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Dynkin Diagram Classification of λ-Minuscule Bruhat Lattices and of *d*-Complete Posets

ROBERT A. PROCTOR rap@math.unc.edu
Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

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Abstract. d-Complete posets are defined to be posets which satisfy certain local structural conditions. These posets play or conjecturally play several roles in algebraic combinatorics related to the notions of shapes, shifted shapes, plane partitions, and hook length posets. They also play several roles in Lie theory and algebraic geometry related to λ-minuscule elements and Bruhat distributive lattices for simply laced general Weyl or Coxeter groups, and to λ-minuscule Schubert varieties. This paper presents a classification of d-complete posets which is indexed by Dynkin diagrams.

Keywords: d-complete poset, minuscule Weyl group element, reduced decomposition, Dynkin diagram

1. Introduction

A poset is "d-complete" if it satisfies certain local structural conditions which are listed in Section 3. In this paper we describe all possible d-complete posets. This includes an explicit listing of all possible "irreducible components" of d-complete posets. Each such irreducible component is indexed by a connected Dynkin diagram which is embedded in the order (Hasse) diagram of the component. The next three paragraphs are addressed to readers familiar with Lie and Coxeter groups and should be skipped by other readers. Except for those paragraphs and Section 15, this paper may be read by anyone who is familiar with basic poset concepts. There are combinatorial motivations for studying d-complete posets which are independent of Lie theory, e.g., the hook length property.

Let W be a simply laced general Weyl group, i.e., the Weyl group associated to some fixed simply laced Kac-Moody algebra. Let λ be a dominant integral weight. Dale Peterson defines $w \in W$ to be λ-*minuscule* if there exists some decomposition $s_{i_k} \cdots s_{i_1}$ of w such that $s_{i_j}(s_{i_{j-1}} \cdots s_{i_1}\lambda) = (s_{i_{j-1}} \cdots s_{i_1}\lambda) - \alpha_{i_j}$ for $1 \leq j \leq k$, where α_i is the simple root associated to s_i. In the companion paper [7] to this paper, we showed that if w is a λ-minuscule element for a simply laced general Weyl group, then the initial interval $[e, w]$ in either the weak or strong Bruhat order on W/W_λ is a distributive lattice. Set $L_w := [e, w]$. These are the "λ-minuscule Bruhat lattices" mentioned in the title of this paper. Any finite distributive lattice L is determined by its subposet P of "join irreducible" elements. In [7] we characterized the posets P which could arise as posets of join irreducibles of

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λ-minuscule Bruhat lattices L_w. This characterization amounted to the satisfaction of several certain local structural conditions; we defined a poset to be "d-complete" if it satisfied all of those conditions. The present paper describes all possible d-complete posets. (However, the definition of d-complete used in this paper is the order dual of the definition given in [7]; the statements made above are made with respect to the [7] definition.) Using a couple of easy translation steps from [7], the description of all d-complete posets given in the present paper can be converted into a description of all possible λ-minuscule elements for simply laced general Weyl groups.

Let W be an arbitrary Coxeter group. Stembridge has studied the elements $w \in W$ for which the weak Bruhat interval $[e, w]$ is a distributive lattice [14]. He showed that this property is equivalent to w being "fully commutative", namely, any reduced decomposition for w can be converted into any other reduced decomposition for w using only relations of the form $s_i s_j = s_j s_i$. It can be seen that every λ-minuscule element w is fully commutative. Section 15 contains further remarks on some Weyl group implications of this paper; also consult the last section of [7] for comments on representation theoretic and geometric implications. In particular, the list of d-complete posets given in this paper index a family of particularly simple Schubert subvarieties of Kac-Moody flag manifolds, and it is hoped that each poset drawn in this paper will embody much useful geometric information for the corresponding variety. Also, the d-complete posets describe the structures of the "minuscule" portions of weight diagrams for integrable representations of simply laced Kac-Moody algebras.

In [10], we re-constructed some special cases of a basis of Lakshmibai [5] for Demazure modules in a poset-theoretic setting. We showed that a colored poset has such a basis for a module associated to it if and only if it is a colored d-complete poset. This was the origin of the notion of d-complete poset. Proposition 8.6 of [7] implies that the notions of d-complete poset and of colored d-complete poset are equivalent.

The introduction for a general audience begins here. Section 3 contains the definition of "d-complete" poset. In Section 4 we show how to decompose an arbitrary connected d-complete poset into a "slant sum" of "irreducible components". In Sections 5 and 6 we derive several facts about irreducible components; the most important being that a combinatorially defined subposet of an irreducible component, its "top tree", must be "Y-shaped". In Sections 9–13 we define 15 exhaustive classes of irreducible components and describe all of the members of each class. In 14 out of the 15 classes, the top tree of an irreducible component must be a Dynkin diagram of "general type E". This listing of possibilities is summarized in Section 7. Combining the theorems of Sections 4, 5, and 7 gives the classification of d-complete posets.

Shapes (Ferrers diagrams) and shifted shapes are diagrams upon which Young tableaux and plane partitions are defined. The boxes of a (shifted) shape may be viewed as the elements of a certain poset. Then shapes and shifted shapes constitute two particular infinite families of posets. These two families of posets essentially form our Classes 1 and 2 of irreducible components. Two other infinite families of posets which are important for this paper are defined in Sections 2 and 4: these consist of "double-tailed diamonds" and "rooted trees". The double tailed diamond posets play a central role in the definition of

d-complete poset. Rooted tree posets are the "trivial" d-complete posets in a certain sense. By introducing the notions of d-complete and slant sum, and by describing the irreducible components in Classes 3–15, this paper may be thought of as "filling out" the category of posets "hinted at" by shapes, shifted shapes, double-tailed diamonds, and rooted trees. "Slant irreducible components" may be thought of as being the result of "weaving together" many double-tailed diamonds in a certain fashion. General connected d-complete posets are obtained by combining slant irreducible components with the slant sum operation.

A poset P is said to be "hook length" if its associated P-partition generating function factors in a certain nice fashion analogous to identities discovered by Euler and Stanley. Until recently, the only known infinite families of hook length posets were shapes [12], shifted shapes [4, 11], rooted trees [12], and double-tailed diamonds. With Dale Peterson, we recently have shown that any d-complete poset is a hook length poset, by combining facts from algebraic geometry and representation theory with the viewpoint of [7]. A corollary to this result is a generalization of the hook product formula for the number of standard Young tableaux on an ordinary shape to a product formula for the number of order extensions of any d-complete poset. This corollary can be viewed as a conversion of Dale Peterson's (long known) hook formula for the number of reduced decompositions of a λ-minuscule element into a combinatorial form analogous to the original Frame-Robinson-Thrall form.

After reading the definition of d-complete poset in Section 3 (with references to Section 2 as needed), browsers should also glance at the definition of slant sum in Section 4. Next they should read the last third of Section 4 (following the Corollary), which contains the theorem for decomposing d-complete posets into their irreducible components. This theorem is illustrated by figure 3. Then they should skim the description of the classification of irreducible components presented in Section 7, while consulting Table 1 and figures 5.1–5.15. Except for Sections 1, 14, and 15, this paper is entirely self-contained.

It is possible [7] to recast the λ-minuscule definition above as a mild modification of the naively expressible "numbers game" on the simple graph G. This game has been studied in several papers, including [1–3, 6]. The paper [1] quickly shifts to such a naive environment. If one preferred, one could regard the Dynkin diagram classification result of this paper as being for certain aspects of a certain numbers game.

In [8] we found all Bruhat orders on parabolic quotients W^J of finite Weyl groups which are distributive lattices. We called the associated posets of join irreducibles "minuscule", since they were exactly the posets of join irreducibles for the distributive lattices arising as weight diagrams for minuscule representations. In Section 14 we use our classification result to show that a d-complete poset is order self-dual if and only if it is a minuscule poset. (The minuscule posets are the only known Gaussian posets [13, p. 288].)

2. Poset definitions

Let P be a poset (finite partially ordered set). If x is covered by y, we write $x \to y$. The *order diagram* of P is the directed graph made with such edges. Two subsets of a poset are *non-adjacent* if they share no elements and if there are no edges joining an element in one to an element in the other. A poset is *connected* if its order diagram is connected. Any poset can be expressed as a direct sum of non-adjacent connected subposets. An *ideal* of P is a subset $I \subseteq P$ such that $y \in I$ and $x \leq y$ imply that $x \in P$. A *filter* of P is a subset $F \subseteq P$ such that $x \in F$ and $y \geq x$ imply that $y \in P$. If $x \in P$, then the *principal ideal* $(x) := \{y : y \leq x\}$. If $x,\ y \in P$, we define *intervals* $[x, y] := \{z : x \leq z \leq y\}$ and $[x,\ y) := \{z : x \leq z < y\}$. A *chain of length n* in P is a sequence of elements $x_0 \to x_1 \to \cdots \to x_n$. The *order dual poset* P^* of P is defined on the same set of elements as P by: $y \leq x$ in P^* if $x \leq y$ in P.

Let $\mathcal{Q}$ denote the set of integral points in the strict fourth quadrant of the plane. Its elements $(i,\ j)$ will be coordinatized as in a matrix, so $i \geq 1$ and $j \geq 1$. We turn $\mathcal{Q}$ into a poset by: $(i_1,\ j_1) \geq (i_2,\ j_2)$ if $i_1 \leq i_2$ and $j_1 \leq j_2$. A *shape* $\lambda = (\lambda_1,\ \lambda_2, \ldots)$ is a finite filter of $\mathcal{Q}$ with λ_i elements of the form (i, j). Note that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ for some maximal $r \geq 0$. The *width* of λ is λ_1 and the *length* of λ is r. Let $\mathcal{O}$ denote the "octant" subposet of $\mathcal{Q}$ formed by taking the weakly upper triangular portion of $\mathcal{Q} : (i, j) \in O$ if $j \geq i$. A *shifted shape* $\mu = (\mu_1,\ \mu_2, \ldots)$ is a finite filter of $\mathcal{Q}$ with μ_i elements of the form (i, j). Note that $\mu_1 > \mu_2 > \cdots > \mu_r > 0$ for some maximal $r \geq 0$. The *width* of μ is μ_1 and the *length* of μ is r. When depicting such posets with graph paper, the "up" direction is Northwest, not North as usual.

Let $\Lambda(r, c)$ denote the set of shapes λ whose length does not exceed r and whose width does not exceed c. Let $\Sigma(r, c)$ denote the set of shifted shapes whose length does not exceed r and whose width does not exceed c. Let $\Sigma(r, c)_M$ denote the maximal element of $\Sigma(r, c)$, namely, the shifted shape with row lengths $c, c-1, c-2, \ldots, c-r+1$. The set $\Sigma(r, c)'$ is defined to consist of all filters of the order dual of $\Sigma(r, c)_M$.

The order diagrams of the "double-tailed diamond" posets $d_k(1)$ are shown in figure 1 for $k = 3,\ 4$, and 5. For $k \geq 3$, the *double-tailed diamond* poset $d_k(1)$ has $2k-2$ elements, of which two are incomparable elements in the middle rank and $k-2$ apiece form chains above and below the two incomparable elements. (The poset $d_k(1)$ is the poset of join irreducibles of the Bruhat lattice $D_k(1)$ for the ω_1 representation of the simple Lie algebra of type D_k. [8]) The $k-2$ elements above the two incomparable elements are called *neck* elements, and when $k \geq 4$ all but the lowest of these are called *strict neck* elements.

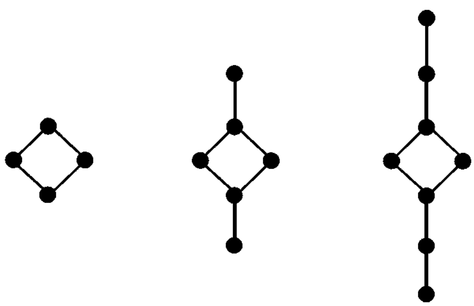


Figure 1.

