifferential is assured to be nonempty only on a dense set). Then $\partial_C f(x)$ is compact, and one has the relations

$$\partial_C f(x) = \mathrm{co}[\partial_L f(x)] = -\partial_C(-f)(x).$$

Should it exist, the *directional derivative* of f at x in the direction v is the quantity

$$f'(x;v) := \lim_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

If there exists $\zeta \in \Re^n$ such that $f'(x;v) = \langle \zeta, v\rangle$ for every v, then ζ is unique, and we say that $\zeta := f'(x)$ is the *derivative* of f at x. (In the finite dimensional and Lipschitz case presently under consideration, one can show that Gâteaux and Fréchet differentiabililties coincide, and so there is no ambiguity in this terminology.) For given $v \in \Re^n$, one defines the *generalized directional derivative* of f at x in the direction v as

$$f^o(x;v) := \limsup_{\substack{t\downarrow 0\\ y\to x}} \frac{f(y+tv)-f(y)}{t}.$$

For any v, one has

$$f^o(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial_C f(x)\}.$$

The *C-tangent cone* to S at $x \in S$ is defined via polarity as the closed convex cone

$$T_S^C(x) := [N_S^C(x)]^* = \{v \in \Re^n : \langle \zeta, v\rangle \leq 0 \ \ \forall\, \zeta \in N_S^C(x)\} = [N_S^L(x)]^*.$$

Therefore $T_S^C(x) = \Re^n$ for $x \in \mathrm{int}(S)$, and S is epi-Lipschitz at $x \in \mathrm{bdry}(S)$ iff the interior of $T_S^C(x)$ is nonempty. It can be shown that

$$T_S^C(x) = \{v \in \Re^n : d_S^o(x;v) = 0\},$$

which is the traditional way of defining the C-tangent cone.

The *D-tangent cone* to S at $x \in S$, referred to also as the Bouligand or contingent tangent cone in some references, is defined to be the closed (but possibly nonconvex) cone

$$T_S^D(x) := \mathrm{cone}\left\{\lim \frac{s_i - x}{\|s_i - x\|} : s_i \in S,\ s_i \to x\right\}.$$

One always has $T_S^C(x) \subseteq T_S^D(x)$. Should equality occur, then we say that S is *regular* at x, and if the property holds at every $x \in S$, then S is said to be a regular set. We require the fact that if S is regular at x, then $N_S^L(x) = N_S^C(x)$.

A function $f : \Re^n \to \Re$ which is Lipschitz near x is said to be *regular* at x if the ordinary directional derivative $f'(x;v)$ exists and $f'(x;v) = f^o(x;v)$ for every $v \in \Re^n$. Regularity at x is guaranteed when f is continuously differentiable at x, or if f is convex on a ball around x. The tie-in beween the geometric and analytic versions of regularity is made via the following fact: Regularity of a Lipschitz function f at x is equivalent to regularity of its epigraph at $(x, f(x))$. In this situation, it follows that $\partial_C f(x) = \partial_L f(x)$.

Let $x \in S$ and $0 \neq \zeta \in N_S^P(x)$. Then for some $r > 0$ one has

$$S \cap \operatorname{int}\left\{x + r\left(\frac{\zeta}{\|\zeta\|} + \overline{B}\right)\right\} = \phi \quad \forall\, y \in S,$$

and we say that ζ is *realized by an r-ball*, or that ζ is *r-realizable*. This simply means that the open ball of radius r centered at $x + r\frac{\zeta}{\|\zeta\|}$ has empty intersection with S. This property can be rephrased as

$$\frac{1}{2r}\|y - x\|^2 \geq \left\langle \frac{\zeta}{\|\zeta\|}, y - x \right\rangle \quad \forall\, y \in S. \tag{10}$$

If this should hold at every $x \in \operatorname{bdry}(S)$, then we say that S is *proximally smooth* of radius r. (Our reference in on proximal smoothness is [14]; see also Vial [28] for earlier work in this regard.) Proximal smoothness of radius $r > 0$ is equivalent to each of the following:

(a) $\partial_P d_S(x) \neq \phi$ for every x in the open tube $U(r) := \{\operatorname{int}(S^r)\}\backslash S$.

(b) For every $\tilde{r} \in (0, r)$ and every x such that $d_S(x) = \tilde{r}$, one has $N^P_{S^{\tilde{r}}}(x) \neq \{0\}$. (This follows directly from (a) and (8)).

(c) $d_S \in C^{1+}$ on $U(r)$; that is, d'_S exists and is locally Lipschitz on $U(r)$. (One can show that then $\partial_P d_S(x) = \partial_C d_S(x) = \{d'_S(x)\}$ on $U(r)$.)

(d) $S = (S^{\tilde{r}})_{\tilde{r}}$ for every $\tilde{r} \in (0, r)$.

Suppose that S is proximally smooth of radius r, and that $\tilde{r} \in (0, r)$. Then $S^{\tilde{r}}$ is C^{1+}-*smooth* (that is, the mapping $x \to N^C_{S^{\tilde{r}}}(x) \cap \Omega$ is single valued and Lipschitz on $\operatorname{bdry}(S^{\tilde{r}})$), and $S^{\tilde{r}}$ is proximally smooth of radius $r - \tilde{r}$. Also,

$$N^C_{S^{\tilde{r}}}(x) \cap \Omega = \{\nabla d_S(x)\} = \{\nabla d_{S^{\tilde{r}}}(x)\}$$

for every x of distance $\tilde{r}$ from S. A necessary condition for S to be proximally smooth is that N_S^P be a closed multifunction on S. (This is not sufficient however; see [14] for a counterexample.) The closedness condition implies

$$N_S^P(x) = N_S^L(x) = N_S^C(x) \quad \forall\, x \in S, \tag{11}$$

which can be used to show that S is regular at x. Of course, (11) implies

$$N_S^P(x) \neq \{0\} \quad \forall\, x \in \operatorname{bdry}(S). \tag{12}$$

The property (12) alone will be termed *mild proximal smoothness*. For an example of $S \subseteq \Re^2$ which is mildly proximally smooth but not proximally smooth, consider the epigraph of the (regular) function f given by $f(x) = -x$ for $x \leq 0$ and $f(x) = -x^{3/2}$ for $x > 0$.

Several of our results will be formulated with respect to the differential inclusion (4). Trajectories of this system are absolutely continuous functions $x(\cdot)$ satisfying (4). Our *standing hypotheses* regarding the multifunction $F : \Re^n \hookrightarrow \Re^n$ will be

(SH): $F(x)$ is compact and convex for every x, and F is Hausdorff continuous on $\Re^n$.

Two other conditions that will be referred to are *linear growth* and a *Lipschitz condition*:

(LG): There exists $c > 0$ such that for every x and every $v \in F(x)$, one has $\|v\| \leq c(1 + \|x\|)$.

(L): F is Lipschitz on $\Re^n$; that is, there exists $K > 0$ such that

$$d\,(F(x), F(y)) \leq K\|x - y\|,$$

for all x, y in S, where $d(\cdot,\cdot)$ denotes Hausdorff distance. (Note that (L) $\Longrightarrow$ (LG).)

The set S is said to be *weakly invariant* with respect to F (or the pair (S, F) is weakly invariant) provided that for each $x_0 \in S$, there exists a trajectory of (4) satisfying $x(0) = x_0$ and $x(t) \in S$ for all $t \geq 0$. Likewise, we say that (S, F) is *strongly invariant* if for each $x_0 \in S$, this occurs for *every* trajectory $x(\cdot)$. When (SH) and (LG) hold, weak invariance is equivalent to

$$h_F(x, \zeta) \leq 0 \;\; \forall\, x \in S, \quad \forall\, \zeta \in N_S^P(x), \tag{13}$$

where h_F is the Hamiltonian

$$h_F(x, \zeta) := \min_{v \in F(x)} \langle \zeta, v \rangle.$$

We also express (13) as

$$h_F(x, N_S^P(x)) \leq 0 \;\; \forall\, x \in S. \tag{14}$$

It is easy to see that an equivalent condition for weak invariance in terms of the L-normal cone is given by

$$h_F(x, N_S^L(x)) \leq 0 \;\; \forall\, x \in S. \tag{15}$$

We also require the fact that when S is regular, this is the same as

$$h_F(x, N_S^C(x)) \leq 0 \;\; \forall\, x \in S. \tag{16}$$

In terms of D-tangency, yet another equivalent condition is

$$T_S^D(x) \cap F(x) \neq \phi \;\; \forall\, x \in S. \tag{17}$$

There is a long history behind invariance theory and its tangential characterizations (see for example [12] [4] [3]). To our knowledge it is Veliov [27] who first clearly expressed and systematically developed the normal characterization (13) (for the special case where the differential inclusion models a control system). Let us pause to note that Hausdorff continuity of F is more than what is needed in the above characterizations of weak invariance—upper semicontinuity will suffice here, but our subsequent results involving set approximations require the stronger hypothesis, so we have included it in (SH) at the outset. Also, should S be compact, then (LG) can be omitted in these characterizations of weak invariance. Hamiltonian and tangent cone characterizations of strong invariance are possible too, when the Lipschitz property (L) holds, but will not be required in this article. In fact, our only need for (L) arises in connection with continuity in initial data for (4); see Lemma 6.1 below.

