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Biprimitive Graphs of Smallest Order

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Abstract. A regular and edge-transitive graph which is not vertex-transitive is said to be *semisymmetric*. Every semisymmetric graph is necessarily bipartite, with the two parts having equal size and the automorphism group acting transitively on each of these parts. A semisymmetric graph is called *biprimitive* if its automorphism group acts primitively on each part. In this paper biprimitive graphs of smallest order are determined.

Keywords: primitive group, semisymmetric graph, biprimitive graph

1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected. For the group-theoretic concepts and notation not defined here we refer the reader to [4, 9].

For a graph X we let $V(X)$, $E(X)$ and Aut X be, respectively, the vertex set, the edge set and the automorphism group of X. A graph X is said to be *vertex-transitive, edge-transitive* and *symmetric,* respectively, if Aut X acts transitively on the set of vertices, edges, or arcs of X, respectively. Moreover, we say that X is *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. We remark that every semisymmetric graph is bipartite with the two parts of equal size and the automorphism group acting transitively on each of these two parts. The study of semisymmetric graphs was initiated by Folkman [8] who gave a construction of several infinite families of such graphs including, among others, a smallest semisymmetric graph on 20 vertices. At the end of his paper several problems were posed, most of which have already been solved (see [1, 2, 10–12, 14]). In all of the semisymmetric graphs given by Folkman [8] the automorphism group acts imprimitively on each of the two bipartition parts. A semisymmetric graph X is called *biprimitive* if Aut X acts primitively on each of the two parts of the bipartition. The first construction of a biprimitive graph is due to Iofinova and Ivanov who gave a classification of cubic biprimitive graphs [10]. It follows from their classification that only five such graphs exist. In 1995, the first author constructed an infinite family of biprimitive graphs by giving, for

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each prime $p \equiv 1 \pmod{48}$, a biprimitive graph with the automorphism group isomorphic to $PSL(2, p)$ (see [6]). Moreover, biprimitive graphs of order $2pq$, where p and q are distinct primes, have been classified in [7].

It is the purpose of this paper to give the construction of biprimitve graphs with the smallest order. More precisely, we shall prove the following result.

Theorem 1.1 *A biprimitve graph with the smallest order has* 80 *vertices and is isomorphic to one of the graphs* U_{80}^4 *and* U_{80}^{36}, *defined in Section* 3, *with respective valencies* 4 *and* 36 *and automorphism groups isomorphic to* $\mathrm{Aut}\, U_4(2)$.

In Section 2 the methods for constructing semisymmetric graphs are given together with some results on semisymmetric graphs to be used in Section 3, where the graphs U_{80}^4 and U_{80}^{36} are defined and the proof of Theorem 1.1 is given.

2. Preliminaries

We first recall the general methods for constructing semisymmetric graphs. Let G be a permutation group on a set V having two orbits U and W of the same cardinality and no other orbits. Furthermore let $\Delta_1, \ldots, \Delta_r$ be the orbits of the action of G on $U \times W$. For any $i \in \{1, \ldots, r\}$, let $X_i = X(G, V, \Delta_i)$ denote the bipartite graph with vertex set V and edges of the form uw, where $(u, w) \in \Delta_i$. Of course, X_i is regular and edge-transitive with bipartition (U, W). Moreover, X_i is semisymmetric if and only if its automorphism group preserves the two orbits of G.

Conversely, every semisymmetric graph can be obtained in the way described above. Namely, let X be a semisymmetric graph with the automorphism group G and bipartition (U, W) of its vertex set V. Take $u \in U$ and $w \in W$ and let $H = G_u$ and $K = G_w$. It may be easily seen that there is a one-to-one correspondence between the orbits of H on W (as well as the orbits of K on U) and the orbits of the action of G on the set $U \times W$, giving us precisely the situation of the previous paragraph.

For a vertex v of a graph X we let $N(v)$ denote the set of neighbors of v in X.