Let P be a poset. A subset $\{w,\ x,\ y,\ z\}$ of P is a *diamond* if z covers x and y, and each of x and y cover w. The *top* and *bottom* of this diamond are z and w, and the *sides* are x and y. An interval $[w, z]$ is a d_3*-interval* if it is a diamond $\{w,\ x,\ y,\ z\}$ for some x and y, or in other words, if $[w,\ z] \cong d_3(1)$. More generally, for $k \geq 3$, we say that an interval $[w\ , z]$ is a d_k*-interval* if is isomorphic to $d_k(1)$. A d_3^-*-interval* $[w; x, y]$ consists of three elements $x,\ y$, and w such that x and y each cover w. For $k \geq 4$, we say that an interval $[w, y]$ is a d_k^-*-interval* if is isomorphic to $d_k(1) - \{t\}$, where t is the maximal element of $d_k(1)$.

3. *d*-Complete posets

Let P be a poset with elements $w,\ x$, and y. Suppose that $[w;\ x,\ y]$ is a d_3^--interval. If there is no $z \in P$ such that $\{w, x, y, z\}$ is a d_3-interval, then $[w; x, y]$ is an *incomplete* d_3^--interval. If there exists $w' \neq w$ such that $[w';\ x,\ y]$ is also a d_3^--interval, then we say that $[w; x, y]$ and $[w'; x, y]$ *overlap*. A poset P is d_3*-complete* if it contains no incomplete d_3^--intervals, if the maximal element of each d_3-interval does not cover any elements outside of that interval, and if it contains no overlapping d_3^--intervals. We have just required:

(D1) Anytime two elements x and y cover a third element w, there must exist a fourth element z which covers each of x and y,
(D2) If $\{w, x, y, z\}$ is a diamond in P, then z covers only x and y in P, and
(D3) No two elements x and y can cover each of two other elements w and w'.

Let $k \geq 4$. Suppose $[w, y]$ is a d_k^--interval in which x is the unique element covering w. If there is no $z \in P$ covering y such that $[w, z]$ is a d_k-interval, then $[w, y]$ is an *incomplete* d_k^--interval. If there exists $w' \neq w$ which is covered by x such that $[w', y]$ is also a d_k^--interval, then we say that $[w, y]$ and $[w', y]$ *overlap*. For any $k \geq 4$, a poset P is d_k*-complete* if:

(D4) There are no incomplete d_k^--intervals,
(D5) If $[w, z]$ is a d_k-interval, then z covers only one element in P, and
(D6) There are no overlapping d_k^--intervals.

A poset P is d*-complete* if it is d_k-complete for every $k \geq 3$. It is easy to see that any filter of a d-complete poset is itself d-complete. Note that D5 implies: Any strict neck element of a d_k-interval (necessarily $k \geq 4$) in a d-complete poset covers exactly one element. Now we list some properties which follow from D1 and D2.

Proposition *Let P be a poset with elements w, x, x', y, z which satisfies* D1. *Then the finiteness of P implies*:
(F1) *Suppose that $x \leq x'$, $x \to y$, and $y \not\leq x'$. Then there exists a $y' \in P$ which covers x'.*
(F2) *If P is connected, then it has a unique maximal element z_0.*

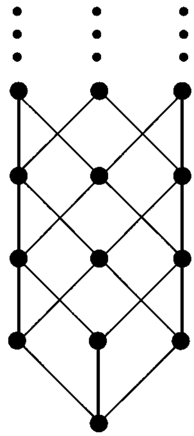


Figure 2.

(F3) *If P is connected, every chain from an element w to z_0 has the same length.*
If in addition P satisfies D2, *then*:
(F4) *If P is connected, each element of P other than z_0 is covered by* 1 *or* 2 *elements.*

Proof: For F1, let $x =: x_0,\ x_1,\ \ldots,\ x_k := x'$ be such that x_i covers x_{i-1} for $0 < i \leq k$. Let $y_0 := y$. For $1 \leq i \leq k$ apply D1 to $[x_{i-1};\ x_i,\ y_{i-1}]$ to see that a y_i exists which covers x_i and y_{i-1}. It is never the case that $y_i = x'$, since $y_i \geq y$ but $x' \not\geq y$. Let $y' := y_k$. For F2, note that P has at least one maximal element since it is finite. Suppose it has at least two maximal elements z_0 and z_1. Since P is connected, there will be a "up/down path" from z_0 to z_1 of shortest possible length. Let y be the earliest element such that $y \not\leq z_0$ and let x be the preceding element on the path: $x \leq z_0$. Applying F1 produces a contradiction of the maximality of z_0, and so z_0 must be unique. For F3, let w be maximal in P such that $w \to a_1 \to \cdots \to a_r = z_0$ and $w \to b_1 \to \cdots \to b_s = z_0$ with $r > s$. Clearly $s > 1$, and so $b_1 < z_0$. Use D1 to construct $c_2,\ c_3, \ldots$ along $a_1 \to \cdots \to a_r = z_0$ such that $a_1 \to c_2$ and $b_1 \to c_2$, then $a_2 \to c_3$ and $c_2 \to c_3$, etc. By the maximality of z_0, there must exist some $t \leq r$ such that $c_t = a_t$. Thus $b_1 \to c_2 \to \cdots \to c_t = a_t \to \cdots \to a_r = z_0$ and $b_1 \to b_2 \to \cdots \to b_s = z_0$ are chains of lengths $r - 1 > s - 1$, contradicting the maximality of w. For F4, note that if an element is covered by three or more other elements, then D1 and D2 force P to be infinite, as illustrated in figure 2. □

4. Slant sum decomposition

It is easy to see that a poset is d-complete if and only if each of the posets defined by the connected components of its order diagram is d-complete. So for our classification, we will consider only connected d-complete posets. The notion of slant sum introduced here will be used to break connected d-complete posets into smaller pieces, as illustrated in figure 3.

In this paper, a *rooted tree* is a poset which has a unique maximal element, and is such that each non-maximal element is covered by exactly one other element. It is easy to see that rooted tree posets are d-complete.

Let P be a connected poset with a unique maximal element z. A *top tree element* $x \in P$ is an element which is covered by at most one other element and is such that every $y \geq x$ is covered by at most one other element. The *top tree* T of P consists of all top tree elements. It is easy to see that T is a filter of P which is a rooted tree under the order inherited from P. Obviously the top tree of a rooted tree T is all of T.

Let P be a connected d-complete poset with top tree T. An element $y \in P$ is *acyclic* if $y \in T$ and it is not in the neck of any d_k-interval for any $k \geq 3$. An element of P is *cyclic* if it is not acyclic. If $y \in P$ is cyclic, then either it is in the neck of some d_k-interval or there exists some $z \geq y$ which is covered by two elements. If $y \in T$, then it is cyclic if and only if it is in the neck of some d_k-interval.

Let P_1 be a d-complete poset containing an acyclic element y. Let P_2 be a connected d-complete poset which is non-adjacent to P_1. By F2, let x denote the unique maximal element of P_2. Then the *slant sum* of P_1 with P_2 at y, denoted $P_1\,{}^y\!\backslash_x\, P_2$, is the poset formed by creating a covering relation $x \to y$. A d-complete poset P is *slant irreducible* if it is connected and it cannot be expressed as a slant sum of two non-empty d-complete posets. Suppose that P is a connected d-complete poset with top tree T. An edge $x \to y$ of P is a *slant edge* if $x, y \in T$ and y is acyclic.

All of the elements of a rooted tree P_1 are acyclic. A one element poset P_2 is d-complete. If $y \in P_1$, then the slant sum $P_1\,{}^y\!\backslash_x\, P_2$ will be a rooted tree. Any iterated slant sum formed with one element posets will be a rooted tree, and every rooted tree can be obtained in this fashion.

All of the edges of a rooted tree are slant edges. One element posets are slant irreducible. Removing all of the edges of a rooted tree produces a disjoint union of slant irreducible posets. What if we remove a slant edge from an arbitrary connected d-complete poset?

Proposition A *Let P be a connected d-complete poset and let $x \to y$ be a slant edge. Let $P_2 := (x)$ and let $P_1 := P - (x)$. Then P is a slant sum $P_1\,{}^y\!\backslash_x\, P_2$ of two non-adjacent connected d-complete posets P_1 and P_2. The top tree of T of P is the slant sum $T_1\,{}^y\!\backslash_x\, T_2$ of the top trees T_1 and T_2 of P_1 and P_2. The acyclic elements of P are acyclic in P_1 or P_2, and slant edges other than $x \to y$ are slant edges in P_1 or P_2.*

Proof: By definition, y is acyclic. Since y is the only element covering x in P, there cannot be any d_k-conditions passing through x and beyond. Suppose that some element $u \neq x$ in (x) is covered by some element v in $P - (x)$. Since x is covered only by y, an argument similar to that used to prove F1 could be used to show that y would be the top element of a diamond. So aside from $x \to y$, the subsets P_1 and P_2 are non-adjacent and any d_k-intervals contained in $(x) \subset P$ are completed within (x). Hence $P_2 := (x)$ is d-complete. Since P_2 is an ideal of P, the subset P_1 is a filter. Hence P_1 is d-complete. Clearly y will remain acyclic in P_1. Obviously x is the maximal element of (x). Removing one edge of a connected poset creates at most two components. So $P = P_1\, {}^{y}\!\backslash_{x}\, P_2$ as claimed. The other statements are easy. $\square$

Going in the other direction, we have:

Proposition B *Let P_1 be a connected d-complete poset with acyclic element y and let P_2 be a connected d-complete poset with maximal element x. Then the slant sum $P := P_1\, {}^{y}\!\backslash_{x}\, P_2$ is a connected d-complete poset. If T_1 and T_2 are the top trees of P_1 and P_2, then $T_1\, {}^{y}\!\backslash_{x}\, T_2$ is the top tree of P. The acyclic elements of P_1 and P_2 are acyclic in P, the slant edges of P_1 and P_2 are slant edges in P, and the edge $x \to y$ is a slant edge in P.*

Proof: The added edge is the only edge joining an element of P_1 to an element of P_2. Since y is not below any element which is covered by two elements and x is not covered by any other element, adding a downward edge at y cannot cause a violation of D1, D4, D3, or D6. Since y is not in the neck of any d_k-interval in P_1, adding this edge cannot cause a violation of D2 or D5. So $P_1\, {}^{y}\!\backslash_{x}\, P_2$ is d-complete. It is given that y is acyclic in P_1; in order for y to not be acyclic in P, the element x would have to be the maximal element of some d_k^--interval in P_2 for some $k \geq 4$. This is impossible since P_2 is d-complete. The other statements are easy. $\square$

These two results can be combined to immediately yield:

Proposition C *Let P be a connected d-complete poset. Then P is slant irreducible if and only if it contains no slant edges. Also, P is slant irreducible if and only if every acyclic element is a minimal element of its top tree.*

We now use this proposition to describe the structure of any connected d-complete poset P. First locate all of its slant edges. These may be erased in any order to produce a collection $P_1, P_2, \ldots$ of uniquely determined smaller non-adjacent connected d-complete posets. No new slant edges are created, and so each of $P_1, P_2, \ldots$ are slant irreducible. We say that $P_1, P_2, \ldots$ are the *slant irreducible components* of P.

Conversely, suppose that $P_1, P_2, \ldots$ are slant irreducible d-complete posets in which we have identified all acyclic elements. In general, many different possible larger connected d-complete posets can be formed from these posets by forming various slant sums. Acyclic elements remain acylic and may be used more than once, but maximal elements cease to be maximal after being used as the bottom of a slant edge. This procedure generalizes the process of producing any rooted tree by forming an iterated slant sum of one element posets. There exist slant irreducible d-complete posets with no acyclic elements; given only such posets we would not be able to form larger connected d-complete posets.

The remaining sections of this paper will be devoted to listing all possible slant irreducible d-complete posets. We will regard the one element poset as the trivial slant irreducible d-complete poset. By *irreducible component* we will mean a slant irreducible d-complete poset which has two or more elements. An irreducible component has a unique maximal

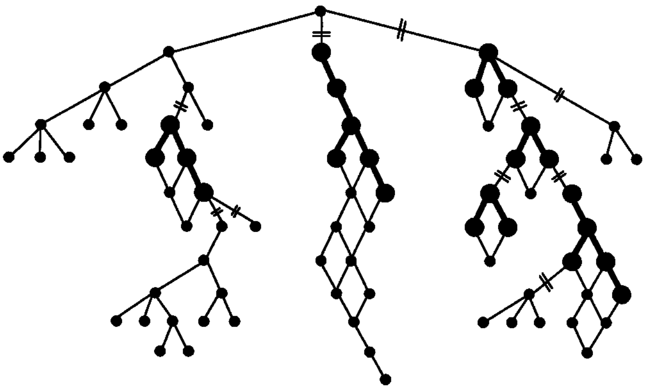


Figure 3.

element and is not a rooted tree. Hence its order diagram contains at least one cycle when viewed as a graph.