Let $f : \Re^n \to \Re$ be a continuous function, and let the multifunction F satisfy (SH), (LG). We say the pair (f, F) is *weakly decreasing* provided that for every $x_0 \in \Re^n$, there exists a trajectory of (4) satisfying $x(0) = x_0$ and such that the function $t \to f(x(t))$ is nonincreasing. One can show that this property is equivalent to weak invariance of the pair $(\mathrm{epi}(f), F \times \{0\})$, and with some effort, that this is in turn equivalent to the condition

$$h_F(x, \partial_P f(x)) \leq 0 \quad \forall\, x \in \Re^n \tag{18}$$

as well as to

$$h_F(x, \partial_L f(x)) \leq 0 \quad \forall\, x \in \Re^n. \tag{19}$$

Should f be locally Lipschitz and regular, then this is in turn equivalent to

$$h_F(x, \partial_C f(x)) \leq 0 \quad \forall\, x \in \Re^n. \tag{20}$$

3. Complementary P-normal formulas

3.1. General case

The following result provides a fundamental relationship between the P-normal cone to a closed set S and the P-normal cone to the closure of its complement. The proof is purely geometric and variational in nature, and is apparently independent of any known results in nonsmooth analysis. It requires only the mild topological condition that x belong to the closure of the interior of S.

Prior to stating the result, we need some further notation. Let $K \subseteq \Re^n$ be a cone. Then given $\delta > 0$, we denote

$$[K]^\delta := \mathrm{cone}\{w + \delta B : w \in K, \|w\| = 1\}.$$

One can think of $[K]^\delta$ as a "directional" approximation of K, as opposed to a metric one. Of course, this approximation agrees with K when $\delta = 0$.

Theorem 3.1. *Suppose that $x \in \mathrm{cl}[\mathrm{int}(S)]$. Then for any $\varepsilon > 0$ and any $\delta > 0$ one has*

$$-N_S^P(x) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} \left[N_{\hat{S}}^P(y) \right]^\delta \right\}. \tag{21}$$

The proof of the theorem is based upon the following proposition, which is of some independent interest:

Proposition 3.2. *Given $r > 0$ such that $S_r \neq \phi$ and $w \in \hat{S}$, let $z \in \mathrm{proj}_{S_r}(w)$. Then for any $\delta > 0$ one has*

$$w - z \in \overline{\mathrm{co}} \left\{ \mathrm{cone} \left[\bigcup_{y \in \mathrm{proj}_{\hat{S}}(z)} (y - z) + \delta r B \right] \right\}. \tag{22}$$

Proof. Clearly $\|z-w\| \geq r$. If $\|z-w\| = r$, then $w \in \text{proj}_{\hat{S}}(z)$ and we are done. We shall assume therefore that

$$\|z-w\| > r. \tag{23}$$

Suppose, by way of contradiction, that (22) did not hold. Then there exist $\gamma > 0$ and $\zeta \in \Re^n$, $\|\zeta\| = 1$, such that

$$\langle \zeta, w-z \rangle > 0$$

and

$$\langle \zeta, \ell \rangle \leq 0 \;\; \forall\, \ell \in \text{co}\left\{ \text{cone}\left[\bigcup_{y \in \text{proj}_{\hat{S}}(z)} (y-z) + \gamma r B \right] \right\}.$$

By taking $\ell = y - z + \gamma r u$, where $y \in \text{proj}_{\hat{S}}(z)$ and $u \in B$, we obtain from the previous inequality that

$$\langle \zeta, y - z + \gamma r u \rangle \leq 0.$$

Since $u \in B$ is arbitrary, it follows that

$$\langle \zeta, y-z \rangle \leq -\gamma r \;\; \forall\, y \in \text{proj}_{\hat{S}}(z).$$

The remainder of the proof requires three lemmas:

Lemma 3.3. *There exists $r_0 > r$ such that*

$$\left.\begin{array}{l} \|v\| = 1 \\ \langle \zeta, v \rangle \geq -\dfrac{\gamma}{2} \\ z + tv \in \hat{S} \end{array}\right\} \Longrightarrow t \geq r_0.$$

Proof. If not, then $z + t_i v_i \in \hat{S}$ where $t_i \downarrow r$ and $\langle \zeta, v_i \rangle \geq \frac{\gamma}{2}$. Taking convergent subsequences yields $z + rv \in \text{proj}_{\hat{S}}(z)$. But then

$$-r\gamma \geq \langle \zeta, (z+rv) - z \rangle = r\langle \zeta, v \rangle \geq -\frac{\gamma r}{2},$$

which is a contradiction. □

Lemma 3.4. *For all $t > 0$ suficiently small, one has*

$$\|w - (z + t\zeta)\| < \|w - z\|.$$

Proof. Upon squaring and expanding, the inequality becomes

$$-2t\langle \zeta, w-z \rangle + t^2\|\zeta\|2 < 0,$$

which is clearly true for small $t > 0$ since $\langle \zeta, w-z \rangle > 0$. □

Lemma 3.5. *For $t > 0$ sufficiently small, one has $z + t\zeta \in S_r$.*

Proof. If not, then there exists a sequence $y_i \in \hat{S}$ such that

$$\|z + t_i\zeta - y_i\| < r,$$

where $t_i \downarrow 0$.
Now, if $\langle \zeta, y_i - z\rangle \geq -\frac{\gamma}{2}\|y_i - z\|$ infinitely often, then by Lemma 3.3 , since

$$z + \|y_i - z\|\frac{(y_i - z)}{\|y_i - z\|} \in \hat{S},$$

we have $\|y_i - z\| \geq r_0$. But this contradicts $\|z + t_i\zeta - y_i\| < r$ for small t_i.
On the other hand, suppose that $\langle \zeta, y_i - z\rangle < -\frac{\gamma}{2}\|y_i - z\|$ for all large i. Squaring and expanding the inequality $\|z + t_i\zeta - y_i\| < r$ gives

$$\|z - y_i\|^2 + 2t_i\langle \zeta, z - y_i\rangle + t_i^2\|\zeta\|^2 < r^2.$$

Since $\langle \zeta, z - y_i\rangle > 0$, this implies that $\|z - y_i\| < r$, contradicting $z \in S_r$. □

Now Lemmas 3.4 and 3.5 imply that for small positive t, $z + t\zeta$ lies in S_r and is nearer to w than z, which is a contradiction. This completes the proof of the proposition. □

We now are in position to prove the theorem:

Proof of Theorem 3.1: Obviously, we only need to consider $x \in \text{bdry}(S)$, since the assertion of the theorem is trivial for interior points. Let $0 \neq \zeta \in N_S^P(x)$. Then for some $t > 0$ chosen sufficiently small, one has $w := x + t\zeta \in \hat{S}$ and $\text{proj}_S(w) = \{x\}$. Since x is by hypothesis the limit of interior points of S, one has that $S_r \neq \phi$ for all small $r > 0$. Consider a sequence $r_i \downarrow 0$, and let $z_i \in \text{proj}_{S_{r_i}}(w)$. Since x is the unique closest point in S to w, we have $z_i \to x$. Upon applying Proposition 3.2, for arbitrary given $\delta > 0$ one obtains

$$w - z_i \in \overline{\text{co}}\left(\text{cone}\left[\bigcup_{y \in \text{proj}_{\hat{S}}(z_i)} (y - z_i) + \delta r_i B\right]\right).$$

Now, if $y \in \text{proj}_{\hat{S}}(z_i)$, then

$$y - z_i + \delta r_i B \in -[N_{\hat{S}}^P(y)]^\delta,$$

and the preceding formula yields

$$w - z_i \in \overline{\text{co}}\left\{\bigcup_{y \in \text{proj}_{\hat{S}}(z_i)} \left[-\{N_{\hat{S}}^P(y)\}^\delta\right]\right\}.$$

This in turn readily yields (21). □

3.2. *The case* $\delta = 0$

In this subsection we will prove the following version of Theorem 3.1 in which extra hypotheses are imposed so as to permit formula (21) to hold true with $\delta = 0$. As always, S is closed:

Theorem 3.6. *Let* $S = \mathrm{epi}(f)$, *where* $f : \Re^n \to \Re$ *is continuous at* x. *Then formula* (3) *holds, that is,*

$$-N^P_S(x, f(x)) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} N^P_{\hat{S}}(y, f(y)) \right\} \quad \forall\, \varepsilon > 0. \tag{24}$$

The proof of Theorem 3.6 will be based upon the following fundamental connection between proximal subgradients and supergradients, which is valid for Hilbert spaces too. In the case of $\Re^n$, a related result was employed by Barron and Jensen [7] in the theory of lower semicontinuous viscosity solutions of Hamilton-Jacobi equations. The proof below is based upon the mean value inequality:

