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The Median Stabilization Degree of a Median Algebra

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Abstract. The median stabilization degree (*msd*, for short) of a median algebra measures the largest possible number of steps needed to generate a subalgebra with an arbitrary set of generators. We determine the value of *msd* of a graphic n-cube Q_n and we derive an estimation of *msd* for the natural median operator of $\mathbb{R}^n$ which is sharp up to one or two units. Interestingly, *msd* of Q_n and of $\mathbb{R}^n$ grows like $\log_{1.5} n$. Finally, we characterize median algebras and median graphs of *msd* ≤ 1 in terms of forbidden subspaces.

Keywords: convex structure, graphic cube, median algebra, median stabilization degree, superextension

1. Introduction

Median algebras arose in the study of distributive lattices and trees by Birkhoff and Kiss [3], Sholander [7, 8], and others. We refer to Bandelt and Hedlíková [1] for a survey on median algebras and to van de Vel [10] for the theory of median convexity.

A median algebra consists of a set M and a *median operator* on M, by which is meant a symmetric function $m : M^3 \to M$ such that

$$m(a, a, b) = a \quad \text{and} \quad m(m(a, b, c), d, c) = m(a, m(b, c, d), c).$$

The last equality can be interpreted as an associative law. A totally ordered set has a natural median operator, which assigns to each triple of points the middle one. On a product of two median algebras (M_1, m_1) and (M_2, m_2) there is a median operator m defined as follows. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$. Then

$$m(a, b, c) = (m_1(a_1, b_1, c_1), m_2(a_2, b_2, c_2)).$$

Medians arise in certain metric spaces as follows (cf. Verheul [12]). Let (X, ρ) be a metric space and let $a, b \in X$. A point $x \in X$ is *in between* a, b provided $\rho(a, x) + \rho(x, b) = \rho(a, b)$. If for each triple of points $a, b, c \in X$ there is a unique point $x \in X$ simultaneously in between a and b, b and c, c and a, then the assignment $(a, b, c) \mapsto x$ determines a median operator on X. For instance, Banach spaces of type L_1 have this property. In particular, $\mathbb{R}^n$

with the "Manhattan" norm

$$\|(x_1, \ldots, x_n)\| = \sum_{i=1}^{n} |x_i|$$

leads to a median operator which agrees with the median operator of the n-fold product of the totally ordered set $\mathbb{R}$ with itself.

A subset A of a median algebra M is *median stable* (or, a *subalgebra*) provided $m(A^3) \subseteq A$. It is known (cf. [1]) that each median algebra occurs as a median stable subset of a distributive lattice under the median operator defined by

$$m(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a).$$

The *median stabilization* $med\,(A)$ of a subset A of a median algebra is the smallest median stable set which includes A. In other words, it is the subalgebra generated by A. It is recursively constructed as follows.

$$med\,(A) = \bigcup_{n=0}^{\infty} A_n, \quad \text{where } A_0 = A \text{ and } A_{n+1} = m\big(A_n^3\big). \tag{$*$}$$

The set A_n represents the nth stage of the stabilization process. The stabilization of a finite set is finite. Indeed, finitely generated free median algebras are finite. The *median stabilization degree*, *msd*, of a median algebra M is defined by the following prescription.

$$msd\,(M) \leq n \quad \text{iff } \forall A \subseteq M : med\,(A) = A_n,$$

where A_n represents the nth stage of the stabilization process (as in ($*$) above).

It is clear that *msd* of a subalgebra does not exceed *msd* of the original algebra. In fact, it can easily be shown that $msd\,(M) \leq n$ iff $msd\,(X) \leq n$ for each finite subalgebra X. In this way, a potential combinatorial method obtains to decide whether a median algebra can be embedded into some other median algebra. A result of Evans [5] can be interpreted as follows. A median algebra has *msd* zero if and only if it is derived from a total order or it is isomorphic to the graphic square. In this paper, we determine the value of *msd* of a graphic n-cube Q_n and we derive an estimation of *msd* for the natural median operator of $\mathbb{R}^n$ which is sharp up to one or two units. Interestingly, *msd* of Q_n and of $\mathbb{R}^n$ grows like $\log_{1.5} n$. Finally, we characterize median algebras and median graphs of $msd \leq 1$ in terms of forbidden subspaces and in terms of the exchange number.