Theorem *Let P be a connected d-complete poset. It may be uniquely* (*up to the order of operations*) *decomposed into a slant sum of one element posets and irreducible components. The top tree of P is an analogous slant sum of the top trees of the irreducible components.*

In order to avoid producing a huge number of one element components during the decomposition, one might want to avoid erasing slant edges which occur within trees. It is possible to describe a smaller set of "cut" edges whose removal will produce a slant sum of maximal subtrees and (non-trivial) irreducible components: Suppose that P is a connected d-complete poset with top tree elements x and y. A slant edge $x \to y$ is a *down cut edge* if y is the side element of some diamond in P; it is an *up cut edge* if x is cyclic. A slant edge may be both a down cut edge and an up cut edge. In figure 3, both kinds of cut edges are denoted with double slash marks. Erasing all of these edges produces our preferred "grapevine" view of an arbitrary connected d-complete poset: Such a poset consists of many irreducible components (viewed as bunches of grapes: any irreducible component contains at least one cycle; view the minimal cycles as grapes) which are connected together with maximal tree portions (portions of the vine outside of the bunches). The only botanically incorrect aspect of this model is that our vine can recontinue from a "corner" of a bunch of grapes. The larger dots and the heavier edges indicate the top trees of the irreducible components of this d-complete poset.

5. The top tree of an irreducible component

From now on, let P denote some fixed irreducible component, namely, a slant irreducible d-complete poset with at least two elements. In this section we learn that the top tree T of P must be "Y-shaped".

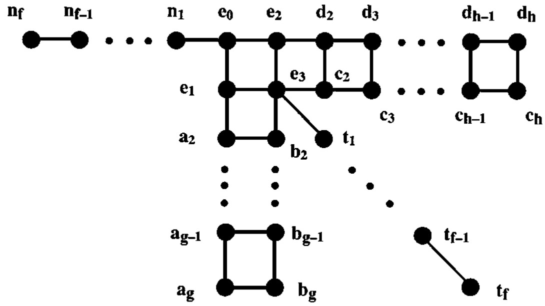


Figure 4.

Lemma *Let T be the top tree of an irreducible component P. Suppose $z, y \in T$ are such that z is the top of a diamond with side element y. Then y is a diamond top in P if and only if it is not a minimal element of T.*

Proof: If y is not minimal in T, it is cyclic in P by Proposition 4.C. Since the bottom of the diamond whose top is z cannot be in T, we know that y covers at least two elements. So it cannot be cyclic by being a strict neck element of a d_k-interval for some $k \geq 4$. Hence y must be a diamond top. Suppose that y is a diamond top in P. If each of the elements covered by y is covered by some element different from y, there would be two distinct elements covering y. But $y \in T$. So one element covered by y is covered by only y, and that element must be in T, contradicting the minimality of y in T. □

We now need to define a particular kind of rooted tree. Let $f \geq 0$ and $h \geq g \geq 0$ be integers. The rooted tree $Y(f; g, h)$ consists of one "branch" element above which a chain of f elements has been adjoined and below which two non-adjacent chains with g and h elements, respectively, have been adjoined toward the left and toward the right, respectively.

From now on, all order diagrams will be rotated 45° counterclockwise before being drawn, so that the "up" direction is "Northwest". With this convention, the order diagram for $Y(f; g, h)$ appears as the $f + g + h + 1$ topmost and leftmost elements in figure 4.

Theorem *Let P be an irreducible component. Then its top tree T is of the form $Y(f; g, h)$ for some $f \geq 0$ and $h \geq g \geq 1$.*

Proof: Consider a non-minimal element z of T. By Proposition 4.C, it is cyclic. Since $z \in T$, it must be in the neck of some d_k-interval for some $k \geq 4$. But elements in the strict neck of a d_k-interval for $k \geq 4$ cover exactly one other element of P. Therefore z can cover two or more other elements of T only if it is the top of some diamond in P. The only way z can cover two or more other elements of T and be a diamond top in P is for the side elements x and y of the diamond to be the other elements of T covered by z. Of course z cannot cover any other elements of P, or of T. So branches in T can only be two-fold branches such that there exists a fourth element of P which is covered by each of the branching elements.

Since P is not a tree, the set $P - T$ is non-empty. Let w be a maximal element of $P - T$. Note that w must be covered by two elements of T, call them x and y. Then there exists an element z of T which covers both x and y. So T has at least one branching.

Now we show that there can only be one such branching. Let z be a minimal such branch node, which covers x and y. Let v be minimal in T such that $v > z$ and such that v is another such branch node. Let t and u be covered by v, and let $s \notin T$ be covered by t and u. One of t and u must be above z in T; suppose that it is u. By the lemma, u is a diamond top. One of the side elements of this diamond must be in T; call it r. Let q be the bottom of this diamond. Application of the lemma can be iterated until an element analogous to r is actually z. For simplicity of notation, depict this with $r = z$. Then not only would z cover x and y as a diamond top, it would also cover q. This is impossible by D2. Therefore, we must have q coinciding with either x or y. But $q \notin T$ and both $x, y \in T$. Therefore, there is exactly one branch in T, and it is a two-fold branch. Draw P so that the longer lower branch of T is to the right. □

6. The top filter of an irreducible component

We continue to consider the fixed irreducible component P. Let f, g, and h be such that the top tree T of P is $Y(f; g, h)$. Let e_0 be the branch node in T. Let e_1 and e_2 be the elements of T covered by e_0, with e_1 being on the left branch (the one of length g). Let $e_3 \notin T$ be the unique element of P covered by e_1 and e_2 whose existence was noted in the proof of the theorem. Successively label the elements of T above e_0 by $n_1, \ldots, n_f$. Label the remaining elements on the two lower branches by $a_2, \ldots, a_g$ and $d_2, \ldots, d_h$. All of these elements named so far are shown in figure 4, as are some other elements whose existence will soon be proved. Given this notation, we can state some additional facts which are apparent from the proof of Theorem 5:

Proposition A *Let P be an irreducible component with top tree $T = Y(f; g, h)$. For $2 \leq i \leq f$, the element n_i covers only n_{i-1}. For $1 \leq i \leq f - 1$, the element n_i is covered only by n_{i+1}. The element e_0 is covered only by n_1 and is a diamond top of a diamond with side elements e_1 and e_2.*

Acyclic elements of irreducible components can only possibly occur at two specific locations:

Proposition B *Let P be an irreducible component with top tree $T = Y(f; g, h)$. The branch elements $a_2, \ldots, a_{g-1}$ and $d_2, \ldots, d_{h-1}$ are diamond tops. The minimal elements a_g and d_h of T cannot be diamond tops, and these are the only two elements of P which could be acyclic elements. Each of these elements is acyclic if and only if it is not in the strict neck of some d_k-interval for some $k \geq 4$.*

Proof: Continuing the proof of Theorem 5, repeated applications of Lemma 5 can be used to show that all but the last element of each lower branch are diamond tops. And

Lemma 5 implies that these last elements are not diamond tops, since they are minimal in T. These minimal elements of T are the only possible acyclic elements of P. Since they cannot be diamond tops, each fails to be cyclic precisely when it lies in the strict neck of some d_k-interval for some $k \geq 4$. □

Before proceeding, we need to state a local structural fact for d-complete posets.

Lemma *Let P be a d-complete poset, with $z, x, y \in P$. Suppose that z covers x and y, each of which is a diamond bottom. Then z can cover at most one other element q. This element q can be covered by no other elements, and any element covered by q can be covered only by q.*

Proof: Let r and s denote the diamond tops covering z which correspond to the diamond bottoms x and y. Let u denote the diamond top covering r and s. If z covered both q_1 and q_2, then $[q_1, u]$ and $[q_2, u]$ would violate D6 with $k = 4$. Suppose q is covered by z. If q is also covered by an element other than z so that D3 is not violated, then a third diamond top t would cover z, violating F4. Let w be covered by q. If w is covered by an element v other than q, there must be a diamond top corresponding to the diamond bottom w. It cannot be z by D2 since z covers x and y. (This would be the case if $v = x$ or $v = y$.) Hence, an element other than z covers q, violating an earlier conclusion. □

We return to the consideration of the fixed irreducible component P with top tree $T = Y(f; g, h)$. Let $b_2, \ldots, b_g$ and $c_2, \ldots, c_h$ denote the bottoms of the diamonds whose existence arose during the proof of Proposition B. If $f \geq 0$ and $h \geq g \geq 1$, define the poset $U(f; g, h)$ by the order diagram of figure 4. We now show that the irreducible component P must contain a filter of this form; we call it the *top filter* of P.

Proposition C *If the irreducible component P has top tree $T = Y(f; g, h)$, then it contains a filter of the form $U(f; g, h)$, which itself contains T. The only elements of $U(f; g, h)$ which can cover elements outside of $U(f; g, h)$ are $b_2, \ldots, b_g, c_2, \ldots, c_h$, $t_1, \ldots, t_f$.*

Proof: The diamond top e_0 cannot be a strict neck element for any d_k-interval. Therefore, in order for the edge $e_0 \to n_1$ to not be a slant edge, the element n_1 must be the top of some d_4-interval with diamond bottom e_3. Hence e_3 covers some element t_1. Apply the lemma with $z = e_3$. So t_1 is covered only by e_3, and e_3 covers no elements aside from b_2, c_2, and t_1. Repeating the slant edge reasoning for n_i with $i \geq 2$ leads to the existence of $t_2, \ldots, t_f$ as shown. Upward diamond propagation implies that t_i can be covered only by t_{i-1}. By F4, none of $b_2, \ldots, b_g$ or e_3 or $c_2, \ldots, c_h$ can be covered by other elements. The diamond tops $a_2, \ldots, a_{g-1}$ and $d_2, \ldots, d_{h-1}$ cannot cover elements other than the two elements shown. If a_g covers another x aside from b_g, then x could be covered only by a_g, or else upward propagation of diamonds would change T. But then $x \in T$, contradicting our assumed T. □

7. The list of possible irreducible components

Recall that an irreducible component is a slant irreducible d-complete poset which contains two or more elements. In Sections 9–13, we will define 15 disjoint classes of irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_{15}$ which will be seen to exhaust the set of all irreducible components. Here we present the resulting list, which is indexed by Table 1.

For each triple of values $f \geq 0$, $g \geq 1$, and $h \geq g$ allowed by the Nth line of Table 1, define the *maximal poset* $M_N = M_N(f; g, h)$ to be the poset defined by the order diagram of figure 5.N. (One is to take f, g, and h large solid dots, respectively, to the left, below, and right of the junction large solid dot.) Also define the *minimal poset* $L_N = L_N(f; g, h)$ to be the filter of M_N consisting of all large solid dots, circled dots, solid dots, solid squares, and boxed squares. (In other words, all elements of M_N other than hollow dots and hollow triangles.) We will prove that the Nth class of posets $\mathcal{C}_N$ consists of all posets which are filters of $M_N(f; g, h)$ containing $L_N(f; g, h)$, as f, g, and h run over all values allowed by the table. These posets will all be distinct, and so each irreducible component will occur exactly once in our listing.

Table 1.