Proposition 3.7. *Let* $f : \Re^n \to \Re$ *be continuous at* x. *Then for any* $\varepsilon > 0$ *one has*

$$\partial_P f(x) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} \partial^P f(y) \right\}$$

and

$$\partial^P f(x) \subseteq \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} \partial_P f(y) \right\}. \tag{26}$$

Proof. Clearly it suffices to prove only (26). To this end, let ζ be a proximal supergradient to f at x. The proximal subgradient inequality then says that

$$\langle \zeta, x - y \rangle - \sigma \|y - x\|^2 \leq f(x) - f(y) \tag{27}$$

for all y near x. In particular, let us take $\delta \in (0, \varepsilon/2)$ so that (27) holds for all $y \in x + \delta B$. Temporarily fix such a y, and consider

$$r := \langle \zeta, x - y \rangle - \sigma \|y - x\|^2 - \delta^2.$$

Then $r < f(x) - f(y)$.
By the mean value inequality, there exist $z \in [x, y] + \delta B$ and $\xi \in \partial_P f(z)$ such that $r < \langle \xi, x - y \rangle$. We can write $y = x - \delta v$ for some $v \in B$. Then

$$\langle \zeta, v \rangle - \sigma\delta - \delta < \langle \xi, v \rangle.$$

Now let us introduce the set

$$S(\varepsilon) := \overline{\mathrm{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} \partial_P f(y) \right\}.$$

Note that $\partial_P f(z) \subseteq S(\varepsilon)$, since $\|z - x\| < 2\delta < \varepsilon$. It follows that for $\delta > 0$ taken sufficiently small and any $v \in B$,

$$\langle \zeta, v \rangle - \sigma\delta - \delta \leq \sup_{\xi \in S(\varepsilon)} \langle \xi, v \rangle.$$

Rearranging terms and letting $\delta \downarrow 0$ leads us to

$$\max_{v \in \overline{B}} \inf_{\xi \in S(\varepsilon)} \langle \zeta - \xi, v \rangle \le 0.$$

In accordance with the lopsided minimax theorem of Aubin [6], we then have

$$\inf_{\xi \in S(\varepsilon)} \max_{v \in \overline{B}} \langle \zeta - \xi, v \rangle = \inf_{\xi \in S(\varepsilon)} \|\zeta - \xi\| \le 0.$$

Consequently $\zeta \in S(\varepsilon)$, which is the assertion (26). □

Proof of Theorem 3.6: Let $\zeta \in N_S^P(x, f(x))$. Then $\zeta = (\eta, \alpha)$ where $\eta \in \Re^n$ and $\alpha \in \Re$. It is readily checked that $\alpha \le 0$.

Case 1: $\alpha < 0$.

In this case, by conicity, we can without loss of generality assume that $\alpha = -1$. Then $\eta \in \partial_P f(x)$ by the definition of the proximal subgradient. By Proposition 3.7 and formula (5), we then have

$$-\zeta \in \left(\overline{\text{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} \partial_P(-f)(x) \right\}, 1 \right)$$

for every $\varepsilon > 0$. Then in view of formula (6) and the fact that $\hat{S} = \text{hyp}(f)$, we arrive at (24).

Case 2: $\alpha = 0$.

By the approximation result of Rockafellar mentioned in the introduction, $\zeta = (\eta, 0) = \lim(\eta_i, \alpha_i)$ where $\alpha_i < 0$, $(\eta_i, \alpha_i) \in N_S^P(x_i, f(x_i))$, with $x_i \to x$. Let $\varepsilon > 0$ be given. Then by Case 1, for each i we have

$$-(\eta_i, \alpha_i) \in \overline{\text{co}} \left\{ \bigcup_{\|y-x_i\|<\frac{\varepsilon}{2}} N_{\hat{S}}^P(y, f(y)) \right\}.$$

Upon considering i so large that $\|x - x_i\| < \varepsilon/2$, it follows that for every $\varepsilon > 0$, $-\zeta$ is a limit of points in

$$\overline{\text{co}} \left\{ \bigcup_{\|y-x\|<\varepsilon} N_{\hat{S}}^P(y, f(y)) \right\}.$$

Then $-\zeta$ is contained in this set as well, which is (24). □

Remark 3.8.

(a) It remains unknown to us whether Theorem 3.1 holds true in general with $\delta = 0$.

(b) Proposition 3.7 goes through in Hilbert space, unchanged.

It is not hard to show that if S is only assumed to be locally isometric to the epigraph of a continuous function, then (21) holds with $\delta = 0$. In particular, this is true in case S is epi-Lipschitz. An alternative proof in the epi-Lipschitz case can be based upon optimization, as follows: For S closed and possibly not epi-Lipschitz, let us introduce a new condition on S:

(CQ): S_r is (nonempty and) epi-Lipschitz for all $r > 0$ sufficiently small.

Condition (CQ) always holds when S is epi-Lipschitz and possesses compact boundary; see proposition 5.1 below).

Claim. *When* (CQ) *holds, then* Theorem 3.1 *is valid with* $\delta = 0$.

It is possible to adapt the proof of Theorem 3.1 in order to verify the above claim. A somewhat simpler optimization based proof goes as follows – here (CQ) lives up to its name, since it plays the role of a constraint qualification:
Let $x \in \mathrm{bdry}(S)$ be as in the theorem; that is, $x \in \mathrm{cl}[(\mathrm{int}(S)]$. Let $0 \neq \zeta \in N_S^P(x)$, $\|\zeta\| = 1$. Then there exists $z \notin S$ such that

$$\zeta = \frac{z - x}{\|z - x\|},$$

where x is the unique closest point in S from z.
For $r > 0$, let us consider the mathematical programming problem (P) given by

$$\begin{aligned} \text{minimize} \quad & \|z - y\| \\ \text{subject to} \quad & d_{\hat{S}}(y) \geq r. \end{aligned}$$

A minimizer x_r exists, and it is not hard to see that $d_{\hat{S}}(x_r) = r$ and $x_r \in x + q(r)B$, where $q(r) \downarrow 0$ as $r \downarrow 0$. Therefore

$$\frac{x_r - z}{\|x_r - z\|} \to -\zeta \tag{28}$$

as $r \downarrow 0$.

The nonsmooth Lagrange multiplier rule (see [10]) for problem (P) implies the existence of $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, such that $\lambda_0 + \lambda_1 = 1$ and

$$0 \in \lambda_0 \frac{x_r - z}{\|x_r - z\|} + \lambda_1 \partial_C(-d_{\hat{S}})(x_r),$$

and since $\partial_C(-d_{\hat{S}})(x_r) = -\partial_C(d_{\hat{S}})(x_r)$, one obtains

$$\lambda_0 \frac{x_r - z}{\|x_r - z\|} \in \lambda_1 \partial_C d_{\hat{S}}(x_r).$$

Now consider any $\tilde{\zeta} \in \partial_C d_{\hat{S}}(x_r)$. One has

$$\tilde{\zeta} \in \mathrm{co}\left\{\lim \zeta_i : \zeta_i \in \partial_P d_{\hat{S}}(x_i),\ x_i \to x_r\right\}.$$

Hence

$$\tilde{\zeta} \in \mathrm{co}\left\{\lim \frac{x_i - y_i}{\|x_i - y_i\|} : x_i \to x_r,\ \mathrm{proj}_{\hat{S}}(x_i) = \{y_i\}\right\},$$

and it follows that

$$\tilde{\zeta} \in \mathrm{co}\left\{ \frac{x_r - y}{\|x_r - y\|} : y \in \mathrm{proj}_{\hat{S}}(x_r) \right\}.$$

Now note that the definition of S_r implies $y - x_r \in N^P_{S_r}(x_r)$ if $y \in \mathrm{proj}_{\hat{S}}(x_r)$, and so we have shown that

$$\partial_C d_{\hat{S}}(x_r) \subseteq \mathrm{co}\{\hat{\zeta} \in -N^P_{S_r}(x_r) : \|\hat{\zeta}\| = 1\}. \tag{29}$$

By (CQ), we may take r small enough a priori so as to ensure that $N^P_{S_r}(x_r)$ is pointed. Then from (29) we readily deduce that

$$0 \notin \partial_C d_{\hat{S}}(x_r).$$

Therefore $\lambda_0 > 0$, and so

$$\frac{x_r - z}{\|x_r - z\|} \in \frac{\lambda_1}{\lambda_0} \partial_C d_{\hat{S}}(x_r) \subseteq N^P_{S_r}(x_r).$$

Now (28) yields (21), proving the claim.