2. Median stabilization degree

2.1. Examples of median algebras and their msd

(1) The natural median of three points in a totally ordered set selects the point which is in between the other two. Therefore, the median stabilization degree of a totally ordered set is zero.

(2) More generally, a *tree* is a (meet) semilattice such that any two points with a common upper bound are comparable in the semilattice order. Both graphic trees and topological trees are within the scope of this definition. If a tree has at least one ramification point, then its *msd* equals 1. In fact, ramification points are medians of non-stable triples.

It follows from the results of [10, Chapter II, Section 4] that a tree with $2n$ endpoints embeds in $\mathbb{R}^m$ for $m \geq n$ only. Since trees have *msd* at most one, *msd* of a median algebra does not give accurate information on embeddability.

(3) In the graphic square Q_2, each subset is median stable. Hence $msd\,(Q_2) = 0$. Next, *msd* of the graphic cube Q_3 equals 1. The simplest way to make this precise is via the observation that a vertex v of Q_3 is the median of three distinct points different from v iff these points are the neighbors of v. For graphic cubes of higher dimension, accurate information will be presented below.

A *median graph* is a connected graph of which the natural metric induces a median operator in a way described in Section 1. There is a natural correspondence between finite median graphs and finite median algebras. It turns out that graphic cubes are the building blocks of median graphs; this allows to regard median graphs as cubical complexes; cf. [10].

(4) The *superextension* $\lambda(\mathbf{n})$ of a finite n-point set is the free median algebra on n generators. Note that such algebras are finite. See Verbeek [11] for an explicit construction and for numerical information. Eckhoff's study [4] of the Radon number of the median algebra $\mathbb{R}^n$ can be used to produce explicit embeddings of superextensions into $\mathbb{R}^n$. The relationship of Eckhoff's work with embeddings of superextensions is explained in [9]. Figure 1 describes the superextensions of a three- and four-point set, embedded in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Now, $\lambda(\mathbf{3})$ is a tree with a ramification point, whence $msd\,(\lambda(\mathbf{3})) = 1$. One can verify that the four generators of $\lambda(\mathbf{4})$ stabilize in two steps

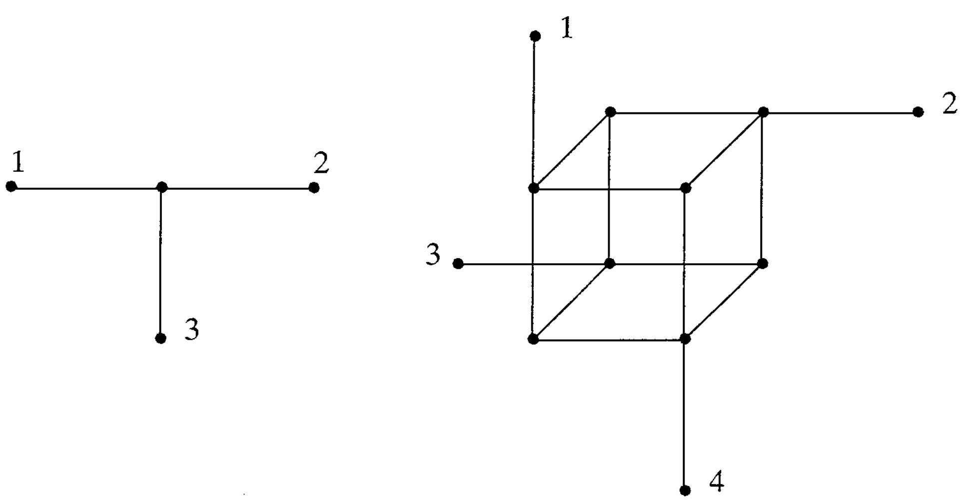

Figure 1. $\lambda(\mathbf{3})$ and $\lambda(\mathbf{4})$ with extreme points as generators.

only. As $\lambda(\mathbf{4})$ embeds into $\mathbb{R}^3$, we conclude from the product theorem below that $msd\,(\lambda(\mathbf{4})) = 2$.