Class	Colloquially	f	g	h	Name	$\lambda \in$	$\mu \in$	Acyclics
1	Shapes	$=0$	≥ 1	$\geq g$	$a_n[0; g, h; \lambda]$	$\Lambda(g-1, h-1)$	—	L, R
2	Shifted shapes	$=1$	$=1$	≥ 1	$d_n[1; 1, h; \mu]$	—	$\Sigma(h, h)^1$	L^2, R
3	Birds	≥ 1	≥ 2	$\geq g$	$y_n[f; g, h]^3$	—	—	L, R
4	Insets	≥ 2	$=1$	≥ 1	$e_n[f; 1, h; 4; \lambda]^4$	$\Lambda(f, h-1)$	—	L,R
5	Tailed insets	≥ 2	$=1$	≥ 2	$e_n[f; 1, h; 5; \lambda, \mu]$	$\Lambda(f-1, 1)$	$\Lambda(1, h-2)$	R
6	Banners	≥ 2	$=1$	≥ 3	$e_n[f; 1, h; 6; \lambda]$	$\Lambda(2, h-3)$	—	R
7	Nooks	≥ 2	$=1$	≥ 3	$e_n[f; 1, h; 7; \lambda]$	$\Lambda(f-2, 2)$	—	R
8	Swivels	≥ 2	$=1$	$=2$	$e_n[f; 1, 2; 8; \lambda]$	$\Lambda(f-1, 4)^5$	—	R^2
9	Tailed swivels	≥ 3	$=1$	$=2$	$e_n[f; 1, 2; 9; \lambda, \mu]$	$\Lambda(f-3, 3)$	$\Lambda(2, 1)$	R^2
10	Tagged swivels	≥ 4	$=1$	$=2$	$e_n[f; 1, 2; 10; \lambda]$	$\Lambda(f-4, 4)$	—	—
11	Swivel shifteds	≥ 4	$=1$	$=2$	$e_n[f; 1, 2; 11; \mu]$	—	$\Sigma(f-3, f+1)'^6$	R
12	Pumps	≥ 3	$=1$	$=2$	$e_n[f; 1, 2; 12; \lambda]$	$\Lambda(f-3, 2)$	—	—
13	Tailed pumps	≥ 3	$=1$	$=2$	$e_n[f; 1, 2; 13; \lambda]$	$\Lambda(f-3, 1)$	—	—
14	Near bats	≥ 3	$=1$	$=2$	$e_n[f; 1, 2; 14]$	—	—	—
15	Bat	$=3$	$=1$	$=2$	$e_7[3; 1, 2; 15]$	—	—	—

[1] In Class 2, the shifted shape μ must contain the element depicted with the boxed square.
[2] In Classes 2, 8, and 9, these elements are acyclic only when the elements depicted with the hollow triangles do not exist.
[3] In Class 3, the name $e_n[1; g, h]$ is used if $f = 1$.
[4] In Class 4, the name $d_n[f; 1, 1]$ is used if $h = 1$.
[5] In Class 8, the shape λ must contain the two elements depicted with solid squares.
[6] In Class 11, the shape μ must contain the four elements depicted with solid squares.

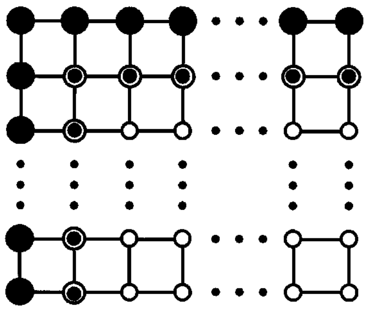


Figure 5.1.

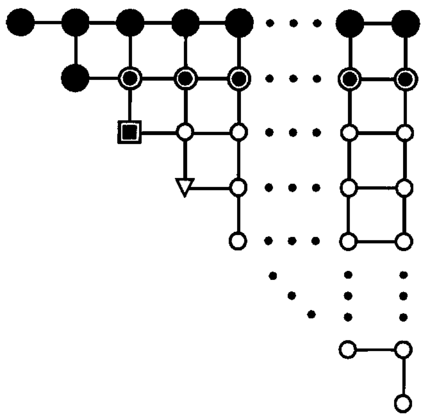


Figure 5.2.

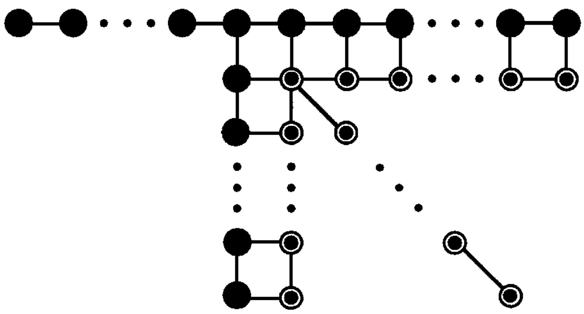


Figure 5.3.

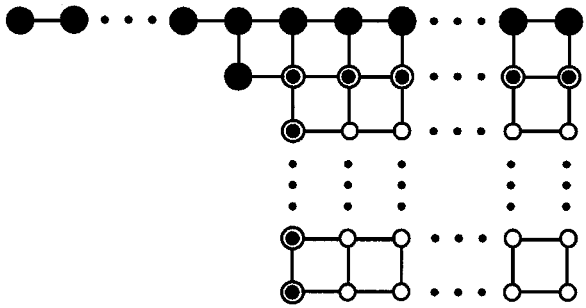
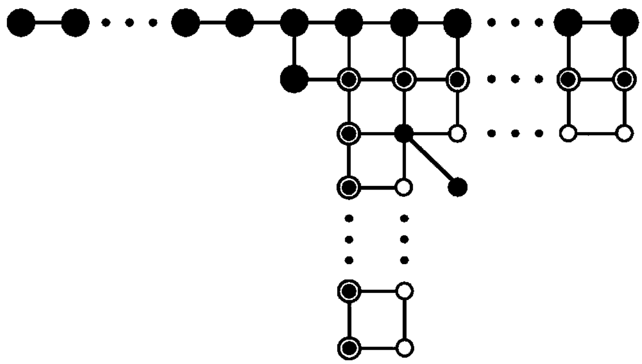


Figure 5.4.



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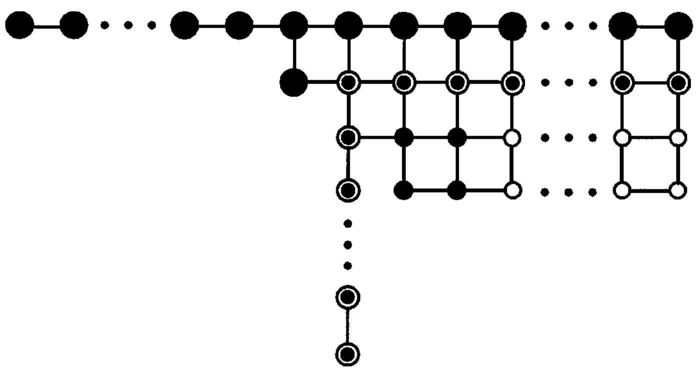

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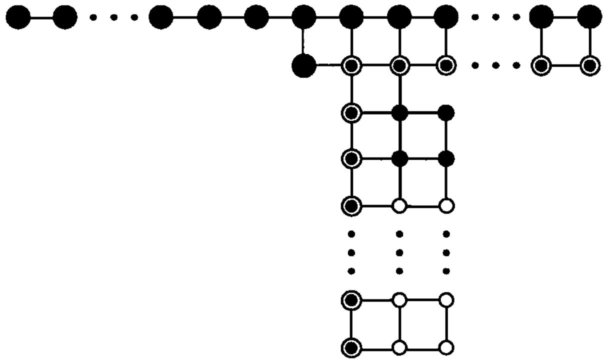

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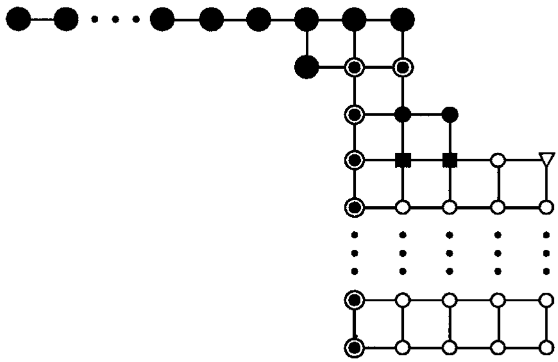

Figure 5.8.

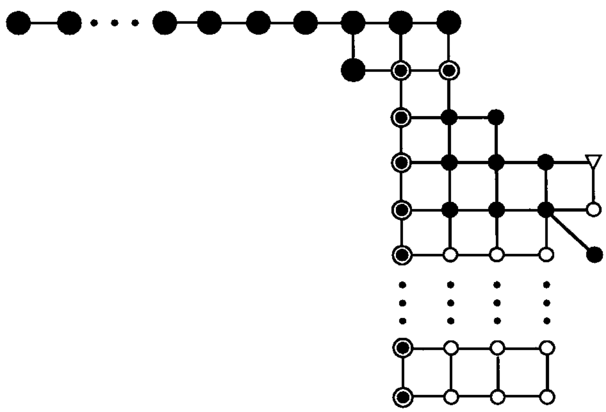

Figure 5.9.

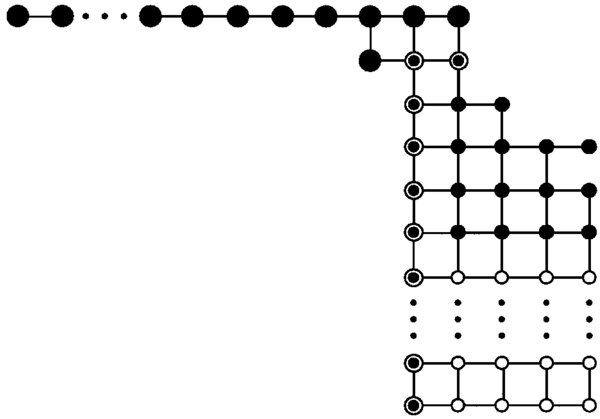

Figure 5.10.

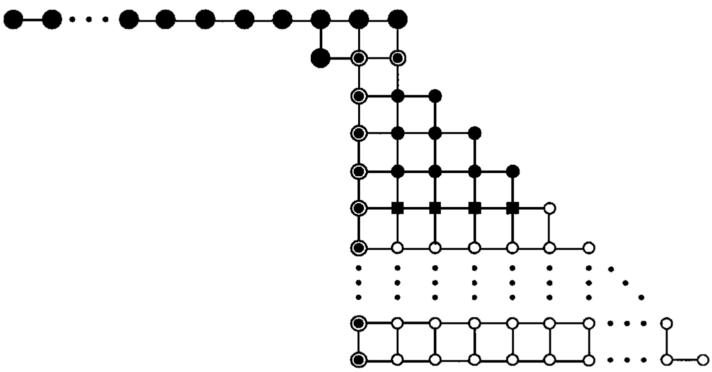

Figure 5.11.

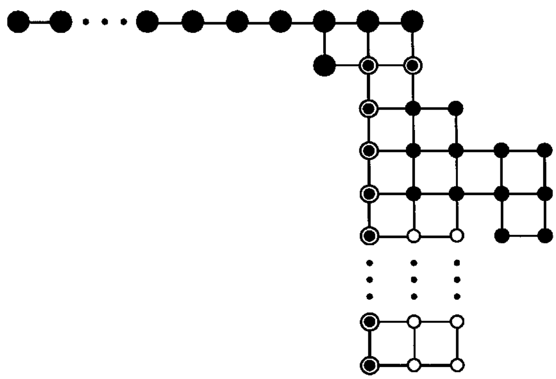

Figure 5.12.

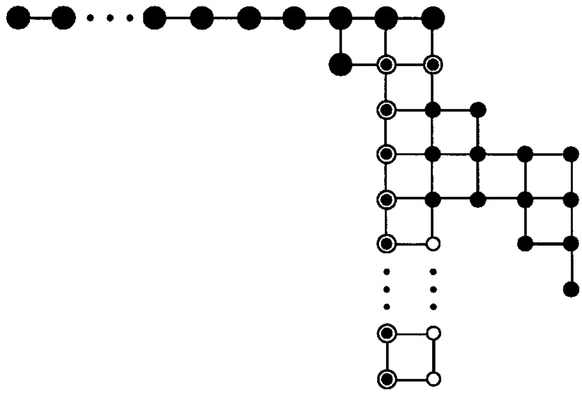

Figure 5.13.

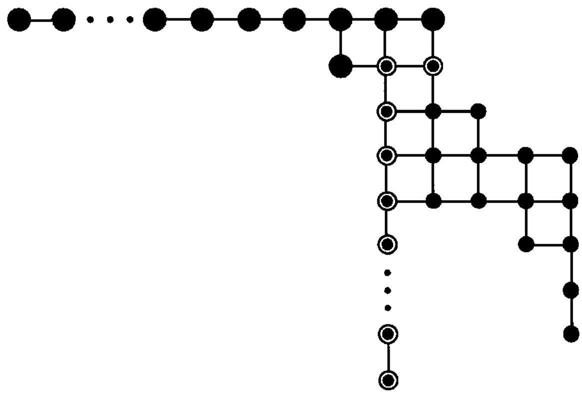

Figure 5.14.