Lemma 2.1 [7] *Let X be a regular bipartite graph with bipartition (U, W) (such that $|U| = |W|$) of its vertex set V and let G be a subgroup of* $\mathrm{Aut}\, X$ *with orbits U and W. Let $u \in U$, $w \in W$, $H = G_u$, $K = G_w$ and $D = \{g \in G \mid w^g \in N(u)\}$. If there exists an element $\sigma \in \mathrm{Aut}\, G$ such that $H^\sigma = K$, $K^\sigma = H$ and $D^\sigma = D^{-1}$ then X is vertex-transitive. In particular,*

(i) *if G is Abelian and acts regularly on each of U and W, then X is vertex-transitive*;

(ii) *if the lengths of the orbits of H on W (or the orbits of K on U) are all distinct then X is vertex-transitive.*

Proof: Under the assumptions, it is easily seen that for any $g \in G$, $w^g \in N(u)$ if and only if $w^{(g^{-1})^\sigma} \in N(u)$. We define now a mapping $\overline{\sigma}$ of $V(X)$ interchanging U and

W by letting

$$(u^g)^{\overline{\sigma}} := v^{g^\sigma} \quad \text{and} \quad (w^g)^{\overline{\sigma}} := u^{g^\sigma}$$

for any $g \in G$. Obviously, $\overline{\sigma}$ is well-defined. Observe that

$$\begin{aligned} u^{g_0}w^{g_1} \in E(X) &\Longleftrightarrow uw^{g_1 g_0^{-1}} \in E(X) \\ &\Longleftrightarrow uw^{(g_0 g_1^{-1})^\sigma} \in E(X) \\ &\Longleftrightarrow u^{g_1^\sigma} w^{g_0^\sigma} \in E(X) \\ &\Longleftrightarrow (u^{g_0})^{\overline{\sigma}}(w^{g_1})^{\overline{\sigma}} \in E(X). \end{aligned}$$

Hence $\overline{\sigma} \in \operatorname{Aut} X$ and so X is vertex-transitive.

Of course, if (i) holds, the conclusion is true by taking $H = K = 1$ and $g^\sigma = g^{-1}$, for any $g \in G$. Similarly, if (ii) holds, the conclusion is true as $|Hg^\sigma K| = |Hg^{-1}K|$, for any $g \in G$, which implies that $D^\sigma = D^{-1}$. □

The next three propositions, extracted from [7, 8, 13], will be used in the proof of Theorem 1.1.

Proposition 2.2 [7] *The smallest biprimitive graphs or order* $2pq$, *where* p *and* q *are distinct primes, have* 110 *vertices.*

Proposition 2.3 [8] *The smallest order of a semisymmetric graph is* 20. *Besides, there exists no semisymmetric graphs of order* $2p$ *or* $2p^2$, *where* p *is prime.*

Proposition 2.4 [13] *Let* G *be a transitive group on a set* V, *let* $H = G_v$ *for some* $v \in V$ *and let* K *be a subgroup of* H. *If the set of* G*-conjugates of* K *which are contained in* H *form* t *conjugacy classes of* H *with representatives* $K_1, K_2, \ldots, K_t$, *then* K *fixes*

$$\sum_{i=1}^{t} |N_G(K_i) : N_H(K_i)|$$

points of V.

The last proposition of this section deals with the automorphism group of a biprimitive graph.

Proposition 2.5 *Let* X *be a biprimitve graph. Then* $\operatorname{Aut} X$ *is not an affine group and its rank* $r(\operatorname{Aut} X)$ *is at least* 3.

Proof: Let (U, W) be the bipartition of the vertex set V of X, let $|U| = n = |W|$ and let $G = \operatorname{Aut} X$. To prove (i) assume on the contrary that G is an affine group. Then it has an elementary Abelian normal subgroup T which is regular on both U and W. By Lemma 2.1 we have that X is vertex-transitive, a contradiction.

As for (ii), suppose that G acts doubly transitively on both U and W. If these two actions are equivalent, then X must be isomorphic either to $K_{n,n}$ or to $K_{n,n}$ minus a 1-factor, which are both vertex-transitive graphs. If on the other hand, these two representations are inequivalent, then [3, Theorem 5.3] implies that G is an almost simple group and the two representations of G can be exchanged by an involution σ in Aut S, where $S = \mathrm{soc}\,\mathrm{G}$. This forces X to be vertex-transitive by Lemma 2.1, completing the proof of Proposition 2.5. □