It is known that $\lambda(\mathbf{5})$ has 81 vertices. The corresponding median graph is a four-dimensional cubical complex, of which a concise description is given by Bandelt and van de Vel [2]. It can be verified that the set of five generators stabilizes in three steps, whence *msd* is at least 3. Since $\lambda(\mathbf{5})$ can be embedded into $\mathbb{R}^5$, the product theorem below yields that *msd* of $\lambda(\mathbf{5})$ is at most 4.

The ten-dimensional cubical complex $\lambda(\mathbf{6})$ with 2,646 vertices can be embedded into $\mathbb{R}^{10}$. The embedded positions of the generators were fed into a computer program which indicated that the whole of $\lambda(\mathbf{6})$ obtains in three steps. However, the result below on graphic cubes shows that *msd* of $\lambda(\mathbf{6})$ is at least 5.

The superextension $\lambda(\mathbf{7})$ is fifteen-dimensional and can be embedded into $\mathbb{R}^{18}$. We do not know yet how many steps are needed to stabilize the seven-point generator set. A straighforward stabilization algorithm has a complexity of magnitude $\#\lambda(\mathbf{7})^3$ and the number of vertices of $\lambda(\mathbf{7})$ is more than 1,400,000. (The exact number has been computed by Brouwer and Verbeek [11].) Finally, $\lambda(\mathbf{8})$ has over 200,000,000,000 vertices; see Mills and Mills [6]. The monograph [10] contains some further information on the corresponding cubical complex.

The first result is a rather rough estimate of *msd* for products of ordered sets.

Proposition 2.2 *Let $X_1, \ldots, X_n$ be totally ordered sets with the natural median. Then*

$$msd\left(\prod_{i=1}^{n} X_i\right) \le n - 1.$$

Proof: Let $X = \prod_{i=1}^{n} X_i$. The result is valid for $n = 1$, and we proceed by induction. Throughout, the ith coordinate of a point $x \in X$ is denoted by x_i. Let F be a finite set in X and let $p \in med(F)$. The projection of F onto $\prod_{i=1}^{n-1} X_i \times \{p_n\}$ stabilizes in at most $n-2$ steps. This implies that there is a point a in X which may differ from p only in the nth coordinate and can be obtained from F in at most $n-2$ steps. Using the $(n-1)$th factor instead, we find a point b in X which may differ from p only in the $(n-1)$th coordinate, and which is obtained from F in at most $n-2$ steps.

We may assume that $p_{n-1} < b_{n-1}$ and $p_n < a_n$. Consider the sets

$$H_i = \{x \in X \mid x_i > p_i\}$$

for $i = n-1, n$. Clearly, $H_{n-1} \cup H_n$ is a median stable set not containing p. Hence there is a point $c \in F/(H_{n-1} \cup H_n)$. The median of a, b, c is easily seen to be p, which therefore obtains from F in at most $n-1$ steps. □

We now find that

$$msd\,(\mathbb{R}) = 0; \quad msd\,(\mathbb{R}^2) = 1; \quad msd\,(\mathbb{R}^3) = 2.$$

The first is clear, and the second follows from the previous proposition in combination with the fact that there exist 3-point sets in the plane that are not median stable. The third one follows from the proposition and the information on $\lambda(\mathbf{4})$ presented earlier. We conclude that $2 \leq msd\,(\mathbb{R}^4) \leq 3$ and (by using the information on $\lambda(\mathbf{5})$) that $3 \leq msd\,(\mathbb{R}^5) \leq 4$.

For each point p of a median algebra, the following prescription determines the so-called *base-point partial order at* p.

$$u \leq_p v \quad \text{iff } u = m(p, u, v).$$

The right-hand equality is usually interpreted as u being *between* p *and* v. The set of points between a and b is called the *interval between* a, b, and is denoted by concatenation, ab. A subset C of a median algebra M is *convex* provided $ab \subseteq C$ whenever $a, b \in C$. In terms of the median operator, this means that $m(C \times C \times M) \subseteq C$.

Lemma 2.3 *Let X be a median algebra and let $S \subseteq X$. Then $p \in X$ is generated by S, that is, $p \in med\,(S)$, iff it is generated by the set*

$$\{x \mid x \geq_p s \text{ for some } s \in S\}.$$

In either situation, the same number of steps is required.