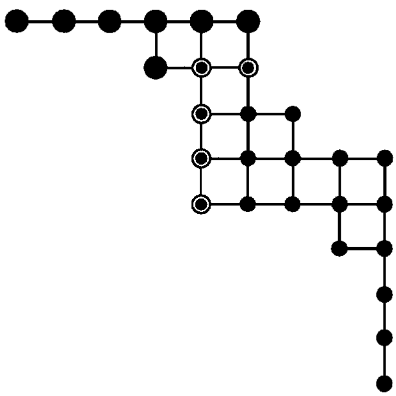

Figure 5.15.

Define the *top tree poset* $T_N = T_N(f; g, h)$ to be the filter of $M_N(f; g, h)$ consisting of all large solid dots. Also define the *top filter poset* $U_N = U_N(f; g, h)$ to be the filter of $M_N(f; g, h)$ consisting of all large solid dots, circled dots, and boxed squares. For each N, it is obvious that $T_N(f; g, h)$ is the top tree of $M_N(f; g, h)$ and that $T_N(f; g, h) = Y(f; g, h)$. It is also obvious that $U_N(f; g, h)$ is the top filter of $M_N(f; g, h)$, i.e., that $U_N(f; g, h) = U(f; g, h)$, as defined in Section 6. The elements of L_N beyond U_N consist of the solid dots and the solid squares. The *optional poset* $O_N = O_N(f; g, h)$ is the ideal of M_N consisting of all hollow dots and hollow triangles, i.e., the complement of L_N in M_N. The optional posets λ and μ listed in Table 1 are usually filters of O_N. For $\mathcal{C}_2$, $\mathcal{C}_8$, and $\mathcal{C}_{11}$, the posets μ and λ are required to contain from one to four elements of L_N as well for notational convenience. Such elements are depicted with boxed or solid squares.

In 14 out of the 15 classes, the top tree $Y(f; g, h)$ of the irreducible components is such that $\min[f, g, h] \leq 1$. When viewed as a Dynkin diagram for a simply laced Kac-Moody algebra, most or all such trees are of type A, D, or E, depending upon how “type E” is defined. The rank of the algebra is $n := f + g + h + 1$. So we will write $X_n(f; g, h)$ instead of $Y(f; g, h)$ when $\min[f, g, h] \leq 1$, where $X \in \{A, D, E\}$. In particular, take $X = A$ when $f = 0$ and take $X = D$ when two of $\{f, g, h\}$ are both equal to 1. Historically it has perhaps been required that $\{f, g, h\} \supseteq \{1, 2\}$ and $\min[\{f, g, h\} - \{1, 2\}] \geq 2$ in order to take $X = E$. But we will more generally require only $1 \in \{f, g, h\}$ and $\min[\{f, g, h\} - \{1\}] \geq 2$ in order to regard $Y(f; g, h)$ as a *Dynkin diagram of general type E*. For the fifteenth class $\mathcal{C}_3$, the only restrictions are $f \geq 1$ and $h \geq g \geq 2$. There we use the letter Y rather than A, D, or E when $f \geq 2$. So each of the top trees appearing in figure 5 can be denoted $X_n(f; g, h)$, where $X \in \{A, D, E, Y\}$ and $n \geq 3$.

Building upon this Dynkin diagram notation, we now introduce a name for each possible irreducible component P. If P has top tree $X_n[f; g, h]$, then it is assigned a name roughly of the form $x_n[f; g, h; N; \lambda, \mu]$. (The lower case “$x$” instead of the upper case “X” continues the convention of [8, 9] of using the lower case letter for the poset of join irreducibles P and the upper case letter for the distributive lattice $J(P)$. This lattice $J(P)$ is a Bruhat order, and in future papers we will denote the Bruhat order corresponding to the irreducible component $x_n[f; g, h; N; \lambda, \mu] =: P$ by $X_n[f; g, h; N; \lambda, \mu] = J(P)$.) If it is not determined by x, f, g, and h, then the number N of the class of which P is a member is displayed. Finally, the parameters λ and μ denote filters of O_N (plus possibly a few elements of L_N) which determine the particular irreducible component P.

Table 1 specifies which of the two minimal elements of T_N are acyclic: Here the presence of “L” means that the element a_g of figure 4 is acyclic, and “R” indicates that the element d_h of that figure is acyclic. For $\mathcal{C}_2$ (respectively, $\mathcal{C}_8$ and $\mathcal{C}_9$), the element a_g (respectively, d_h) is not acyclic when the element of O_N depicted with the hollow triangle is present.

Combining the knowledge of the irreducible components and their acyclic elements with the procedure given at the end of Section 4 enables one to generate all connected d-complete posets with a given top tree.

Theorem *If P is an irreducible component, then it is described in exactly one of the lemmas below, for some* $1 \leq N \leq 15$.

The proof of the theorem is provided by the narrative of Sections 9–13: As we proceed, we explain how the wordings of the definitions of the classes together with various local structural conditions imply that no irreducible components are being missed. The conditions on f, g, and h which appear in Table 1 are incorporated into the definitions of the classes $\mathcal{C}_N$ as we proceed.

Lemma 7 N, $1 \leq N \leq 15$. *Let f, g, and h be fixed parameter values allowed by the Nth row of Table* 1. *Then the poset $M_N(f; g, h)$ is an irreducible component. There exist no extensions of $M_N(f; g, h)$ to a larger irreducible component. A poset P is in the class of irreducible components $\mathcal{C}_N$ and has top tree $Y(f; g, h)$ if and only if P contains $L_N(f; g, h)$ and is a filter of $M_N(f; g, h)$. All such filters of $M_N(f; g, h)$ are distinct. Specification of posets λ and/or μ from the possibilities listed in Table* 1 *corresponds to choosing one member of $\mathcal{C}_N$. The acyclic elements for each $P \in \mathcal{C}_n$ are listed in Table* 1.

The least routine aspects of each of these 15 lemmas will be confirmed in Sections 9–13 following the definition of the corresponding class of irreducible components. Our extension arguments will imply that each M_N is a slant irreducible d-complete poset. For each class we will leave several routine verifications to the reader. The non-existence of slant irreducible extensions of M_N will follow from Lemma 8.A. If P is a filter of $M_N(f; g, h)$, then it is d-complete. If it contains $L_N(f; g, h)$ as well, then it can be seen that it is slant irreducible and has top tree $Y(f; g, h)$. Verifying that any filter P of M_N which contains L_N satisfies the other particular defining conditions of $\mathcal{C}_N$ for each N will be left to the reader.

The key step is the converse: Suppose that P is in the class $\mathcal{C}_N$ of irreducible components and has top tree $Y(f; g, h)$. We will argue that P must then be a filter of M_N. (It will be obvious that if $P \in \mathcal{C}_N$, then $P \supseteq L_N$.) The precise specification of members of $\mathcal{C}_N$ with λ and/or μ will be performed as we proceed. A consequence of this specification will be the fact that the members of each class for fixed values of f, g, and h are distinct. The convention $g \leq h$ guarantees that members of the same class for different parameter values will never coincide. The reader may confirm the list of acyclic elements for each case using Proposition 6.B.

8. Extending *d*-complete posets

Let P_0 be a fixed irreducible component. It will begin with a top tree $Y(f; g, h)$ and a top filter $U(f; g, h)$ for some values of $f \geq 0$, $g \geq 1$, and $h \geq g$. If we fix these values for f, g, and h and exhaustively list all possible irreducible components P which begin with $U(f; g, h)$, our fixed P_0 will eventually appear in the list of possible P's which we generate. In this section we establish the mechanics which will be used for this process in Sections 9–13.

Let F be a d-complete poset. A d-complete poset P is an *extension* of F if F is a filter of P. An element $x \notin F$ is a 1-*extension* with respect to F if $F \cup \{x\}$ is d-complete. A *dangle extension* x of F is a 1-extension of F such that w is covered by one element in F. A *wedge extension* of F is a 1-extension of F such that w is covered by two elements in F. Since elements of d-complete posets are never covered by three or more elements, every 1-extension of F is either a dangle extension or a wedge extension. The following lemma follows immediately from the definition of d-complete poset.

Lemma A *Let F be a d-complete poset with top tree T.*

1. *Suppose that $x \notin F$ is covered by only $y \in F$, and $y \notin T$. Then x is a dangle extension for F if and only if it is the bottom element of some d_k-interval $[x, z]$ for some $z \in F$ and some $k \geq 4$ such that z covers no element in F other than one element of $[x, z]$, the element y is not in the neck of a d_k-interval, and any element in F covered by y must be covered by at least one other element besides y.*
2. *Suppose that $w \notin F$ is covered by $x, y \in F$. Then w is a wedge extension for F if and only if there exists some $z \in F$ which covers only x and y, there is no element in F which is covered by both x and y, and neither x nor y is in the neck of a d_k-interval for any $k \geq 3$.*

We will start with $F_0 = U(f; g, h)$ and repeatedly do the following procedure: For each $i \geq 1$ list all possible ways of extending each F_{i-1} found at the previous stage by one element meeting the requirements of the lemma. In order to focus on one set of values for f, g, and h at a time, we will not consider 1-extensions of F_{i-1} which extend its top tree. (These would be dangles beneath a_g or d_h.) So each F_i produced will be an irreducible component with top filter $U(f; g, h)$.

Let P_0 be a fixed irreducible component with top filter $U(f; g, h)$. Set $F_0 := U(f; g, h)$ and $F_t := P_0$, where $t := |P_0 - U(f; g, h)|$. Let $F_0 \subset F_1 \subset \cdots \subset F_t$ be any sequence of filters starting with $U(f; g, h)$ and increasing to P_0 one element at a time. Since all of these posets are connected, by F4 we see that each $x_i := F_i - F_{i-1}$ for $1 \leq i \leq t$ is either a dangle or a wedge extension with respect to F_{i-1}. So repeated adjunctions of dangle and wedge extensions beginning with $U(f; g, h)$ are guaranteed to eventually produce P_0.

Suppose that a dangle (respectively, wedge) extension is adjoined at some stage. Then in that extension and in all later extensions, that element will be covered by exactly one (respectively, two) elements.

One quickly tires of considering all possible orders in which 1-extensions may be adjoined. Let x and y be two 1-extensions with respect to a d-complete poset F. Although it will be relatively rare, it is possible for y to fail to be a 1-extension with respect to $F \cup \{x\}$. And then x will fail to be a 1-extension with respect to $F \cup \{y\}$. It can be seen that this situation will arise only when there is some element $z \in F$ which covers a 1-extension $x \notin F$ and z is such that $[y, z]$ is a d_k-interval in $F \cup \{y\}$ for some $y \notin F$ and some $k \geq 3$. If this cautionary note is ignored, then any violation of a d-complete requirement created by adjoining both elements will remain a violation if further elements are adjoined. Hence, any such error would eventually be detected by noting that one of our "maximal irreducible components" M_N defined in Section 7 is not d-complete. So we can be casual concerning the order in which 1-extensions are adjoined. In addition, If $x \notin F$ is not a 1-extension of F, then it will never be a 1-extension with respect to any extensions of F. So once we rule out a conceivable 1-extension, we do not need to keep checking on it as a possibility.

Let F be a d-complete poset. The following local situation depicted in figure 6 will often arise in Sections 9–13 when we are seeking all possible extensions of F: Suppose that for some $s \geq 1$ and some $t \geq 1$ we have elements $a_s \to \cdots \to a_1 \to z$ and $b_t \to \cdots \to b_1 \to z$ such that a_i covers only a_{i+1} for $1 \leq i < s$ and such that b_j covers only b_{j+1} for $1 \leq j < t$. Also suppose that z covers only a_1 and b_1 and that a_s and b_t are

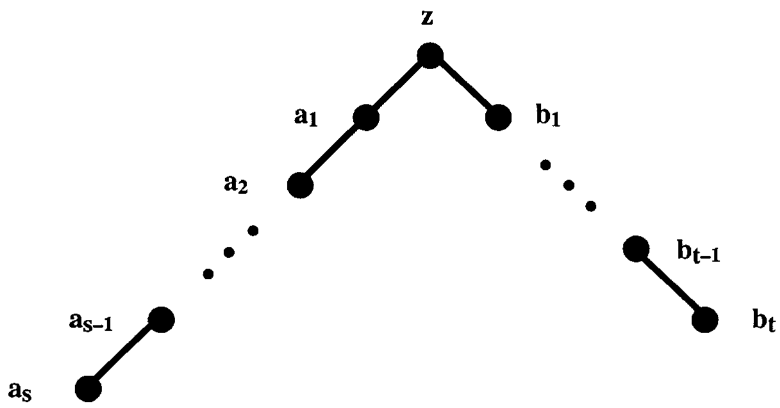

Figure 6.

minimal in F. Finally, assume that no a_i, $1 \leq i \leq s$, and no b_j, $1 \leq j \leq t$, is a neck element in any d_k-interval, for any $k \geq 3$.