4. The complementary C-normal formula revisited

The next proposition provides useful directional information about the distance function at a point where S is epi-Lipschitz. This result is known, and is part of Theorem 2.5.8 in [10]. We refer the reader there for its proof, which is not reliant upon proximal methods, but rather upon the C-calculus:

Proposition 4.1. *Let $x \in \mathrm{bdry}(S)$, and let $v \in \mathrm{int}\left\{T^C_S(x)\right\}$ (so that in particular, S is epi-Lipschitz at x). Then there exists $\gamma > 0$ such that*

$$d_C(y + tw) \leq d_C(y) \quad \forall\, y \in x + \gamma B, \quad \forall\, w \in v + \gamma B, \quad \forall\, t \in (0, \gamma]. \tag{30}$$

The following global topological condition on S is now introduced:

(T): $\quad S = \mathrm{cl}[\mathrm{int}(S)]$.

Let us state, without proof, an elementary topological lemma involving (T), and which bears upon the remainder of the article:

Lemma 4.2. *For any nonempty closed set $S \subseteq \Re^n$, the following inclusions all hold:*

(a) $\mathrm{bdry}(\hat{S}) \subseteq \mathrm{bdry}(S)$.
(b) $\mathrm{cl}[\mathrm{int}(S)] \subseteq S$.
(c) $\mathrm{cl}[\mathrm{comp}(\hat{S})] \subseteq S$.

Furthermore, if equality holds in any one of these inclusions, then the others hold as equalities as well; in particular, (T) *holds.*

Upon taking $y = x$ in (30), a useful specialization of the Proposition 4.1 ensues:

Corollary 4.3. *Under the hypotheses of* Proposition 4.1, *there exists $\gamma > 0$ such that*

$$x + tv \in \mathrm{int}(S) \quad \forall\, t \in (0, \gamma]. \tag{31}$$

In particular, $x \in \mathrm{cl}[\mathrm{int}(S)]$*, and if* S *is epi-Lipschitz, then* (T) *holds.*

In the next proposition, directions in $-\,\mathrm{int}\{T_S^C(x)\}$ are considered. Unlike Proposition 4.1 and its corollary, however, its proof relies on proximal arguments:

Proposition 4.4. *Under the hypotheses of* Proposition 4.1*, there exists* $\gamma > 0$ *such that*

$$y - tv \notin S \quad \forall\, y \in \{x + \gamma B\} \cap \mathrm{bdry}(S), \quad \forall\, t \in (0, \gamma]. \tag{32}$$

Furthermore, $x \in \mathrm{bdry}(\hat{S})$*.*

Proof. The assumption on v implies the existence of $\Delta > 0$ such that

$$\langle \zeta, v \rangle \leq -\Delta \|\zeta\| \quad \forall\, \zeta \in N_S^C(x).$$

Then there exists $\beta > 0$ such that

$$\langle \zeta, v \rangle \leq -\frac{\Delta}{2} \|\zeta\| \quad \forall\, \zeta \in N_S^P(y), \quad \forall\, y \in S \cap \{x + \beta B\}. \tag{34}$$

Suppose that, contrary to the assertion of the proposition, $x_i - t_i v \in S$ for every i along sequences $x_i \to x$, $x_i \in \mathrm{bdry}(S)$, $t_i \downarrow 0$. The fact that x_i is a boundary point of S implies that for each i there exists $w_i \in \mathrm{comp}(S)$ arbitrarily close to x_i; we shall in fact insist that

$$\|w_i - x_i\| < \frac{\Delta}{2} t_i. \tag{35}$$

Since $d_S(w_i) - d_S(x_i - t_i v) > 0$ for each i, the mean value inequality implies the existence of

$$z_i \in [w_i, x_i - t_i v] + t_i B$$

such that for some $\zeta_i \in \partial_P d_S(z_i)$,

$$\langle \zeta_i, w_i - x_i + t_i v \rangle > 0. \tag{36}$$

Then for large i, $\|\zeta_i\| > 0$, implying that $z_i \notin \mathrm{int}(S)$. Hence for large i, $\zeta_i \in N_S^P(u_i)$, where $u_i = \{\mathrm{proj}_S(z_i)\}$ is a point in the boundary of S; this follows from (7) if $z_i \notin S$ or from (9) if $z_i \in \mathrm{bdry}(S)$. Note that $u_i \to x$, and $\|\zeta_i\| \leq 1$ (since the distance function is Lispchitz of rank 1). In view of (34) and (36), for large i we therefore arrive at

$$\|w_i - x_i\| > \frac{\Delta}{2} t_i,$$

which contradicts (35).
The "furthermore" part of the assertion follows directly from (31) and (32). □

We are now in position to provide an intrinsically geometric proof of the complementary normal formula for the C-normal cone to an epi-Lipschitz set, as well as a dual form in terms of the C-tangent cone:

Theorem 4.5. *Let* S *be epi-Lipschitz at* $x \in \mathrm{bdry}(S)$*. Then the following hold:*

$$T_S^C(x) = -T_{\hat{S}}^C(x). \tag{37}$$

$$N_S^C(x) = -N_{\hat{S}}^C(x). \tag{38}$$

Proof. By polarity, (37) and (38) are equivalent, and so we shall only prove the former. To this end, let $v \in \text{int}\,\{T_S^C(x)\}$.
We first claim that

$$-v \in \left[N_{\hat{S}}^P(w)\right]^* \tag{39}$$

for all $w \in \hat{S}$ sufficiently near x. To see this, consider $0 \neq \psi \in N_{\hat{S}}^P(w)$. Then

$$w \in \text{bdry}(\hat{S}) \subseteq \text{bdry}(S).$$

By the proximal normal inequality, there exists $\sigma > 0$ such that

$$\langle \psi, w' - w \rangle \leq \sigma \|w' - w\|^2 \quad \forall\, w' \in \hat{S}.$$

But by (32), $w' = w - tv \in \hat{S}$ for small positive t and $w \in \text{bdry}(S)$ near x, in which case $\langle \psi, -tv \rangle \leq \sigma t^2$ for such t and w. Letting $t \downarrow 0$ gives $\langle \psi, -v \rangle \leq 0$, which proves (39). Then it follows that

$$-v \in \left[N_{\hat{S}}^L(x)\right]^* = T_{\hat{S}}^C(x),$$

and so we have shown that

$$\text{int}\left\{T_S^C(x)\right\} \subseteq -T_{\hat{S}}^C(x). \tag{40}$$

In particular,

$$\text{int}\{T_{\hat{S}}^C(x)\} \neq \phi. \tag{40}$$

Since, by the present epi-Lipschitz hypothesis, the closed convex cone $T_S^C(x)$ has nonempty interior, it satisfies (T). Hence (40) gives the inclusion

$$T_S^C(x) \subseteq -T_{\hat{S}}^C(x). \tag{42}$$

For the reverse inclusion, we have $x \in \text{bdry}(\hat{S})$ (by Proposition 4.4). Therefore, in view of (41), we can let $\hat{S}$ play the role of S in the first half of the proof, using (31) in place of (32). Upon doing so, we obtain

$$T_{\hat{S}}^C(x) \subseteq -T_S^C(x),$$

which verifies (37) and completes the proof. □

An immediate corollary of the preceding proof is now noted:

Corollary 4.6. *S is epi-Lipschitz at $x \in \text{bdry}(S)$ iff $\hat{S}$ is epi-Lipschitz at x.*

5. Geometry of outer and inner approximations

For a closed convex nonzero cone $K \subseteq \Re^n$, we denote the *maximal angle* of K by

$$\Theta(K) := \arccos\left[\min\{\langle u, v\rangle : u \in K,\ v \in K,\ \|u\| = \|v\| = 1\}\right],$$

where we restrict $0 \leq \Theta(K) \leq \pi$. The primary use of the maximal angle is in characterizing pointedness; in particular, it is clear that K is pointed iff $\Theta(K) < \pi$ (or equivalently, $\cos(\Theta(K)) > -1$). Observe that S is Lipschitz at $x \in \mathrm{bdry}(S)$ iff $\Theta(N_S^C(x)) < \pi$.
We define the *Lipschitz index* of a closed set S as

$$\Theta_S := \sup\left\{\Theta[N_S^C(x)] : x \in \mathrm{bdry}(S)\right\}.$$

In view of the closedness of the mapping N_S^C on S when S is epi-Lipschitz, it follows that in this case the function $\Theta(N_S^C(\cdot))$ is upper semicontinuous on $\mathrm{bdry}(S)$. So if S has compact boundary and is epi-Lipschitz, then bearing in mind that $\mathrm{bdry}(S) = \mathrm{bdry}(\hat{S})$ (since (T) holds), Corollary 4.6 and formula (38) imply that $\hat{S}$ is epi-Lipschitz and

$$\begin{aligned}\Theta_{\hat{S}} = \Theta_S = {} & \max\{\Theta[N_S^C(x)] : x \in \mathrm{bdry}(S)\} \\ = {} & \max\{\Theta[N_{\hat{S}}^C(x)] : x \in \mathrm{bdry}(S)\}.\end{aligned}$$

Note also that a compact set S is epi-Lipschitz iff $\Theta_S < \pi$.
The next result provides basic information on the Lipschitz indices of inner and outer approximations:

Proposition 5.1. *Let S be epi-Lipschitz and assume that* $\mathrm{bdry}(S)$ *is compact. Then one has*

$$\limsup_{r \downarrow 0} \Theta_{S^r} \leq \Theta_S. \tag{43}$$

and

$$\limsup_{r \downarrow 0} \Theta_{S_r} \leq \Theta_S. \tag{44}$$

In particular, both S^r and S_r are epi-Lipschitz for all r sufficiently small.