Proof: This is a direct consequence of the fact that, if $y_i \leq_p x_i$ for $i = 1, 2, 3$, then $m(y_1, y_2, y_3) \leq_p m(x_1, x_2, x_3)$. □

Theorem 2.4 *For $n \geq 4$, the median stabilization degree of the graphic n-cube Q_n satisfies*

$$msd(Q_n) \geq 2 + \log_{3/2} n/4.$$

In fact, let $q_0 = 2$ and, recursively, $q_{i+1} = \lfloor 3q_i/2 \rfloor$. Then $msd(Q_n)$ equals the least i such that $n \leq q_i$.

Proof: The members of Q_n are regarded as subsets of a fixed n-set. The median of three elements A, B, C is then given by

$$m(A, B, C) := (A \cap B) \cup (B \cap C) \cup (C \cap A).$$

Consider the set $S \subseteq Q_n$ consisting of all 2-sets. Note that singletons are generated in step one. The empty set is then generated at latest in step two. Assume that after step i, all subsets of cardinality k have been generated, and no set of cardinality $>k$ is generated. If $\#A, \#B, \#C \leq k$, then $\#m(A, B, C) \leq 3k/2$ by a simple counting argument. Hence, in step $i + 1$, we can only generate sets of cardinality $\leq 3k/2$. If $n \geq 3k/2$, then evidently each set of cardinality $3k/2$ can be obtained as a median of three sets of cardinality k. The initial generation progress is illustrated by Table 1.

An inductive argument starting at $k = 4$ and $i = 2$ now shows that one cannot arrive at the full n-set in step i unless $n \leq (3/2)^{i-2} \cdot 4$, which gives the desired formula.

Table 1. Growth of *msd* for Q_k.

msd	0	1	2	3	4	5	6	7	8	9
$k\leq$	2	3	4	6	9	13	19	28	42	63

It is evident from the above argument that if i is the least integer with $n \leq q_i$, then the *msd* is at least i. To see that this estimate is sharp, let $S \subseteq Q_n$ and $p \in \mathit{med}(S)$. By Lemma 2.3, we may replace S by the set

$$\{x \mid \exists s \in S : x \geq_p s\}.$$

Without loss of generality, p is represented by the full n-set. If the (enlarged) collection S does not contain a certain 2-set r, then the entire $n-2$-cube spanned by p and r is disjoint with S. However, Q_n minus a convex $(n-2)$-subcube is median stable, so S would not generate p, a contradiction. As the first part of the proof indicates, we need at most i steps to get from the 2-sets to p. □

The next result is a useful tool in determining *msd* of general spaces. Recall that a subset C of a median algebra X is *gated* provided for each $x \in X$ there exists a (necessarily unique) point $\pi(x) \in C$ such that $\pi(x) \in xc$ for all $c \in C$. This point $\pi(x)$ is called the *gate of x in C*, and the resulting function $\pi : X \to C$ is called the *gate map* of C. Gated sets in a median space are convex, and conversely, nonempty finite convex sets are always gated; see [10] for general information.

The *convex neighborhood* of a point p in a finite median algebra is the convex hull of p together with all its neighbors.

Proposition 2.5 *In a finite median graph G, the following assertions are equivalent for each $n < \infty$.*
(1) $\mathit{msd}(G) \leq n$.
(2) *For each vertex the convex neighborhood has msd at most n.*

Proof: The implication (1) $\Rightarrow$ (2) is trivial. Suppose $\mathit{msd}(G) > n$. Then there is a set $S \subseteq G$ and a point $q \in \mathit{med}(S)$ which is not obtained from S in n or fewer steps. Let U be the convex neighborhood of q, and let $\pi : G \to U$ be the gate map. As π is median-preserving, $q = \pi(q)$ is also generated by $\pi(S)$. Suppose q is obtained in k steps from $\pi(S)$. For each member of $\pi(S)$ we fix one pre-image in S. This gives a set $S' \subseteq S$ and a bijection $\pi : S' \to \pi(S)$. Consider a sequence of k operations, leading from $\pi(S)$ to q. In each operation, we formally replace the involved members of $\pi(S)$ by the corresponding members of S'. Then, at each stage of the process, the resulting point maps to the corresponding point of the original process. In this way, the pre-image process ends in a point q' of G with $\pi(q') = q$. However, as U contains all neighbors of q we have $q' = q$. Since q is obtained from S in k steps, we conclude that $k > n$ and $\mathit{msd}(U) > n$. □

Table 2. Lower bound i and upper bound j of *msd* $(\mathbb{R}^n)$

n	i	j
4	2	4
5, 6	3	5
7, 8, 9	4	6
10–13	5	7
14	6	7
15–19	6	8
20, 21	7	8
22–28	7	9

Note that, in fact, $msd(G) \leq n$ iff for each $q \in G$, each subset of the convex neighborhood of q generates q in at most n steps.