Let F_M be the poset formed by successively adjoining st wedges in a grid pattern in the region between the chains $a_s \to \cdots \to a_1$ and $b_t \to \cdots \to b_1$ below F. It is clear that F_M will be d-complete, as will any filter P of F_M which contains F. (Such posets P are obtained by successively adjoining some shape $\lambda \in \Lambda(s, t)$ of wedges below F.)

What about other 1-extensions to F or a successor poset which are below z? First, some terminology. A dangle extension of F beneath an a_i, a b_j, or z will be called a *first generation dangle*. Any wedge extension of F or of a later F' which is beneath an a_i and/or a b_j and which has both parents below z will be called a *first generation wedge*. Later wedges both of whose parents are beneath z will be called *higher generation wedges*. Wedges which have one parent which is below z and one parent which is not below z will be called *outer wedges*. Any dangle extension of a later F' which is beneath a first generation wedge will be called a *second generation dangle*. In any poset P, an element x is an *only child* if it is covered by some element of P which covers only x.

Here is the procedure we will follow when listing all possible extensions of F below z. First note all first generation dangles. If one of these is used, then exit this procedure and consider that situation separately. Similarly, note all outer wedges and consider implementing them at another time. By D1 and D3 it can be seen that the only possible first or higher generation wedges must be the obvious grid filling wedges noted above. After choosing some first generation wedges, look for possible second generation dangles using the following result:

Lemma B *Let u be a first generation wedge extension. It can have one second generation dangle q beneath it if and only if the diamond top v corresponding to the diamond bottom u is an only child with respect to an element other than an a_i or a b_j. There are no dangle extensions beneath higher generation wedges.*

Proof: Let y be such that y covers only v. Then $[q, y]$ will be a d_4-interval, and adjoining one dangle q beneath u creates no d-complete violations. If no such y exists, then $[q, v]$ will be an incomplete d_4^--interval. Let u be a higher generation wedge extension and let v be the corresponding diamond top. Then v is an earlier wedge extension which we adjoined, and none of these which became diamond tops by some later obvious wedge extensions are only children. So as above, the element u can have no dangles beneath it. □

Outer wedge possibilities will rarely arise. By D1, the two parents must already be covered by a mutual element. This situation will arise only when the parent below z is one of the a_i or b_j and the grandparent is not below z.

Whenever the situation of figure 6 arises in Sections 9–13, we will assume that the reader will assist us in performing the procedure described above: Note all first generation dangles and outer wedges for future consideration. Add some first generation wedges. Using Lemma B, note all second generation dangles for future consideration. Add some higher generation wedges to complete an adjoined shape $\lambda \in \Lambda(s, t)$ of wedges between the two chains. Lemma B excuses us from checking for later dangles. Altogether, this routine will be referred to as the "Grid Filling Procedure", or "GFP".

From Sections 5 and 6, recall that any irreducible component P has a top tree $Y(f; g, h)$ for some $f \geq 0$ and $h \geq g \geq 0$ and contains the top filter $U(f; g, h)$. The classes $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are defined to consist of all irreducible components having certain values for f, g, and h. To describe these classes we need to generate all extensions of $U(f; g, h)$. If $N \geq 4$, the class $\mathcal{C}_N$ is defined to consist of all irreducible components which contain a specified poset V_* as a filter, which have the same top tree as V_*, and which satisfy certain other conditions. Then we need to generate all extensions of V_* which satisfy the specified conditions. By Proposition 6.C, at the beginning of the extension process we only need to consider extensions beneath the elements $b_2, \ldots, b_g$, $c_2, \ldots, c_h$, and $t_1, \ldots, t_f$.

9. Classes 1–3: Shapes, shifted shapes, and birds

In this section we define the three simplest classes of irreducible components and confirm the corresponding Lemma 7.N's. If the top tree of an irreducible component P is T, let f, g, and h be such that $T = Y(f; g, h)$. Recall that $f \geq 0$, $g \geq 1$, and $h \geq g$ in general. To be thorough, in the definition of each class we will restate all known constraints on f, g, and h.

Either $f = 0$ or $f > 0$. Let $\mathcal{C}_1$ consist of all irreducible components for which $f = 0$, $g \geq 1$, $h \geq g$.

To prove Lemma 7.1, note that the top filter $U(0; g, h)$ for any $P \in \mathcal{C}_1$ is as shown in figure 7. By Proposition 6.C, all extensions will be beneath e_3. Following GFP, it can be seen that the only possible extensions P of $U(0; g, h)$ are the filters of M_1 which contain $L_1 = U(0; g, h)$. Each such filter is determined by its intersection with O_1; such intersections are elements λ of $\Lambda(g-1, h-1)$. (Note that each poset $a_n[0; g, h; \lambda] \in \mathcal{C}_1$ is a shape whose first two columns are of length $g+1$ and whose first two rows are of length $h+1$.)

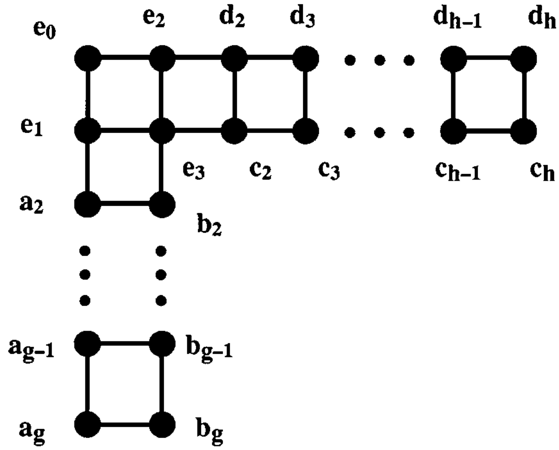

Figure 7.

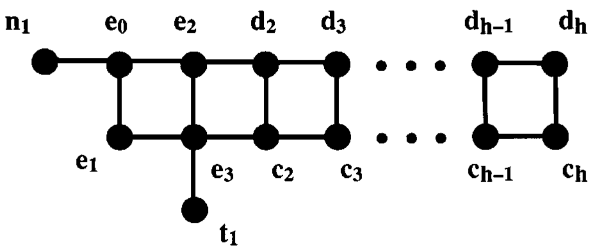

Figure 8.

Now assume that $f = 1$. Either $g = 1$ or $g \geq 2$. Let $\mathcal{C}_2$ consist of all irreducible components for which $f = 1$, $g = 1$, $h \geq 1$.

To prove Lemma 7.2, note that the top filter $U(1; 1, h)$ for any $P \in \mathcal{C}_2$ is as shown in figure 8. By Proposition 6.C, all extensions will be beneath e_3. Following GFP, the only near term wedge extensions are first generation wedge extensions; adjoin up to $h - 1$ of these, starting from the left. Note that e_3 is an only child of e_1. If at least one first generation wedge was chosen, then there is a second generation dangle beneath the first one. If we do not use this dangle, we are done. If we use it, then the situation is now equivalent to the one with which we started, provided that $h \geq 3$. Repeat this reasoning. Eventually, there will not be another element such that a wedge can be adjoined beneath it and the most recent dangle. Hence, we see that the only possible extensions P of $U(1; 1, h)$ are the filters of M_2 which contain L_2. Each such filter is determined by its intersection with O_2; such intersections are elements μ of $\Sigma(h, h)$, once the element t_1 has been adjoined. (Note that each poset $d_n[1; 1, h; \mu] \in \mathcal{C}_2$ is a shifted shape with at least three rows and whose first two rows have lengths $h + 2$ and $h + 1$.)

Next suppose that $g \geq 2$. We still have $f \geq 1$ and $h \geq g$. The next class will handle the remaining $f = 1$ cases and the simplest cases for $f > 1$. Let $\mathcal{C}_3$ consist of all irreducible components for which $f \geq 1,\ g \geq 2,\ h \geq g$.

To prove Lemma 7.3, note that the top filter $U(f; g, h)$ for any $P \in \mathcal{C}_3$ is depicted in figure 4. None of $b_g, \ldots, b_2, c_h, \ldots, c_2$ have dangles beneath them. Since $f \geq 1$, the element t_1 exists. None of $t_f, \ldots, t_1$ have dangles or wedge extensions beneath them. A wedge extension beneath any two of $b_2,\ c_2$, and t_1 is ruled out by D2, and so no wedge extensions exist. So there are no extensions of $U(f; g, h)$, and every P in $\mathcal{C}_3$ is of the form $M_3 = L_3$. (Lemma 7.3 can be paraphrased as: If all three parameters $f,\ g$, and h are "non-minimal", then any irreducible component P must be "minimal", i.e., it consists only of $L_3 = U(f; g, h)$.)

10. Classes 4–6: Inset-type classes

We now define and describe the simplest three classes of irreducible components P amongst the remaining cases of $f \geq 2$ and $g = 1$. For fixed $f \geq 2$ and $h \geq 1$, let $V_4 := U(f; 1, h)$. The only 1-extension with respect to V_4 is the wedge p beneath t_1 and c_2, as shown in figure 9. Once p is adjoined, we next consider several possible ways to adjoin a few more elements one or two steps below p, see figures 10 and 11. The most notable aspect of such extensions is whether there exists some element q which is covered only by p.

Lemma *Let $f \geq 2,\ g = 1$, and $h \geq 2$. Let $V := V_4 \cup \{p\}$. There exist at most three* 1*-extensions of V: a wedge r beneath t_2 and p, a dangle q beneath p, and, if $h \geq 3$, a wedge s beneath p and c_3. It is possible to extend V by any of the* 2^3 *combinations of these three. Next consider extensions of $V_5 := V_4 \cup \{p, q\}$. There is never a dangle beneath q. If s is*

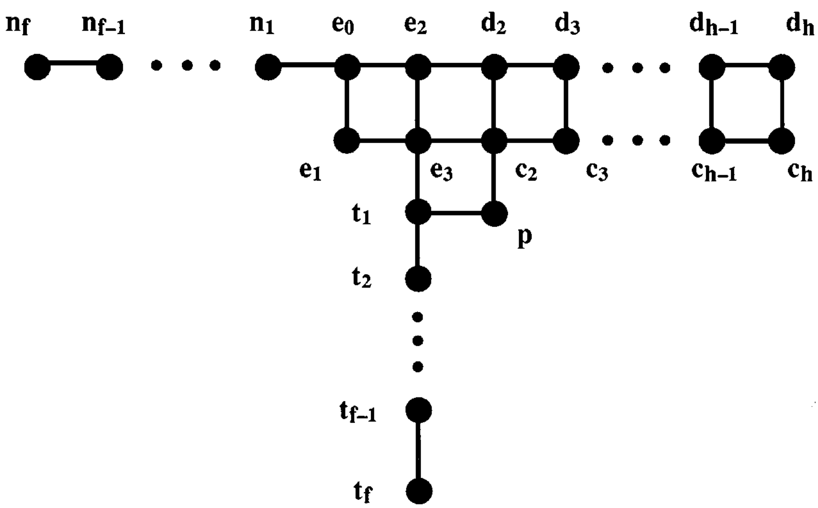

Figure 9.

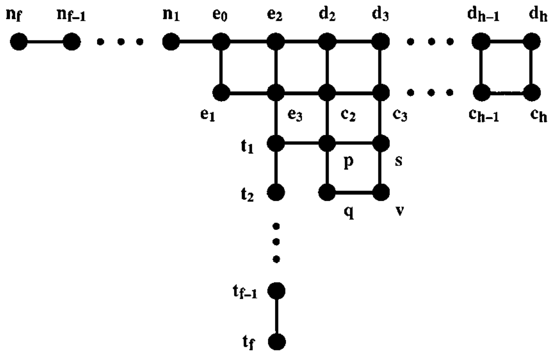

Figure 10.