Note: In general, the limits superior in (43)–(44) are not equal, and the asserted inequalities can hold strictly. Consider for example the situation where $S \subseteq \Re^2$ is a square. Then $\Theta_{S^r} = 0$ for all $r > 0$, but $\Theta_S = \frac{\pi}{2} = \Theta_{S_r}$ for all small $r > 0$.

Proof. We first will verify (43). Suppose to the contrary that there exist sequences $r_i \downarrow 0$, $z_i \in S^{r_i}$, $\delta > 0$, and unit C-normals u_i, v_i to S^{r_i} at z_i, such that $\langle u_i, v_i\rangle > \Theta_S + \delta$ for each i. It is not difficult to verify that

$$N_{S^{r_i}}^L(z_i) \subseteq \bigcup_{y \in \mathrm{proj}_S(z_i)} N_S^P(y). \tag{45}$$

Therefore u_i and v_i are contained in the set

$$Q(z_i) := \mathrm{co}\left\{\bigcup_{y \in \mathrm{proj}_S(z_i)} N_S^C(y)\right\}.$$

Since bdry(S) is compact, we may assume that $z_i \to z \in \text{bdry}(S)$, whence $d(Q(z_i), z) \to 0$. Also, since $\|u_i\| = \|v_i\| = 1$, we can assume $u_i \to u$ and $v_i \to v$. We now claim that the limit vectors u and v are contained in $N_S^C(z)$; the argument is provided for u. By Carathéodory's theorem,

$$u = \lim u_i = \lim \sum_{j=1}^{n+1} \alpha_i^j \zeta_i^j, \tag{46}$$

where $\zeta_i^j \in N_S^C(y_i^j)$, $\|\zeta_i^j\| = 1$, $\alpha_i^j \geq 0$, and $y_i^j \to z$ as $i \to \infty$ for each $j = 1, 2, \ldots, n+1$. If $u \notin N_S^C(z)$, then since $N_S^C(z)$ is closed, convex and pointed, a standard separation argument yields the existence of a vector p and scalar $\gamma > 0$ such that $\langle p, u \rangle < 0$ and $\langle p, \zeta \rangle > \gamma$ for all unit vectors $\zeta \in N_S^C(z)$. Now, the closedness of the multifunction N_S^C implies the upper semicontinuity of $N_S^C \cap \Omega$. Consequently, $\langle p, \zeta_i^j \rangle > \frac{\gamma}{2}$ for each ζ_i^j in (46), for all i sufficiently large. This violates the stated separation property of p; hence u and v are contained in $N_S^C(z)$. Now, $\langle u, v \rangle > \Theta_S + \delta$, which contradicts the definition of Θ_S, and so (43) has been verified. What is more, this implies that S^r is epi-Lipschitz for sufficiently small r.

In order to verify (44), one repeats above the argument with S replaced by $\hat{S}$. This leads to

$$\limsup_{r \downarrow 0} \Theta_{(\hat{S})^r} \leq \Theta_{\hat{S}} = \Theta_S$$

and $(\hat{S})^r$ being epi-Lipschitz for sufficiently large r. But then cl[comp$((\hat{S})^r)$] $= S_r$, and so $\Theta_{(\hat{S})^r} = \Theta_{S_r}$, and (44) follows. □

We require a pair of technical lemmas. The first of these provides an elementary "half-angle inequality":

Lemma 5.2. *Let v and w be nonzero vectors in $\Re^n$ and let*

$$\Theta := \arccos \left[\frac{\langle v, w \rangle}{\|v\| \, \|w\|} \right]$$

be the angle between them. Then

$$\cos\left(\frac{\Theta}{2}\right) \leq \frac{\|v + w\|}{\|v\| + \|w\|}. \tag{47}$$

Proof. Upon squaring and using the cosine half-angle formula, we see that (47) is equivalent to

$$\frac{1 + \cos(\Theta)}{2} \leq \frac{\|v\|^2 + \|w\|^2 + 2\langle v, w \rangle}{\|v\|^2 + \|w\|^2 + 2\|v\| \, \|w\|}.$$

Now apply the definition of Θ and regroup terms in order to rewrite this as

$$\langle v, w \rangle \left[\|v\| - \|w\|\right]^2 \leq \|v\| \, \|w\| \; \left[\|v\| - \|w\|\right]^2.$$

The validity of the last expression follows from the Cauchy-Schwarz inequality, completing the proof. □

The next lemma's proof is inductive in nature:

Lemma 5.3. *Assume $\Theta_S < \pi$, and let $x \in \mathrm{bdry}(S)$. Suppose that $0 \neq \zeta \in N_S^P(x)$ and $r > 0$ are such that*

$$\zeta \in \mathrm{co}\left\{\hat{\zeta} \in N_S^P(x) : \hat{\zeta} \text{ is realized by an } r\text{−ball}\right\}. \tag{48}$$

Then ζ is realized by a ball of radius $r\left[\cos\left(\frac{\Theta_S}{2}\right)\right]^n$.

Proof. Let ζ be as in (48). Then $\zeta = \sum_{i=1}^s \alpha_i\zeta_i$, where the ζ_i are r-realizable P-normals to S at x, and where the α_i are nonnegative scalars summing to 1.
First suppose that $s = 2$. Upon appealing to (10), we have

$$\frac{\|\zeta_i\|}{2r}\|y-x\|^2 \geq \langle \zeta_i, y-x\rangle \quad \forall\, y \in S,\; i = 1, 2.$$

Then

$$\frac{\alpha_1\|\zeta_1\| + \alpha_2\|\zeta_2\|}{2r\|\alpha_1\zeta_1 + \alpha_2\zeta_2\|}\|y-x\|^2 \geq \langle \zeta, y-x\rangle \quad \forall\, y \in S,\; i = 1, 2.$$

Clearly $\Theta \leq \Theta_S < \pi$, where Θ is the angle between ζ_1 and ζ_2 (and so between $\alpha_1\zeta_1$ and $\alpha_2\zeta_2$ as well). Then by Lemma 5.2 we obtain

$$\frac{1}{2r\cos\left(\frac{\Theta_S}{2}\right)}\|y-x\|^2 \geq \left\langle \frac{\zeta}{\|\zeta\|}, y-x\right\rangle \quad \forall\, y \in S,\; i = 1, 2.$$

Hence ζ is realizable by a ball of radius $r\cos\left(\frac{\Theta_S}{2}\right)$.
Suppose now that $s = 3$. Then ζ can be expressed as a convex combination of two P-normals $\tilde{\psi}$ and $\tilde{\xi}$, where $\tilde{\psi}$ is r-realizable and where $\tilde{\xi}$ is a convex combination of two P-normals, each r-realizable. In particular, $\tilde{\psi}$ is realizable by a ball of radius $\tilde{r} := r\cos\left(\frac{\Theta_S}{2}\right)$, and by the case $s = 2$, so is $\tilde{\xi}$. Applying the $s = 2$ case again then allows us to conclude that ζ is realizable by a ball of radius $\tilde{r}\cos\left(\frac{\Theta_S}{2}\right) = r\left[\cos\left(\frac{\Theta_S}{2}\right)\right]^2$. Continuing in this way, until $s = n+1$, shows that ζ is realizable by a ball of radius $r\left[\cos\left(\frac{\Theta_S}{2}\right)\right]^n$ for each $s = 1, 2, \ldots, n+1$. □

Proposition 5.1 and Lemma 5.3 yield the following useful consequence regarding inner approximations:

Proposition 5.4. *Suppose that $r > 0$ is such that $S_r \neq \phi$ and $\Theta_{S_r} < \pi$. Then S_r is proximally smooth of radius*

$$\rho_r := r\left[\cos\left(\frac{\Theta_{S_r}}{2}\right)\right]^n.$$

Proof. Let $0 \neq \zeta \in N^P_{S_r}(x)$. Then certainly ζ is a C-normal to S_r at x, and therefore

$$-\zeta \in N^C_{(\hat{S})^r}(x) = \mathrm{co}\left[N^L_{(\hat{S})^r}(x)\right].$$

Consider any $\eta \in N^L_{(\hat{S})^r}(x)$.
Now, as in (45), we have $\eta \in N^P_{\hat{S}}(y)$ and $y \in \mathrm{proj}_{\hat{S}}(x)$. But then $\|x - y\| = r$, and so also $x \in \mathrm{proj}_{S_r}(y)$. Hence $-\eta \in N^P_{S_r}(x)$, and each such P-normal is realized by an r-ball. Now apply Lemma 5.3 (to S_r). □