Corollary 2.6 *Let $L_n \subseteq \mathbb{R}^n$ be the lattice $\{-1, 0, 1\}^n$. Then msd $(\mathbb{R}^n) =$ msd (L_n).*

Proof: We noted before that *msd* is determined by the behavior of finite sets. If $F \subseteq \mathbb{R}^n$ is finite, then *med* (F) is part of a finite lattice of type

$$F_1 \times F_2 \times \cdots F_n,$$

where F_i is the projection of F onto the ith axis. The convex neighborhood of a point in F_i contains at most three points. The result now follows from Proposition 2.5. □

Corollary 2.7 *The median stabilization degree of $\mathbb{R}^n$ is in between two integers i, j, determined as follows: i is the smallest integer satisfying $n \leq q_i$ and j is the smallest integer satisfying $2n \leq q_j$.*

Proof: Q_n is a subalgebra of $\mathbb{R}^n$ and L_n can be embedded into Q_{2n}. □

Table 2 may give an impression of the values and of the (un)sharpness of the estimates.[1] Note that if $n = 4, 5$ then the indicated value of j can be improved with the aid of Proposition 2.2.

Remark.

Since $\lambda(\mathbf{4})$ is a free algebra on 4 generators stabilizing in two steps, it follows that *any* 4-point set in *any* median algebra stabilizes in at most two steps. See Example 2.1 (4). Therefore, in regard to Corollary 2.6, one has to consider all subsets of the lattice L_4 with at least 5 points in order to verify whether *msd* $(L_4) = 3$. Similarly, $\lambda(\mathbf{6})$ is a free algebra on 6 generators stabilizing in three steps, whence *any* 6-point set in *any* median algebra stabilizes in at most three steps. To verify whether *msd* $(L_5) = 4$ requires investigating all subsets of L_5 with at least 7 points. An upper bound for the cardinality of subsets will be presented next. With no further information at hand, these tasks take far too much computer time.

We need a few concepts from the theory of convex invariants; see [10], Chapter II. The *Carathódory number* c of a convex structure X with convex hull operator *co* is the smallest number k such that for each finite set $F \subseteq X$ with $\#F > k$,

$$co(F) = \bigcup_{x \in F} co(F/\{x\});$$

$c = \infty$ if no such k exists. The *exchange number* e of X is the smallest number k such that for each finite set $F \subseteq X$ with $\#F > k$ and for each $p \in F$,

$$co(F/\{p\}) \subseteq \bigcup_{x \in F, x \neq p} co(F/\{x\});$$

$e = \infty$ if no such k exists. Informally, each "face" of $co(F)$ is covered by the other faces if the number of points in F exceeds the exchange number of X.

Both numbers c, e are closely related: $c = e - 1$ in case $c \geq 3$ (cf. [10], Chapter II, Section 1.9). Trees with a ramification point and totally ordered sets with at least three points have $c = 2$, whereas all trees and all ordered sets with more than one point have $e = 2$. In fact, median algebras of exchange number ≤ 2 are precisely the trees. The exchange number of a median graph is one larger than the dimension of the corresponding cubical complex.

Theorem 2.8 *Let M be a non-empty median algebra with a finite exchange number e and Carathódory number c.*

(1) *If a point of M is generated by a set $S \subseteq M$, then it is generated by a subset of S with at most $(e-1) \cdot c + 1$ points.*
(2) $msd\,(M) \leq 1 + \lceil \log_2 c \rceil + msd\,(Q_{e-1})$.