3. Proof of Theorem 1.1

We now give the definition of the two graphs U^4_{80} and U^{36}_{80}. Let $V(4, 4)$ be the four-dimensional unitary space over $GF(4)$. Let U and W be, respectively, the set of bases and the set of non-isotropic points of the projective space PG(V(4, 4)) of $V(4, 4)$ and let $G = U_4(2)$. Then $|U| = 40 = |W|$ and G acts primitively on both U and W. Take an arbitrary unitary base $B = \{e_1, e_2, e_3, e_4\}$ of $V(4, 4)$, where $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$ are two hyperbolic planes of $V(4, 4)$ such that for each element $w = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in V(4, 2)$ we have $(w, w) = x_1x_2^2 + x_2x_1^2 + x_3x_4^2 + x_4x_3^2$. Letting $H = G_B$ we have that H is isomorphic to $Z_3^3 : S_4$, an extension of Z_3^3 by S_4. Then H has precisely two orbits on W. The first one, call it D_1, has cardinality 4 and consists of all those $\langle w\rangle \in W$ for which $(w, w) = 1$ and either $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$. The second one, call it D_2, has cardinality 36 and consists of all those $\langle w\rangle \in W$ for which $(w, w) = 1$ and $x_i = 1$ for precisely one $i \in \{1, 2, 3, 4\}$. Now let $u = B \in U$, let $w \in W$ and let $K = G_w$. The graphs U^4_{80} and U^{36}_{80} are defined to be, respectively, the graphs $X(G, V(4, 4), \Delta_1)$ and $X(G, V(4, 4), \Delta_2)$, with Δ_1 and Δ_2 corresponding, respectively, to D_1 and D_2 in the one-to-one correspondence between the orbits of the action of G on $U \times W$ and the orbits of H on W, as mentioned in the second paragraph of the previous section. Of course, the graphs U^4_{80} and U^{36}_{80} are bipartite and regular of order 80 and with respective valencies 4 and 36 and, moreover, G acts transitively on their edge sets.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Letting $Y \in \{U^4_{80}, U^{36}_{80}\}$, we first show that Y is biprimitive and that Aut Y is as claimed. Let $A = \mathrm{Aut}\,U_4(2) \cong U_4(2) : Z_2$. Then by [5, Table 2] we have that A is a maximal uniprimitive subgroup of S_{40} having two representations of degree 40. It follows that A is the maximal subgroup of Aut Y preserving U and W. Supposing that Y is vertex-transitive we have $[\mathrm{Aut}\,Y : A] = 2$. Let $\tau \in \mathrm{Aut}\,Y\backslash A$. Since A is a complete group, it follows that $\tau \in C_{(\mathrm{Aut}\,Y)}(A)$ and so τ is an involution and thus $\mathrm{Aut}\,Y \cong A \times Z_2$. But this implies that A has two equivalent representations on U and W, a contradiction. Hence Y is semisymmetric and so biprimitive.

It remains to be seen that no biprimitive graph on at most 80 vertices, other than U^4_{80} and U^{36}_{80}, exists. For that purpose we now let X be a regular bipartite graph with bipartition (U, W), such that $|U| = n = |W|$, where $n \leq 40$, admitting a group G having orbits U and W on $V(X)$ and, moreover, acting primitively on U and W and transitively on $E(X)$. Clearly, X is connected. By Propositions 2.2 and 2.3 we may assume that $10 \leq n$ and that n is not of the form p, p^2 or pq, where p and q are distinct primes. Furthermore, $r(G) \geq 3$

Table 1.

Row	G^*	Degree of G^*	Point stabilizer in G^*	Rank of G^*
1	$U_4(2)$	27	$2^4 : A_5$	3
2	$U_4(2)$	36	S_6	3
3	$U_4(2)$	40	$3_+ : 2A_4$	3
4	$U_4(2)$	40	$3^3 : S_4$	3
5	$U_3(3)$	36	$L_2(7)$	3
6	A_8	28	S_6	3
7	A_9	36	S_7	3
8	$L_2(8)$	28	D_{18}	4
9	$L_2(8)$	36	D_{14}	5
10	$A_6.2$	36	D_{20}	4
11	$A_6 \times A_6$	36	$A_5 \times A_5$	3
12	$A_5 \times A_5$	36	$D_{10} \times D_{10}$	3

and G is not an affine group in view of Proposition 2.5. In addition, soc G acts transitively on both U and W.