6. Applications to invariance issues

We now set ourselves the task of investigating the degree to which complements, approximations and associated smoothings of S inherit invariance properties posited for S.
First, complementary strong invariance will be dealt with. Our main result in this regard depends upon the following lemma on continuity in in initial data; this is well known (see e.g. Deimling [20], Aubin and Cellina [4] or [17]), but we sketch the proof for the sake of completeness:

Lemma 6.1. *Suppose that F satsifies* (SH), (L) *and let $x(\cdot)$ be a trajectory of* (4) *on* $[0, T]$. *Then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds: For any $y_o \in x(0) + \delta B$, there exists a trajectory $y(\cdot)$ on $[0, T]$ satisfying $y(0) = y_o$ and $y(T) \in x(T) + \varepsilon B$.*

Proof. Consider the selection of F given by

$$f(t, y) := \mathrm{proj}_{F(y)}(\dot{x}(t)).$$

The assumptions on F imply that f is Lebesgue measurable in t, continuous in y, and satisfies linear growth in y. Hence for any $y_o \in \Re^n$, there exists a trajectory $y(\cdot)$ of the differential equation $\dot{y}(t) = f(t, y(t))$ on $[0, T]$ satisfying $y(0) = y_o$. Furthermore, in view of (L),

$$\|\dot{y}(t) - \dot{x}(t)\| \leq K\|y(t) - x(t)\|$$

on $[0, T]$, where K denotes the Lipschitz rank of F. An application of Gronwall's inequality now provides the result. □

Theorem 6.2. *Assume that S satisfies* (T)*, and that conditions* (SH), (L) *hold for the multifunction F. Then (S, F) is strongly invariant iff $(\hat{S}, -F)$ is strongly invariant.*

Proof. In view of (T), $\mathrm{cl}[\mathrm{comp}(\hat{S})] = S$. Hence we only need to show that strong invariance of (S, F) implies strong invariance of $(\hat{S}, -F)$. Suppose, by way of contradiction, that $(\hat{S}, -F)$ was not strongly invariant. Then there exists a trajectory of $\dot{x}(t) \in -F(x(t))$ satisfying $x(0) \in \mathrm{bdry}(\hat{S}) = \mathrm{bdry}(S)$ (by (T)) and $x(T) \in \mathrm{int}(S)$ for some $T > 0$. Now, by the preceding lemma, there exists a trajectory satisfying $\dot{y}(t) \in -F(y(t))$, $y(0) \in \mathrm{comp}(S)$, and $y(T) \in \mathrm{int}(S)$. A straightforward time reversal argument implies that $z(t) := y(T-t)$

satisfies $\dot{z}(t) \in f(z(t))$, $z(0) \in \text{int}(S)$, and $z(T) \notin S$, violating the strong invariance of (S, F). □

Let us now consider the characterization of complementary forms of weak invariance. A first result in this regard is the following:

Proposition 6.3. *Assume that F satisfies* (SH), (LG), *and that S is regular and epi-Lipschitz. Suppose that (S, F) is weakly invariant. Then $(\hat{S}, -F)$ is weakly invariant.*

Proof. Because of regularity, the weak invariance of (S, F) may be characterized by the condition

$$T_S^C(x) \cap F(x) \neq \phi \quad \forall\, x \in \text{bdry}(S).$$

In view of (37), the epi-Lipschitz hypothesis implies that we then have

$$T_{\hat{S}}^C(x) \cap \{-F(x)\} \neq \phi \quad \forall\, x \in \text{bdry}(\hat{S}),$$

which implies

$$T_{\hat{S}}^D(x) \cap \{-F(x)\} \neq \phi \quad \forall\, x \in \text{bdry}(\hat{S}).$$

Therefore $(\hat{S}, -F)$ is weakly invariant. □

We require the following geometric lemma dealing with certain points in $\{\text{bdry}(S)\} \cap \text{bdry}(\hat{S})$:

Lemma 6.4. *Let $0 \neq \zeta \in N_{\hat{S}}^P(x)$ and assume that $N_S^P(x) \neq \{0\}$. Then*

$$N_{\hat{S}}^P(x) = -N_S^P(x) = \{\alpha\zeta : \alpha \geq 0\}. \tag{49}$$

Proof. From the definition of proximal normals, there exists $\alpha > 0$ such that the closed ball

$$B_1 := x + \alpha\zeta + \alpha\|\zeta\|\overline{B}$$

meets $\hat{S}$ only at x. By hypothesis, there exists $0 \neq \eta \in N_S^P(S)$. Hence, for some $\beta > 0$, the closed ball

$$B_2 := x + \beta\eta + \beta\|\eta\|\overline{B}$$

meets S only at x. Clearly $B_1 \cap B_2$ has empty interior, which implies that

$$\left\langle \frac{\zeta}{\|\zeta\|}, \frac{\eta}{\|\eta\|} \right\rangle = -1.$$

The arbitrariness of η therefore implies that $-N_S^P(x)$ is the half-line on the right of (49) A similar argument shows that this set also equals $N_{\hat{S}}^P(x)$. □

Remark 6.5. The preceding lemma may be used in order to prove the following:

(a) *If S satisfies the topological hypothesis* (T), *then*

$$N_{\text{bdry}(S)}^P(x) = N_S^P(x) \cup N_{\hat{S}}^P(x) \quad \forall\, x \in \text{bdry}(S).$$

This leads to a conclusion regarding weak invariance of the boundary of S:

(b) *Suppose that* (SH) *and* (LG) *hold. Then under the assumption that* (T) *holds,* bdry(S) *is weakly invariant iff both* S *and* $\hat{S}$ *are weakly invariant.*

The lemma yields a counterpart to Proposition 6.3 , in which mild proximal smoothness replaces "regular and epi-Lipschitz". Note that these sets of hypotheses are independent; in fact, the following result does not even require the topological assumption (T) (which always holds in the epi-Lipschitz case):

Proposition 6.6. *Assume that* F *satisfies* (SH) *and* (LG). *Suppose that* S *is mildly proximally smooth and that* (S, F) *is weakly invariant. Then* $(\hat{S}, -F)$ *is weakly invariant.*

Proof. Let ζ be a nonzero P-normal to $\hat{S}$ at x. In view of the proximal characterization of weak invariance of (S, F), the lemma then tells us that $h_F(x, -\zeta) \leq 0$. But then $h_{-F}(x, \zeta) \leq 0$, and the weak invariance of $(\hat{S}, -F)$ follows. □

Remark 6.7. Proposition 6.3 is false if regularity is not assumed, and Proposition 6.6 is false if there exists $x \in$ bdry(S) with $N_S^P(x) = \{0\}$ (i.e. if mild proximal smoothness does not hold). A single simple example which illustrates both of these phenomena is the following: Let S be a compact subset of $\Re^2$ which near the origin coincides with the epigraph of the function $-|x|$, and take $F(x) = \{(x_1, -1) : |x_1| \leq 1\}$ for all x near the origin.

We now turn our attention to the question of inheritance of weak invariance by approximations. For outer approximations we have the following:

Proposition 6.8. *Suppose that* S *is compact,* (SH) *holds, and that* (S, F) *is weakly invariant. Then for any given* $\varepsilon > 0$, $(S^r, F + \varepsilon\overline{B})$ *is weakly invariant for all* $r > 0$ *sufficiently small.*

Proof. Let $\zeta \in N_{S^r}^P(x)$, and let $y = \{\text{proj}_S(x)\}$. Then $\zeta \in N_S^P(y)$, and the weak invariance assumption implies $h_F(y, \zeta) \leq 0$. Then since F is uniformly Hausdorff continuous on a neighborhood of the compact set S, it is readily noted that r may be taken small enough to ensure that $h_{F+\varepsilon\overline{B}}(x, \zeta) \leq 0$. □

We now consider inner approximations, in which case further assumptions on S are required.

Theorem 6.9. *Suppose that* (SH) *holds. Assume that* S *is compact, regular, epi-Lipschitz, and that* (S, F) *is weakly invariant. Then for any given* $\varepsilon > 0$, $(S_r, F + \varepsilon\overline{B})$ *is weakly invariant for all* $r > 0$ *sufficiently small.*

Proof. We proceed by way of contradiction. Suppose that the assertion was false. Then there would exist $\varepsilon > 0$ and sequences $r_i \downarrow 0$, $x_i \in S_{r_i}$, and $\zeta_i \in N_{S_{r_i}}^P(x_i) \cap \Omega$, such that for each i

$$h_{F+\varepsilon\overline{B}}(x_i, \zeta_i) > 0. \tag{50}$$

The fact that $\|\zeta_i\| = 1$ implies that

$$h_F(x_i, \zeta_i) > \varepsilon. \tag{51}$$

Since $S_{r_i}(x_i) \subseteq S$ and S is compact, we may extract a subsequence (not relabeled) such that $x_i \to x \in \text{bdry}(S)$ and $\zeta_i \to \zeta$. We claim that

$$\zeta \in N_S^C(x). \tag{52}$$

This will provide the requisite contradiction, because the Hausdorff continuity of F then implies $h_F(x, \zeta) \geq \varepsilon$, violating (16), which characterizes weak invariance of (S, F) when S is regular.