Proof: Let $S \subseteq M$ be finite, and let $p \in med\,(S)$. We will estimate the number of steps needed to generate p from S. Without loss of generality, we may assume $M = med\,(S)$. To begin with, fix a point $0 \in S$ and consider the corresponding base-point order $\leq_0$:

$$x \leq_0 y \quad \text{iff } m(0, x, y) = x, \text{ iff } x \in 0y.$$

The partially ordered set $(M, \leq_0)$ is a *median semilattice*, that is: a (meet) semilattice in which every principal ideal is a distributive lattice and any three elements have an upper bound whenever each pair is bounded above (cf. [1]). Moreover, the assumption on the exchange number of M implies that each sublattice of M has breadth $e - 1$. Note that $e = 1$ iff M is a one-point set, in which case the theorem is valid. We assume that $e > 1$. According to [10], Chapter I, Section 6.34, a point $q \in M$ is in $med\,(S)$ iff there exist finite sets $F_1, \ldots, F_k \subseteq S$ with

$$\bigcap_{i=1}^{k} co\,(F_i) = \{q\}.$$

It is not difficult to deduce that

$$q = \left(\bigwedge F_1\right) \vee \cdots \vee \left(\bigwedge F_k\right).$$

By definition, we may assume that $\# F_i \leq c$ for each i. As the interval $0q$ is of breadth $e-1$, we may assume that $k \leq e-1$. In particular, q is generated by the set

$$\bigcup_{i=1}^{k} F_i \cup \{0\},$$

establishing (1). The meet of two points in M can be obtained as a median of these points with 0. Hence it takes at most $\lceil \log_2 \cdot c \rceil$ many steps to generate an arbitrary meet of members of S.

Let $I \subseteq M$ be the set of all *join-irreducible elements*: $a \in I$ iff $a = x \vee y$ implies $a = x$ or $a = y$. Each join-irreducible element obtains as a meet of elements in S. Let T consist of all joins of pairs in I. Since M is finite, each element in M is a join of join-irreducible elements. Moreover, for every join-irreducible element $a \leq x \vee y$ we have $a \leq x$ or $a \leq y$.

If $t \in T$ then $M/\uparrow(t)$ is a subalgebra by the following argument. Suppose $t = a \vee b \leq m(x, y, z)$, where $a, b \in I$. The median of x, y, z can be obtained in lattice terms as

$$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Then, as a, b are join-irreducible, one of x, y, z is a common upper bound of a, b and hence it also bounds t from above.

We conclude that $\uparrow(t)$ meets S for each $t \in T$, and hence the entire set T can be generated from S in at most one extra step. As the breadth of the lattice $0p$ is the same with respect to its meet or its join, we obtain a minimal set

$$J = \{j_1, \ldots, j_m\} \subseteq I$$

of $m \leq e-1$ points such that $p = \bigvee J$. Let $e_1, \ldots, e_m \in \{0,1\}^m$ be the standard unit vectors. The irreducibility of J implies that the correspondence $e_i \mapsto j_i$ $(i = 1, \ldots, m)$ extends in a canonical way to a lattice isomorphism of $\{0,1\}^n$ with the interval joining $\bigwedge J$ and $\bigvee J$. This interval includes the set T, whence p is generated from T in at most $msd\,(Q_{e-1})$ additional steps. □

The estimation of the number of points needed to generate a particular point seems to be sharp in low dimensions. We do not know whether it is sharp in general. Comparing the lower bound of Theorem 2.8(2) with the estimates of Table 2 one has the impression that the correction term

$$1 + \lceil \log_2 c \rceil$$

on $msd\,(Q_{e-1})$ is somewhat too large.

2.10. Examples

(1) We have $c(L_n) = n$ if $n \geq 2$ and $e(L_n) = n + 1$ for all n. Hence in L_4 and in L_5 one needs to consider subsets of at most 17 and 26 points, respectively, in order to compute *msd*. Alternatively, one can use Lemma 2.3: if a set S generates a point p, then the minima of $(S, \leq_p)$ generate p too. In case of L_n, this seems to lead to larger estimates. Part (2) of the previous result yields $msd\,(L_4) \leq 4$ and $msd\,(L_5) \leq 6$, which are a bit too large.

(2) $\lambda(\mathbf{7})$ is a fifteen-dimensional cubical complex with $e = 16$ and $c = 15$, and which can be embedded in $\mathbb{R}^{18}$. Estimating via Theorem 2.8, we find $msd\,(\lambda(\mathbf{7})) \leq 11$. Estimating via Table 2 yields $6 \leq msd\,(\lambda(\mathbf{7})) \leq 8$.