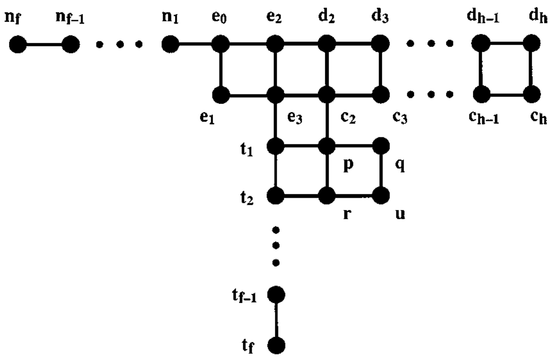

Figure 11.

adjoined beneath p and c_3, there is one wedge v beneath q (with s). If r is adjoined beneath t_2 and p, there is one wedge u beneath q (with r). If both r and s are adjoined, then q can cover no other elements.

Proof: For the last statement, note that dangles beneath q have already been ruled out, and then apply Lemma 6. □

Further consider the two cases of adjoining both q and s or both q and r after adjoining p. In the former (latter) case, one may then choose to adjoin v (respectively, u) or not. The subcase in which v is adjoined is depicted in figure 10 and is the starting point for $\mathcal{C}_6$. The subcase in which u is adjoined is depicted in figure 11 and leads to $\mathcal{C}_7$–$\mathcal{C}_{15}$. Considering these two subcases for each of these two of the original 2^3 cases yields six cases and four subcases for the local structure immediately below p.

The four cases in which no dangle is adjoined beneath p will form the fourth class: Let $\mathcal{C}_4$ consist of all irreducible components for which $f \geq 2,\ g = 1,\ h \geq 1$, and which are extensions of V_4 in which there do not exist elements p and q such that q is covered only by p.

Note that the existence of any of $d_2,\ c_2$, or p is not being assumed, and so the only condition on h is $h \geq 1$. If $h \geq 2$ but p does not exist, then the irreducible component is in $\mathcal{C}_4$. This is also the case when $h = 1$ and c_2 and d_2 do not exist. Then it can be seen that the only irreducible component is $U(f; 1, 1) \cong d_{f+3}(1)$. So the posets $d_k(1)$ are in $\mathcal{C}_4$ when $k \geq 5$. (The posets $d_3(1)$ and $d_4(1)$ are the smallest members of Classes 1 and 2, respectively.)

To prove Lemma 7.4 when $h \geq 2$, consult figure 9: now d_2 and c_2 must exist. The consideration of the only second generation dangle, q beneath p, has been deferred to later classes. Following GFP, it can be seen that the only possible extensions P of $U(f; 1, h)$ here are the filters of M_4 which contain L_4. Each such filter is determined by its intersection with O_4; such intersections are elements λ of $\Lambda(g-1, h-1)$.

In the remaining cases, both p and a dangle q beneath p exist. So necessarily $h \geq 2$. Now either q does not cover any other elements, or it does. The two subcases and the two cases in which q covers no other elements will form the fifth class: Let $\mathcal{C}_5$ consist of all irreducible components for which $f \geq 2,\ g = 1,\ h \geq 2$ and which are extensions of V_5 in which q is minimal.

To prove Lemma 7.5, adjoin q beneath p in figure 9 at a 45° angle. There can be no other dangles beneath p, and there are no elements beneath q. We are left with two simple grid filling situations, which yield filters of M_5 containing L_5.

In the remaining cases, q covers at least one other element. The lemma described the only two scenarios in which q covers another element; for the next class we take the first subcase mentioned there. For fixed $f,\ g$, and h, define the poset $V_6 := V_4 \cup \{p, q, s, v\}$. It is depicted in figure 10. Since s is assumed to exist, we will necessarily have $h \geq 3$. Let $\mathcal{C}_6$ consist of all irreducible components for which $f \geq 2,\ g = 1,\ h \geq 3$, and which are extensions of V_6.

To prove Lemma 7.6, first note that there are no dangle extensions beneath V_6. Initially, the only wedge extension is beneath s and c_4. Follow GFP to produce all filters of M_6 containing L_6.

11. Class 7: Nooks

All of the remaining cases for $f \geq 2$ and $g = 1$ begin with the second subcase described in Lemma 10. So from now on we will only consider extensions of the poset $V_7 := V_4 \cup \{p, q, r, u\}$ depicted in figure 11. Our next juncture is $h \geq 3$ or $h = 2$. Let $\mathcal{C}_7$ consist of all irreducible components for which $f \geq 2,\ g = 1,\ h \geq 3$, and which are extensions of V_7.

To prove Lemma 7.7, note that the existence of c_3 rules out putting a dangle beneath u. In fact, there are no dangles anywhere beneath V_7, and initially the only wedge extension is beneath t_3 and r. Follow GFP to produce all filters of M_7 containing L_7.

12. Classes 8–11: Swivel-type classes

We are left with extensions of V_7 which have $h = 2$. To conserve letters, rename $e_4 := d_2,\ e_5 := c_2,\ e_6 := p$, and $e_7 := q$. Note that there is a dangle extension beneath u; call

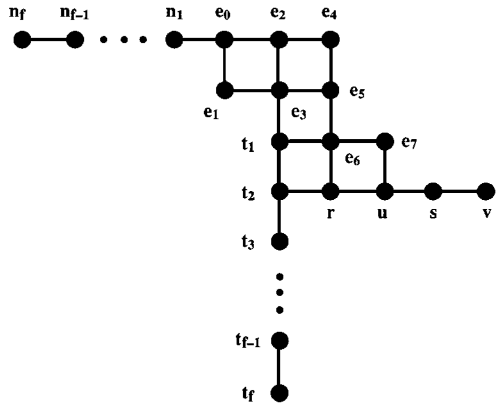

Figure 12.

it s. And if s is adjoined, then there is a dangle extension beneath it; call it v, see figure 12. Let $\mathcal{C}_8$ consist of all irreducible components for which $f \geq 2$, $g = 1$, $h = 2$, and which are extensions of V_7 which use no dangle extensions except possibly for s and v.

To prove Lemma 7.8, note that there are initially no wedges other than the one beneath t_3 and r, and follow GFP. For notational convenience, include the elements r and u with the optional elements from $M_8 - L_8$ when defining λ.

We are left with extensions of V_7 with $h = 2$ and in which at least one dangle extension aside from s and v is used. First suppose that s does not exist. Then it can be seen that no extensions of V_7 have dangle extensions. So from now on, the element s must be present. There are no dangles for $V_7 \cup \{s\}$ aside from v. The only extension is a wedge beneath t_3 and r; call it b_2 (see figure 13). We must adjoin it. Let b_3 denote the wedge beneath b_2

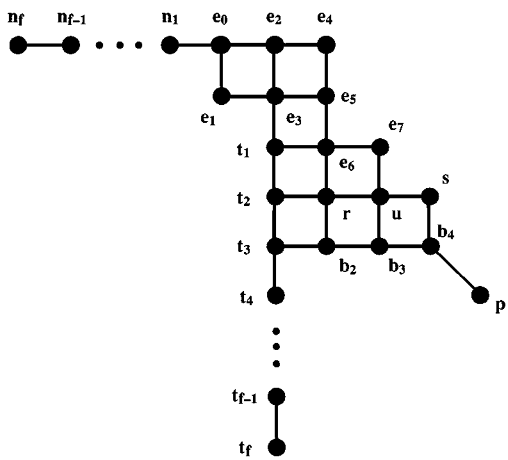

Figure 13.

and u. If we do not adjoin b_3, it can be seen that no dangles will ever arise. So we must adjoin b_3 as well. Similar reasoning forces us to adjoin a wedge b_4 beneath b_3 and s. Now there exists a dangle beneath b_4; call it p. It can be seen that if we never adjoin p, then there will be no future dangles aside from v, whether we adjoin v or not. (If p is not adjoined, all possibilities have already appeared in $\mathcal{C}_8$.)

All of the remaining cases must have the dangle p adjoined in addition to b_2, b_3, and b_4. So from now on we consider only extensions of the poset $V_9 := V_7 \cup \{s, b_2, b_3, b_4, p\}$, which is depicted in figure 13. The existence of s implies that $h = 2$; the existence of b_2 implies that $f \geq 3$. Either p does not cover any other elements, or it does. Let $\mathcal{C}_9$ consist of all irreducible components for which $f \geq 3$, $g = 1$, $h = 2$, and which are extensions of V_9 in which p is minimal.

To prove Lemma 7.9, note that the only dangle extension is v. The only possible extension activity arises as grid filling between the t_i and b_j chains, and, if v is adjoined, as grid filling between b_4 and v. Here the shape μ is defined to contain v if it is adjoined.

We are left with extensions of V_9 in which p covers at least one other element. Let q denote an element covered by p. To avoid creating an incomplete d_5^--interval, the element q must be covered by some other element, say w. Then w and p must be covered by one element, say x. But since p is a dangle, we must have $x = b_4$. Repeating this reasoning rules out having a second element covered by p. To avoid violating D6, the element w must be covered by something besides b_4, say y. Then b_4 and y would both have to be covered by one element, say z. Since b_4 is already covered by two elements, by F4 we must have $z = b_3$ or $z = s$.

Let P be a connected d-complete poset with unique maximal element z_0. Let $x \in P$ be such that the length of any chain (by F3) in P from z_0 to x is n. Then the *depth* $\delta(x)$ of x in P is defined to be $-n$.

First suppose that $z = b_3$, as shown in figure 14. If y is not covered by another element besides b_3 which is less than r, then $[y, r]$ would be an incomplete d_4^--interval. So some

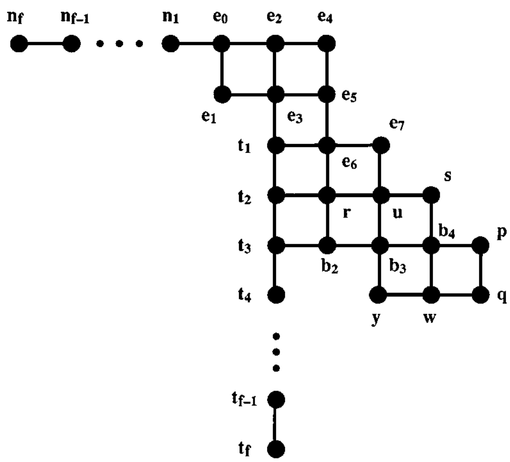

Figure 14.

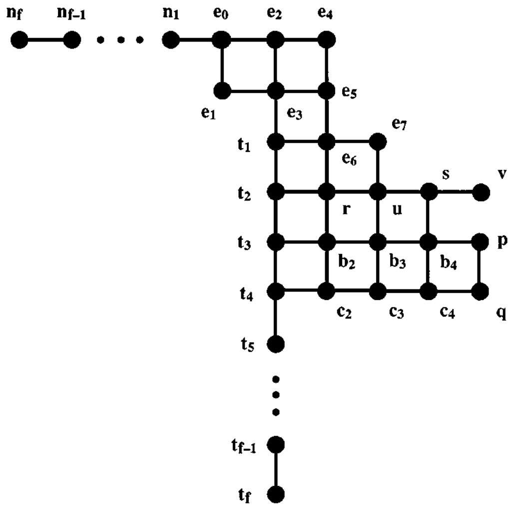

Figure 15.

$m \neq b_3$ must cover y. Note that $\delta(m) = \delta(p) + 2 = \delta(b_3) = \delta(s)$. Clearly $m \neq s$. The only extension of V_9 or a successor of depth $\delta(b_3)$ is the element c_2 shown in figure 15. So $m = c_2$. Rename $c_3 := y$ and $c_4 := w$. Summarizing, the assumption that $z = b_3$ forces the existence of the new elements c_4, c_3, and c_2 in $V_{11} := V_9 \cup \{c_2, c_3, c_4, q\}$. (These elements do not coincide with any preexisting elements, and there can be no edges emanating upward from these elements besides the edges shown.) Now either the dangle v exists, or it does not. Assume that it does, and set $V_{10} := V_{11} \cup \{v\}$. This poset is shown in figure 15. In order for c_2 to exist, we must have $f \geq 4$. Let $\mathcal{C}_{10}$ consist of all irreducible components for which $f \geq 4$, $g = 1$, $h = 2$, and which are extensions of V_{10}.