We now turn to the verification of (52). Let us first note that, by Proposition 5.1 , the sets S_{r_i} are epi-Lipschitz for i sufficiently large, and so $\text{cl}[\text{comp}(S_{r_i})] = (\hat{S})^r$. Then Remark 3.8 (b) implies that for i taken large enough,

$$-\zeta_i \in \overline{\text{co}} \left\{ \bigcup_{\|y - x_i\| < i^{-1}} N^P_{(\hat{S})^r}(y) \right\}. \tag{53}$$

Now, if $\eta \in N^P_{(\hat{S})^r_i}(y)$, then $\eta \in N^P_{\hat{S}}(w) \subseteq N^C_{\hat{S}}(w)$, where $\{w\} = \text{proj}_{\hat{S}}(y)$. Hence (38) implies $-\eta \in N_S^C(w)$. Upon noting that

$$\|w - x\| \leq \|w - y\| + \|y - x_i\| + \|x_i - x\| \to 0$$

as $i \to \infty$, it follows from (53) that

$$\zeta = \lim_{i \to \infty} \left[\sum_{j=1}^{n+1} \alpha_i^j \xi_i^j \right], \tag{54}$$

where the α_i^j are nonnegative and $\xi_i^j \in N_S^C(w_i^j) \cap \Omega$, $\lim_{i \to \infty} w_i^j = x$. Now, since N_S^C is a closed multifunction, we may assume that $\xi_i^j \to \xi^j \in N_S^C(x) \cap \Omega$ as $i \to \infty$, for each j. Suppose (52) did not hold. Then, in view of the pointedness of $N_S^C(x)$, there would exist a vector $p \in \Re^n$ and a scalar $\delta > 0$ such that $\langle \zeta, p \rangle < 0$ and $\langle \xi, p \rangle > \delta$ for every $\xi \in N_S^C(x) \cap \Omega$. It follows that for i sufficiently large, $\langle \xi_i^j, p \rangle > \frac{\delta}{2}$ for each $j = 1, 2, \ldots, n$. But then (54) implies $\langle \zeta, p \rangle \geq 0$, a contradiction. Hence (52) holds and the proof is completed. □

Remark 6.10. The example in Remark 6.7 shows that the regularity hypothesis is crucial in Theorem 6.9.

A result now ensues which asserts the existence of C^{1+}-smooth inner approximations S which inherit the weak invariance property from S (albeit for a "slightly enlarged" velocity map F). This weakly invariant "inner smoothing" is a double approximation of the form $(S_r)^\delta$; that is, an outer approximation of an inner approximation. The smoothness properties of such sets were originally studied by Benoist [8].

Theorem 6.11. *Let S be compact, epi-Lipschitz and regular. Let F satisfy* (SH), *and assume that (S, F) is weakly invariant. Then given any $\varepsilon > 0$, there exists a set*

$Q \subseteq \operatorname{int}(S)$ such that $d(Q,S) < \varepsilon$, where Q is C^{1+}-smooth, and such that $(Q, F + \varepsilon\overline{B})$ is weakly invariant.

Proof. By Theorem 6.9 , $(S_r, F + \varepsilon\overline{B})$ is weakly invariant for small r. Furthermore, by Propositions 5.1 and 5.4, S_r is proximally smooth for small positive r. Hence, upon fixing such an r and bearing Proposition 6.8 in mind, the set $Q = (S_r)^\delta$ has the required properties for sufficiently small $\delta > 0$. □

Next on the agenda is the existence of a Lipschitz feedback law which achieves penetration of sets S satisfying certain hypotheses including a strict tangentiality or inwardness condition:

Theorem 6.12. *Let S be compact, epi-Lipschitz, and proximally smooth. Assume that* (SH) *holds and that there exists $\alpha > 0$ such that for every $x \in S$ one has*

$$h_F(x,\zeta) \leq -\alpha\|\zeta\| \quad \forall\, \zeta \in N_S^P(x). \tag{55}$$

Then there exist $\delta > 0$ and $\gamma > 0$ satisfying

$$S \subseteq \operatorname{int}\left[(S_\gamma)^\delta\right] \tag{56}$$

and the following: For any $\varepsilon > 0$, there exists a Lipschitz function v on $(S_\gamma)^\delta \backslash S_\gamma$ such that

$$v(x) \in F(x) + \varepsilon\overline{B}, \tag{57}$$

and such that for any $x_0 \in (S_\gamma)^\delta \backslash \operatorname{int}(S)$, the solution to $\dot{x}(t) = v(x(t))$ emanating from $x(0) = x_0$ satisfies $x(T) \in \operatorname{int}(S)$ for some $T > 0$.

Proof. Let $r \in (0,\beta)$, where β is the radius of proximal smoothness of S. Then $S = (S^r)_r$, and S_γ is epi-Lipschitz for small γ (and in particular satisfies (T)). Armed with this (as well as the fact that S^r is epi-Lipschitz since it is C^{1+}), we deduce that $S_\gamma = (S^r)_{r+\gamma}$. Also, by Propositions 5.1 and 5.4, we see that for sufficiently small γ, there exists $\delta > 0$ so that (56) holds, with $d_{S_\gamma} \in C^{1+}$ on $(S_\gamma)^\delta \backslash S_\gamma$. Furthermore, $r > 0$ can initially be chosen small enough a priori to ensure that

$$h_F(x, \nabla d_{S_\gamma}(x)) = \min_{v \in F(x)} \langle \nabla d_{S_\gamma}(x), v\rangle \leq -\frac{\alpha}{2} \quad \forall\, x \in (S_\gamma)^\delta \backslash S_\gamma. \tag{58}$$

Indeed, if $x \in (S_\gamma)^\delta$, and $d_{S_\gamma}(x) = \rho > 0$, then $\nabla d_{S_\gamma}(x) \in N^P_{(S_\gamma)^\rho}(x) \cap \Omega$ as a consequence of proximal smoothness, and condition (55) leads to (58) via an argument similar to the proof of Proposition 6.8.

Denote the minimizing set of v in (58) by $\Gamma(x)$. It is readily checked that $\Gamma(x)$ is compact and convex. Also, the multifunction Γ is upper semicontinuous on $(S_\gamma)^\delta \backslash S_\gamma$, which follows readily from the continuity of ∇d_{S_γ} and Hausdorff continuity of F. It follows that for every $\tau > 0$ there exists a locally Lipschitz function v such that $\operatorname{gr}(v) \subseteq \operatorname{gr}(\Gamma) + \tau B$ (where "gr" denotes graph); see e.g. [4] for this "approximate selection" result . (Since $(S_\gamma)^\delta$

is compact, v is in fact globally Lipschitz on that set.) Any such v, for τ small enough, satisfies

$$\langle \nabla d_{S_\gamma}(x), v(x) \rangle \leq -\frac{\alpha}{4} \quad \forall\, x \in (S_\gamma)^\delta \backslash S_\gamma.$$

Hence starting at any point $x(0) = x_0 \in (S_\gamma)^\delta \backslash \{\mathrm{int}(S)\}$, the (unique) solution of $\dot{x}(t) = v(x(t))$ satisfies

$$\frac{d}{dt} d_{S_\gamma}(x(t)) \leq -\frac{\alpha}{4} \quad \forall\, t \geq 0$$

on its maximal interval of existence. Consequently, $x(\cdot)$ is a trajectory of the differential inclusion (4) which enters the interior of S. This completes the proof. □

7. Remarks on analytic versions

Let $f : \Re^n \to (-\infty, \infty]$ be a lower semicontinuous extended real valued function. For $r > 0$, let g_r denote the infimal convolution of f with the function

$$k_r(u) := \begin{cases} -\sqrt{r^2 - \|u\|^2} & \text{if } \|u\| \leq r \\ \infty & \text{otherwise} \end{cases}$$

That is,

$$g_r(x) := \inf_{y \in \Re^n} \{f(y) + k_r(x - y)\}.$$

Then it is not difficult to show that

$$\mathrm{epi}(g_r) = \{\mathrm{epi}(f)\}^r$$

and

$$\mathrm{dom}(g_r) = \{\mathrm{dom}(f)\}^r.$$

Seeger [26] and Ioffe [21] studied the smoothing properties of this type of convolution operation in case f is convex. Unfortunately, for discontinuous f, g_r need not be continuous; furthermore, one can exhibit a continuous f such that g_r is not locally Lipschitz. Hence the convolution g_r does not possess the nice regularization properties of the Moreau-Yosida infimal convolution

$$\inf_{y \in \Re^n} \{f(y) + r\|x - y\|^2\}$$

when f is only required to satisfy a mild coercivity condition; see Attouch [1]. On the other hand, as will be made clear, the "geometric convolution" g_r is the right tool for formulating analytic versions of the invariance results of the preceding sections. Before proceeding, let us note that if f is Lipschitz of rank K on $\Re^n$, then so is g_r, for any $r > 0$. To see this, let $\eta = (\zeta, -1)$ be a P-normal to $\mathrm{epi}(g_r) = \{\mathrm{epi}(f)\}^r$ at $(x, g_r(x))$, where x is arbitrary. Then η is a P-normal to $\mathrm{epi}(f)$ (at the unique point closest to $(x, g_r(x))$), implying $\|\zeta\| \leq K$. But then g_r is Lipschitz of rank K.