3. Spaces of low *msd*

In this section, we determine all median spaces of *msd* equal to 0, 1. The characterization of zero *msd* is actually a reformulation of a result of Evans [5].

Theorem 3.1 [5] *A median algebra has msd zero iff either it is embeddable in a totally ordered set* (*as a subalgebra*), *or it is a graphic square.*

3.1. Simplex graphs

In order to characterize general median algebras of *msd* equal to 1, we first concentrate on the special class of so-called simplex graphs. (cf. Bandelt and van de Vel [2]). We recall that the *simplex graph* $\hat{F}$ of a graph F consists of all simplices (complete subgraphs) of F, where two simplices σ_1, σ_2 form an edge iff their symmetric difference has at most one point. These graphs have been classified as the median graphs G with a "central" vertex v, such that all maximal graphic cubes in G contain v. In a true simplex graph, the empty simplex is such a central vertex. Figure 2 presents the simplex graphs of cycles having length 3, 4, or 5, respectively.

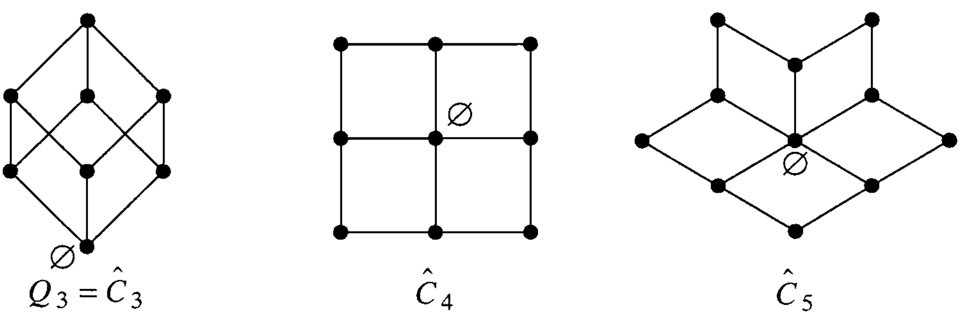

Figure 2. Three simplex graphs.

Lemma 3.2 *Let G be a median graph and $p \in G$. Then the convex neighborhood of p equals the union of all cubes of G which contain p. In particular, this neighborhood is a simplex graph.*

Proof: If $Q \subseteq G$ is a cube containing p and of dimension >1, then Q is the convex hull of all neighbors of p in Q. This implies that the union of all cubes at p is included in the convex neighborhood.

Suppose that some point q in the convex neighborhood of p is in no cube with p. We assume q is closest to p with this property. Pick the last point $r \neq q$ on a geodesic from p to q. Now $p, r \in Q$ for some cube Q of G. Note that the interval pr is a cube Q'. Choose a half-space H (i.e., a convex set with a convex complement) containing q but disjoint with Q. As q is in the hull of all proper neighbors of p, there is an edge ps with $s \in H$. It follows that the mutual gate mappings $Q' = pr \to qs$ and $qs \to pr$ map p and r to s and q respectively, and vice versa. Therefore, these sets are mutual nearest point sets which are one edge away. By virtue of the "amalgamation theorem" [9], it follows that $pr \cup qs$ is another cube, a contradiction. □

Proposition 3.3 *Let G be the simplex graph of a finite graph F. Then $msd\,(G) \leq 1$ iff $F = C_3$, or F has no subgraph of type C_3 or C_5.*

Proof: First, observe that $msd\,(\hat{C}_3) = 1$. If F properly includes a C_3, then G admits an induced median subgraph consisting of a 3-cube and with an additional edge pending one of the cube's vertices. Figure 3 describes the stages $0, 1, 2$ of a stabilizing process in this subgraph, showing that $msd\,(G) \geq 2$. If F has an induced C_5 then $msd\,(G) \geq msd\,(\hat{C}_5) = 2$ (the five corner points of $\hat{C}_5$ generate Ø at the second step). This shows that if $msd\,(G) \leq 1$, then G is as described.