To prove Lemma 7.10, note that there are no extensions of V_{10} or its successors not between the two chains with $f - 4$ and 4 elements, respectively. Apply the GFP.

Continue to assume that $z = b_3$, but now suppose that v does not exist. Erase v in figure 15 to depict V_{11}. Let $\mathcal{C}_{11}$ consist of of all irreducible components for which $f \geq 4$, $g = 1$, $h = 2$, and which are extensions of V_{11} in which s covers no dangle.

To prove Lemma 7.11, note that there can be a dangle, call it c_6, beneath $c_5 := q$. This is the only possible extension of V_{11} outside of the obvious grid region (and aside from a dangle beneath s, which is treated in $\mathcal{C}_{10}$). If c_6 is not adjoined, then only obvious grid filling wedges can be adjoined. If c_6 is adjoined, then a slightly wider grid region becomes available. Adjoining first generation wedges $d_1, \ldots, d_6$ beneath $c_2, \ldots, c_6$ then creates the possibility of a dangle, call it d_7, beneath d_6. It can be seen that this process can continue as long as there are remaining t_i's, and that there are no other possibilities. For notational convenience, require that the elements c_2, c_3, c_4, c_5 be included with the optional elements of M_{11} to produce a filter of $\Sigma(f-3, f+1)^*_M$.

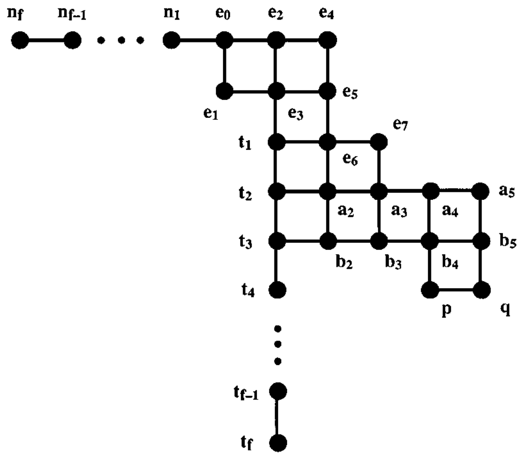

Figure 16.

13. Classes 12–15: Pump- and bat-type classes

We are left with the cases in which $z = s$. After renaming $a_4 := s$, figure 16 shows the poset determined by the $z = s$ assumption. (Also, the renamings $a_2 := r$ and $a_3 := u$ have been implemented.) Here $b_5 := w$ and $a_5 := y$ are the new elements which arose leading up to the $z = s$ option. (These elements could not have coincided with any preexisting elements, and there cannot be any additional edges emanating up from either of them.) It can be seen that this poset is a legitimate extension of V_9. So we are left to consider extensions of the poset $V_{12} := V_9 \cup \{a_5, b_5, q\}$ depicted in figure 16. All remaining irreducible components must be extensions of V_{12}. Necessarily $f \geq 3$.

It can be seen that the only possible 1-extensions of V_{12} and its successors are wedge extensions between the chains $t_3 \leftarrow \cdots \leftarrow t_f$ and $t_3 \leftarrow b_2 \leftarrow b_3$ and (possibly iterated) dangle extensions beneath q. Notice that some of these extensions may conflict with each other. By now, the proofs of the Lemmas 7.N have become routine, and there are no wrinkles in the proofs for the last four classes. So no comments will be made for the proofs of Lemmas 7.12–7.15.

Either q does not cover any other elements, or it does. Let $\mathcal{C}_{12}$ consist of all irreducible components for which $f \geq 3,\ g = 1,\ h = 2$, and which are extensions of V_{12} in which q is minimal.

Now suppose that q covers some other element, say r. It can be seen that r must be a dangle. Either r does not cover any other elements, or it does. Set $V_{13} := V_{12} \cup \{r\}$. Let $\mathcal{C}_{13}$ consist of all irreducible components for which $f \geq 3,\ g = 1,\ h = 2$, and which are extensions of V_{13} in which r is minimal.

Now suppose that r covers some other element, say s. It can be seen that s must be a dangle. Either s does not cover any other elements, or it does. Set $V_{14} := V_{13} \cup \{s\}$. Let $\mathcal{C}_{14}$ consist of all irreducible components for which $f \geq 3,\ g = 1,\ h = 2$, and which are extensions of V_{14} in which s is minimal.

Now suppose that s covers some other element, say u. It can be seen that u must be a dangle and that t_4 cannot exist. Hence $f = 3$. Set $V_{15} := V_{14} \cup \{u\}$. Let $\mathcal{C}_{15}$ consist of all irreducible components for which $f \geq 3$, $g = 1$, $h = 2$, and which are extensions of V_{15}.

Actually, the irreducible component V_{15} cannot be extended, and so it is the only member of $\mathcal{C}_{15}$. We have argued between the definitions of the classes that those definitions have exhausted all possibilities for irreducible components. Hence, the proof of Theorem 7 is complete, if it is accepted that the figures 5.N describe all of the possibilities within each class.

14. The self-dual *d*-complete posets are the minuscule posets

A poset P is *self-dual* if it is isomorphic to its order dual P^*. In this section we use the classification theorem to identify the self-dual d-complete posets. We can immediately reduce to the connected case. The global tree structure of a self-dual slant sum must be that of a chain. A connected d-complete poset has a unique maximal element. Notice that an acyclic element of an irreducible component cannot be a minimal element of that component. So if an irreducible component appears as the "upper" poset in a slant sum, the order dual of that slant sum will have more than one maximal element. Hence only the trivial slant irreducible d-complete poset, the one element poset, could appear as the upper poset in a self-dual slant sum. To be self-dual, the bottommost poset must also be the one element poset. So chains are the only self-dual slant reducible connected d-complete posets. Obviously, the one element d-complete poset is self-dual.

We are left to consider irreducible components which are self-dual by themselves. Such a poset must have a unique minimal element. Searching all filters of the 15 maximal irreducible components first for the property of having a unique minimal element and then for the property of being self-dual yields one poset for each set of $(f; g, h)$-values in classes $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_{15}$. Also, one poset is produced for each $f \geq 2$ when $h = 1$ in $\mathcal{C}_4$, and one poset arises in $\mathcal{C}_8$ when $f = 2$.

In [8] it was observed that the weight diagrams of irreducible minuscule representations of simple Lie algebras are always distributive lattices. There an *irreducible minuscule poset* was defined to be a poset which arises as the poset of join irreducibles for such a distributive lattice. If an irreducible minuscule representation of the simple Lie algebra of type X_n had highest weight ω_j, then the corresponding minuscule poset was denoted $x_n(j)$. Some distinct minuscule representations give rise to identical minuscule posets. All irreducible minuscule posets were depicted in figure 2 of [9].

Comparison of the conclusions of the first two paragraphs of this section with figure 2 of [9] produces the following theorem:

Theorem 1 *Let P be a connected d-complete poset which is self-dual. Then P is one of the irreducible minuscule posets* $a_n(j)$, $n \geq 1$ *and* $1 \leq j \leq [n/2]$, $d_n(n)$, $n \geq 4$, $d_n(1)$, $n \geq 5$, $e_6(6)$, *and* $e_7(7)$. *Every irreducible minuscule poset appears once in this list.*

When $n = 1$, the poset $a_n(1)$ is the one element poset. When $j = 1$ or $j = n$, the poset $a_n(j)$ is the n-element chain. The remaining cases are irreducible components. When $2 \leq j \leq [n/2]$, the poset $a_n(j)$ is denoted $a_n[j-1, n-j; (j-2) \times (n-j-1)]$ in this paper, where $(j-2)\times(n-j-1)$ denotes the shape with $j-2$ rows of length $n-j-1$. When $n \geq 4$, the poset $d_n(n)$ is denoted $d_n[1;1,n-3;J(2\times(n-4))]$ here, where $J(2\times(n-4))$ denotes the shifted shape with row lengths $n-3,\ n-4,\ldots,2,\ 1$. When $n \geq 5$, the poset $d_n(1)$ is denoted $d_n[n-3;1,1]$ here. The posets $e_6(6)$ and $e_7(7)$ are denoted by $e_6[2;1,2;8;(4)]$ and $e_7[3;1,2;15]$ in this paper, where (4) is the shape consisting of one row of length 4. Order diagrams for these five families of posets appeared in figure 2 of [9]. In that paper, it was shown that the irreducible minuscule posets are the only connected posets whose elements can be labelled with numbers such that a certain system of linear equations determined by the structure of the poset (which implied the existence of a nice $\mathrm{sl}_2(\mathbb{C})$ representation) is satisfied.

15. Weyl group comments

The results of [7] can be combined with the results of this paper to obtain a listing of all of the λ-minuscule elements in any simply laced general Weyl group W. Let G be the simple graph with node set N which specifies W. Fix a dominant integral weight λ. The definition of the "λ-minuscule" property for elements $w \in W$ was given in Section 1.

Using [7], every concept developed in this paper for d-complete posets can be translated into an analogous concept for λ-minuscule Weyl group elements. This includes 'connected poset', 'slant sum', and 'irreducible component'. So this paper can be viewed as a classification of a certain kind of Weyl group element. Here is an overview: Any reduced decomposition of a λ-minuscule element of W corresponds to an increasing sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ of colored d-complete posets such that $F_j - F_{j-1}$ consists of one element of color $i_j \in N$. Ignoring the colors, we obtain a d-complete poset F_k corresponding to w. The "support" of any λ-minuscule w in N consists of a union of tree subgraphs of G. Suppose that G is connected. To generate all connected λ-minuscule elements of W which use each generator s_i at least once, one would successively consider all "rooted spanning trees" of G. When translated to the (order dualized) context of [7], the list of possible d-complete posets for each such tree provided by the present paper would, when colored, become the list of the λ-minuscule elements w whose bottom trees are the fixed rooted spanning tree. All of the reduced decompositions for each such w could then be formed by finding all of the order extensions of the corresponding colored d-complete poset.

In this way it can be seen that there is no infinite sequence $i_1, i_2, \ldots$ from N such that $s_{i_k}\cdots s_{i_2}s_{i_1}$ is λ-minuscule for every $k \geq 1$. (But there *are* such sequences, e.g., from figure 8 of [14] (also see Section 2 of [7]), if we had not required λ to be "dominant" in the definition of λ-minuscule.) Theorem 7 implies that every λ-minuscule slant irreducible component w arises as an initial subword of one of the (finite) maximal irreducible components w_N, $1 \leq N \leq 15$, which can be based upon a fixed top tree.

All of the preceding can be recast in the context of a restricted version [7] of the numbers game of Mozes [6] rather than in the context of λ-minuscule elements. This restricted game is played on the possible labellings of the nodes of a simple graph with integers. The possible moves at any stage correspond to nodes which have -1 labels. In this context, the present paper classifies the possible evolutions of all such games which begin with all labels non-positive. We do not know of any overlap between our "non-existence of λ-minuscule elements of arbitrary length for a fixed G" result just stated and the various terminating numbers games results of [1–3, 6].

Suppose that we start with a given uncolored d-complete poset P with top tree T. Let N consist of the nodes of T, and let G be any simple graph which contains T as a subgraph. Each element of P can be uniquely "colored" with one of the colors from N, as described in Proposition 8.6 of [7]. Then the sequence of colors produced by reading off the colors of some or all of the elements of P from the top down will specify a λ-minuscule element $w = s_{i_k} \cdots s_{i_2} s_{i_1}$ in the simply laced general Weyl group whose Dynkin diagram is G.

Let us continue the discussion of [14] from Section 1. To see that a λ-minuscule element w is fully commutative, use Theorem A of [7] and Theorem 2.2 of [14]. The d-complete poset F_k we associated to $w \in W$ four paragraphs above is the "heap" of a fully commutative element w, by Lemma 2.1 of [14]. Our environment is more specialized than that of [14]: Stembridge does not restrict to the simply laced case, and even there he knows of many elements w which are not λ-minuscule but are such that $[e, w]$ is a distributive lattice in the Bruhat order. So he has many heaps P for Coxeter group elements w such that $J(P) \cong [e, w]$ is a distributive lattice, in addition to the d-complete posets P. Is there some way of characterizing the λ-minuscule elements amongst all fully commutative elements using only Coxeter theoretic notions?

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