Since g_r is a lower approximation of f, we are enticed to define an upper approximation as the supremal convolution

$$h_r(x) := \sup_{y \in \Re^n} \{f(y) - k_r(x - y)\}.$$

This function has the same domain as g_r, and

$$\text{hyp}(h_r) = \{\text{hyp}(f)\}^r.$$

We can also consider the double convolutions

$$w_{r,s}(x) := \inf_{y \in \Re^n} \{h_r(y) + k_s(x-y)\}$$

and

$$\hat{w}_{r,s}(x) := \sup_{y \in \Re^n} \{g_r(y) - k_s(x-y)\}.$$

As is the case with g_r, all these functions inherit the Lipschitz rank of f should f be globally Lipschitz.

Lasry and Lions [22] and Attouch and Azé [2] considered double convolutions built from the Moreau-Yosida infimal convolution. In spite of the fact that the convolutions g_r and h_r fall short of the Moreau-Yosida convolution as regularizers of nonconvex functions, it turns out that when f is Lipschitz, double convolutions of the type $w_{r,s}$ and $\hat{w}_{r,s}$ do in fact exhibit similar smoothing properties to those derived in [22] and [2]. For example, when f is globally Lipschitz, we have

$$\text{epi}(h_r) = \{\text{epi}(f)\}_r. \tag{59}$$

Since h_r is globally Lipschitz, one has $\Theta_{\text{epi}(h_r)} < \pi$, and therefore by Proposition 5.4, $\text{epi}(h_r)$ is proximally smooth. Then for small $s > 0$, the set $[\{\text{epi}(f)\}_r]^s = \text{epi}(w_{r,s})$ is C^{1+}-smooth. Furthermore, it is quite transparent that when f is also assumed to be regular, then $(\text{epi}(w_{r,s}), (F \times \{0\})^\varepsilon)$ inherits weak invariance from $(\text{epi}(f), F \times \{0\})$ in the sense of Theorem 6.11, when the underlying differential inclusion satisfies (SH), (LG) and a uniform Hausdorff continuity assumption. (One needs to posit these extra hypotheses in order to make the earlier arguments work, because the sets presently under consideration, i.e. epigraphs, are unbounded.) Since $(\nabla w_{r,s}(x), -1)$ is a proximal normal to $\text{epi}(w_{r,s})$ at $(x, w_{r,s}(x))$ for any $x \in \Re^n$, weak invariance of $(\text{epi}(w_{r,s}), (F \times \{0\})^\varepsilon)$ is equivalent to the condition

$$\min_{(v,u) \in (F(x) \times \{0\})^\varepsilon} (-u + \langle \nabla w_{r,s}(x), v \rangle) \le 0 \quad \forall\, x \in \Re^n.$$

After recapping this discussion in Hamiltonian terms, one arrives at the following result:

Proposition 7.1. *Let $f : \Re^n \to \Re$ be globally Lipschitz and regular. Assume that F is uniformly Hausdorff continuous on $\Re^n$ and that* (SH), (LG) *hold. Suppose that (f, F) is weakly decreasing, or equivalently, the proximal Hamiltonian inequality*

$$h_F(x, \partial_P f(x)) \le 0 \quad \forall\, x \in \Re^n \tag{60}$$

holds. Let $\tilde{\varepsilon} > 0$ be given. Then for each $r > 0$ sufficiently small, $s > 0$ can be in turn chosen sufficiently small (depending on r), such that $w_{r,s}$ is a C^{1+} function satisfying

$$w_{r,s}(x) \ge f(x) \quad \forall\, x \in \Re^n, \tag{61}$$

$$\|f(x) - w_{r,s}(x)\| < \tilde{\varepsilon} \quad \forall\, x \in \Re^n, \tag{62}$$

as well as the smooth Hamilton-Jacobi inequality

$$h_{F+\tilde{\varepsilon}\overline{B}}(x, \nabla w_{r,s}(x)) \leq \tilde{\varepsilon} \quad \forall\, x \in \Re^n. \tag{63}$$

Another smoothing result in a similar vein to Proposition 7.1 (but one involving only a single convolution, and where the smoothing function is majorized by f) can be obtained by similar reasoning if the assumptions on f are strengthened to *σ-weak convexity*; that is, there exists $\sigma > 0$ such that $f(x) + \sigma\|x\|^2$ is a convex function. The discussion in §5 of [14] shows that epi(f) is then proximally smooth (of radius $\frac{1}{2\sigma}$).

Proposition 7.2. *Let F satisfy the hypotheses of* Proposition 7.1. *Assume that f is σ-weakly convex and satisfies the Hamilton-Jacobi inequality* (60). *Let $\varepsilon > 0$ be given. Then for all small r, one has*

$$g_r(x) \leq f(x) \quad \forall\, x, \tag{64}$$

$$\|f(x) - g_r(x)\| < \varepsilon \quad \forall\, x, \tag{65}$$

and

$$h_{F+\varepsilon\overline{B}}(x, \nabla g_r(x)) \leq 0 \quad \forall\, x \in \Re^n \qquad (66).$$

We can use the smoothing function $w_{r,s}$ in Proposition 7.1 to construct a "universal" Lipschitz feedback law (i.e. one operative on all of $\Re^n$) which "nearly" achieves monotone behavior along trajectories:

Theorem 7.3. *Let the hypotheses of* Proposition 7.1 *hold, and let $\gamma > 0$ be given. Then there exists a locally Lipschitz function $\tilde{v}$ on $\Re^n$ satisfying $\tilde{v}(x) \in F(x) + \gamma\overline{B}$, and such that*

$$f(x(t)) \leq f(x(0)) + \gamma t + \gamma \quad \forall\, t \geq 0 \tag{67}$$

along solutions of $\dot{x}(t) = \tilde{v}(x(t))$.

Proof. Let $\varepsilon > 0$ be given, and take r, s so that $w_{r,s}$ is as in Proposition 7.1. From (63) we obtain

$$\min_{v \in F(x)+\varepsilon\overline{B}} \langle \nabla w_{r,s}(x), v\rangle \leq \varepsilon \quad \forall\, x \in \Re^n. \tag{68}$$

We now proceed similarly to the proof of Theorem 6.12. Denote the set of minimizers in (68) by $\Gamma(x)$. Then Γ is an upper semicontinuous compact convex valued multifunction on $\Re^n$. Consequently, for any $\tau > 0$, there exists a locally Lipschitz function $\tilde{v} : \Re^n \to \Re$ satisfying

$$\mathrm{gr}(\tilde{v}) \in \mathrm{gr}(\Gamma) + \tau B \subseteq \mathrm{gr}(F) + \varepsilon\overline{B} + \tau\overline{B} \quad \forall\, x \in \Re^n.$$

Let the (global) Lipschitz rank of f be K. Then $\|\nabla w_{r,s}(x)\| \leq K$ on $\Re^n$, and therefore from (68) we obtain

$$\langle \nabla w_{r,s}(x), \tilde{v}(x)\rangle \leq \tau K + \varepsilon. \tag{69}$$

Hence, along solutions of $\dot{x}(t) = \tilde{v}(x(t))$ one has

$$\frac{d}{dt} w_{r,s}(x(t)) \leq \tau K + \varepsilon,$$

implying that

$$w_{r,s}(x(t)) \leq w_{r,s}(x(0)) + (\tau K + \varepsilon)t.$$

This, combined with (62), yields (67) upon taking ε and τ sufficiently small, and completes the proof. □

Remark 7.4. Another reference in which the "Lipschitz plus regular" hypothesis was used in constructing "invariant feedback"—but in a geometric setting (as opposed to an analytic one as in the present section)— is Clarke, Ledyaev and Stern [18], which dealt with the existence of equilibria in nonconvex epi-Lipschitz sets. Specifically, the proof techniques of that article show that if a compact homeomorphically convex set $S \subset \Re^n$ is weakly invariant with respect to a multifunction F satisfying (SH) (where now F need only be defined on S), and if S is regular, then for any given $\gamma > 0$ there exists a Lipschitz function v on S, with $\text{gr}(v) \subseteq \text{gr}(F) + \gamma$, such that S is invariant (in the classical sense) with respect to the differential equation $\dot{x}(t) = f(x(t))$. Examples in [18] demonstrate that both the regularity and epi-Lipschitz hypotheses are crucial in this assertion.

Remark 7.5. An alternate method of proving Theorem 7.3, based upon C-calculus (as opposed to smoothing ideas) is to replace (60) with (20)—which is permissible by the regularity hypothesis—and then apply Theorem 10.2 of [12].

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