Henceforth, we assume that F has no induced C_3 nor C_5 but $msd\,(G) \geq 2$. In regard to Proposition 2.5 and Lemma 3.2, there is a simplex graph $\hat{F} \subseteq G$ and a set $S \subseteq \hat{F}$ such that the central vertex Ø of $\hat{F}$ is generated in two steps. We assume that F, S are minimal with these properties. If an edge of F is not in S, then the simplex graph of F minus this edge is

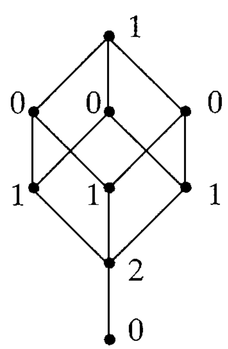

Figure 3. *msd* of a 3-cube plus edge.

a median subalgebra of G including S, so we may as well drop it from F. Suppose $v \in F$ is a vertex of degree one and uv is the only edge at v. Then $\hat{F}$ is an amalgamation along Ø, u of a square and the simplex graph of $F/\{v\}$. If the singleton $\{v\}$ is not in S, necessarily the doubleton $\{u, v\}$ belongs to S. When substituting this by the singleton $\{u\}$, its gate in the simplex graph of $F/\{v\}$, then the generation pattern will not change. Summarizing, we may assume that S includes all edges and endpoints of F.

A first conclusion is that F has at most two components, for otherwise S has three doubletons constituting pairwise non-adjacent edges of F, and the median of such a triple is Ø. If F is not connected, then none of the two components can have two disjoint edges (same reason as above). As F is triangle-free, each component must be a star. But if there are three or more endpoints altogether, then Ø obtains as their median. So F consists of two isolated vertices, and S cannot generate Ø unless Ø $\in S$!

The conclusion so far is that F is connected. If the diameter of F is at least three, then consider a geodesic v_1, v_2, v_3, v_4 of length three. The first point v_1 is either an endpoint of F, or there is an edge e_1 at this point not incident with any other vertex of the geodesic. Similarly, the fourth vertex v_4 is an endpoint, or is incident with an edge e_4 not incident with any other vertex of the geodesic. Note that if both e_1 and e_4 emerge, then they are not incident since v_1 and v_4 are at distance three. In any case, we get three points of S with median Ø, namely, $\{v_2, v_3\}$ (the middle edge of the given geodesic in F), either v_1 or the edge e_1, and either v_4 or the edge e_4.

So F is a connected graph of diameter 2 and without C_3, C_5. Then F is a complete bipartite graph $K_{m,n}$ (say: $m \leq n$). If $m \geq 3$, there exist three pairwise disjoint edges, and Ø obtains from them in one step. Suppose $m = 2$ and $n \geq 3$. Observe that the set of all edges, together with the two points of one color, yield a median stable set of $\hat{F}$. As S has to generate Ø, some vertex v of the second color must occur. As $n \geq 3$, it is possible to find two disjoint edges, not incident with v. The median of this triple is Ø, however.

This leaves us with two types: $K_{2,2} = C_4$, and the stars $K_{1,n}$. In case of C_4, the corresponding simplex graph is $\{-1, 0, 1\}^2$ which has *msd* equal to one. In case of a $K_{1,n}$, note that $n \geq 3$ gives us three endpoints which are in S and generate Ø at once. The case $n = 1$ being a triviality, we consider the two-path $K_{1,2}$. Then S either contains all vertices (and generates Ø in one step) or it contains Ø. □

It is shown in [10] that the exchange number (see Section 2) of a median graph is at least $n + 1$ iff the graph has an induced cube of dimension n. The "cube-free" condition of the previous argument can therefore be expressed by the inequality $e \leq 3$. Combining Lemma 3.2 with Propositions 2.5 and 3.3, we arrive at the following results.

Theorem 3.4 *A median algebra has msd at most* 1 *iff either it is Q_3, or its exchange number satisfies $e \leq 3$ and it does not contain $\hat{C}_5$ as a subalgebra.*

Theorem 3.5 *A median graph has msd at most* 1 *iff either it is Q_3, or its exchange number satisfies $e \leq 3$ and neither Q_3 nor $\hat{C}_5$ occur as convex subgraphs.*

It would be desirable to have similar characterizations in the case of larger *msd*.

Notes

1. The second author has been able to determine *msd* (L_n). Consult this journal.

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