The starting point for our analysis of the structure of the group G is the classification of primitive permutation groups of degree less than 1000, with the exception of the affine groups given in [5]. Those groups among them which satisfy all of the conditions of the previous paragraph are listed in Table 1. An explanation on the contents of this table is in order. Primitive groups are partitioned into *cohorts*, where two groups lie in the same cohort if and only if they have the same degree and their respective socles are permutation isomorphic. Now given any primitive group of degree $10 \leq n \leq 40$, with socle T and N being the normalizer of T in S_n, it follows from [5, Lemma 3] that every other primitive group between T and N must lie in the same cohort, with N as the unique maximal element of that cohort. Now Table 1 gives the primitive group G^* as the minimal element of the cohort, its degree, a description of the point stabilizer in G^*, and finally the rank of G^*.

Let us now use the information gathered in Table 1 to analyze the group G. (Hereafter by a row we mean a row of Table 1.) If the representions of G on U and W are inequivalent, then it follows by Table 1 (see rows 3 and 4) that soc $(G) = U_4(2)$, giving rise to the two graphs U_{80}^4 and U_{80}^{36}. We may therefore assume that the representations of G on U and W are equivalent. There are vertices $u \in U$ and $w \in W$ such that the vertex stabilizers G_u and G_w coincide. Let us denote them by H. We shall now prove that X is vertex-transitive. In view of Lemma 2.1, it is sufficient to prove that all the nontrivial suborbits of G are self-paired. Morever, for each cohort, if all the nontrivial suborbits of its minimal element are self-paired, the same holds for any other member of that cohort. Hence in what follows we may assume that G is one of the minimal elements G^* in Table 1.

Each group G^*, with the exception of those in rows 1, 8, 9 and 10, has rank 3 and even degree. Therefore the two nontrivial suborbits have distinct lengths and are thus self-paired.

Next, the group in row 1 is of rank 3 and its point stablizers are isomorphic to $2^4 : A_5$, which has no subgroup of index 2. It follows that the two nontrivial suborbits of G^* have distinct lengths, and are therefore self-paired.

As for the group G^* in row 8, let H be one of its point stabilizers, and thus isomorphic to D_{18}. Let $\mathcal{H} = \{Hg \mid g \in G^*\}$ and consider the right multiplication of G^* on $\mathcal{H}$. It follows by Proposition 2.4 that any subgroup of H of order greater than 2, fixes only one element of $\mathcal{H}$, that is the coset H. Therefore, for each nontrivial suborbit, the corresponding stabilizer in H must be a subgroup of order 2 of H. In particular G^* has three nontrivial suborbits, all of length 9. Let K be such a subgroup of order 2 in H. Since $N_H(K) = K$ and $N^*_G(K) \cong Z_2^3$, it follows by Proposition 2.4 that K fixes four elements of $\mathcal{H}$, and moreover, K fixes precisely one element of each nontrivial suborbit. Since, for any two fixed elements of K, there is an element in $N^*_G(K)$ interchanging them, [15, Theorem 16.4] implies that all suborbits are self-paired.

The group G^* in row 9 is dealt with in an analogous way. Firstly, it may be seen that its subdegrees are 1, 7, 7, 7, 14. The proof that the three suborbits of length 7 are self-paired now follows almost word-by-word the above proof of self-pairedness of the suborbits of length 9. We omit the details.

Finally, we are left with the group G^* in row 10. Let H be a point stabilizer D_{20} and let $\mathcal{H} = \{Hg \mid g \in G^*\}$ and consider the right multiplication of G^* on $\mathcal{H}$. Proposition 2.4 implies that any nontrivial subgroup of H other than Z_2 or $Z_2 \times Z_2$ fixes only one element of $\mathcal{H}$, that is the coset H. Hence, for each nontrivial suborbit, the corresponding stabilizer in H must be the identity group or Z_2 or $Z_2 \times Z_2$. It follows that the possible lengths of the three nontrivial suborbits of G^* are 5, 10 and 20, and in order for the lengths of the suborbits to add up to 36, the subdegrees must be precisely 1, 5, 10, 20. Thus the suborbits are all self-paired, completing the proof of Theorem 1.1. □

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