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Duality Maps of Finite Abelian Groups and Their Applications to Spin Models

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Abstract. Duality maps of finite abelian groups are classified. As a corollary, spin models on finite abelian groups which arise from the solutions of the modular invariance equations are determined as tensor products of indecomposable spin models. We also classify finite abelian groups whose Bose-Mesner algebra can be generated by a spin model.

Keywords: spin model, finite abelian group, quadratic form, association scheme

1. Introduction

Although the duality property of Bose-Mesner algebras of commutative association schemes was essentially given by Kawada [11], Jaeger [6] was probably the first to regard self-duality as the existence of a so-called duality map. The group algebra of a finite abelian group X can be thought of the Bose-Mesner algebra of its group association scheme. In this case a duality map is essentially equivalent to an isomorphism $\psi : x \mapsto \psi_x$ from X to its character group $\hat{X}$ such that $\psi_x(y) = \psi_y(x)$ for any $x, y \in X$. In other words, a duality map is an isomorphism from X onto $\hat{X}$ such that, if we arrange elements of $\hat{X}$ and X in rows and columns of the character table according to the correspondence ψ, the chracter table becomes a symmetric matrix. A duality map always exists in a finite abelian group. Indeed, when X is cyclic, then any isomorphism of X and $\hat{X}$ is a duality map. If X is not cyclic, then X can be decomposed into a product of cyclic groups, and one can obtain a duality map of X as the natural product of those of cyclic factors (see [5], 2.10.7). The purpose of this paper is to classify duality maps of finite abelian groups. If the group X has an odd order, it turns out that any duality map of X is a product of duality maps of factors of some cyclic decomposition of X. If the group X has even order, the situation is more complicated, but the classification reduces to the case where X is the product of two cyclic 2-groups of the same order.

Motivation of this work comes from spin models on finite abelian groups. If a spin model generates a Bose-Mesner algebra in a certain sense, then it comes from a solution of the modular invariance equation with respect to a duality map, as shown in [4]. Also in [4], all solutions of the modular invariance equations for the Bose-Mesner algebras of finite abelian groups were determined. However, the modular invariance equation depends on the duality map of the finite abelian group. Thus the nature of solutions and the structure of the resulting spin models depend on the duality map. We shall show that, if the duality map is a product of duality maps of the direct product decomposition, then the solutions of the modular invariance equation are tensor products of the solutions of the modular invariance equations in the direct product components, hence the resulting spin models are tensor products of spin models on direct product components. Since the partition function of a tensor product of spin models is the product of the partition functions of each spin model, our work has fundamental importance on the determination of link invariants defined by spin models on finite abelian group. We also classify spin models on finite abelian groups which generate the Bose-Mesner algebras of the groups. It turns out that a spin model can generate the Bose-Mesner algebra of a finite abelian group if the group is cyclic or the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2. Duality maps of Bose-Mesner algebras

A Bose-Mesner algebra $\mathfrak{A}$ is a commutative subalgebra of the full matrix algebra $M_n(\mathbb{C})$ which is closed under the Hadamard (entrywise) product, is closed under the transposition map, and contains the all one matrix J. A duality map Ψ of $\mathfrak{A}$ is a linear isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}$ such that

$$\begin{aligned} \Psi(AB) &= \Psi(A) \circ \Psi(B) \quad A, B \in \mathfrak{A}, \\ \Psi^2 &= n\tau \end{aligned}$$

hold, where $\circ$ denotes the Hadamard product and τ denotes the transposition map. In this paper we shall only consider Bose-Mesner algebras of finite abelian groups defined as follows. Let X be a finite abelian group of order n. For each element $x \in X$, define the adjacency matrix A_x by

$$(A_x)_{y,z} = \begin{cases} 1 & \text{if } y - z = x \\ 0 & \text{otherwise} \end{cases}$$

These matrices span a subalgebra $\mathfrak{A}$ of $M_n(\mathbb{C})$ of dimension n, isomorphic to the group algebra of X. Clearly $\mathfrak{A}$ becomes a Bose-Mesner algebra, and its primitive idempotents are given by

$$E_\chi = \frac{1}{n} \sum_{x \in X} \chi(x) A_x$$

where χ runs through the character group $\hat{X}$ of X. The first condition of duality map is equivalent to $\Psi(E_\chi) = A_x$ for some $x \in X$. Thus Ψ determines a bijection $\psi : X \to \hat{X}$, $x \mapsto \psi_x$ via the rule $\Psi(E_{\psi_x}) = A_x$. The second condition of duality map is then equivalent to

$$\psi_x(y) = \psi_y(x) \quad \text{for any } x, y \in X. \tag{1}$$

This condition implies that ψ is an isomorphism from X onto $\hat{X}$.

These observations lead us to define duality maps of finite abelian groups as follows. A dualtiy map ψ of a finite abelian group X is an isomorphism from X to $\hat{X}$ satisfying (1).

If ψ is a duality map of the finite abelian group X, then we call $P = (\psi_x(y))_{x,y\in X}$ the character table of X associated with ψ. Of course, the character table of X is uniquely determined up to permutation of rows and permutation of columns, but the character table of X associated with a duality map is determined uniquely up to *simultaneous* permutation of rows and columns. Note that the character table associated with a duality map is a symmetric matrix, and indeed, a duality map is precisely a bijection between $\hat{X}$ and X according to which the arrangement of rows and columns of a character table of X makes the character table symmetric.

It is well known and easy to see that the character table of a direct product of groups is the tensor product of character tables of each group. It is also obvious that the tensor product of symmetric matrices is again symmetric. Thus a natural question arises: when is the character table of a finite abelian group associated with a duality map the tensor product of character tables of direct product components? The answer to this question will be given in the next two sections.

3. Duality maps and direct products

A homocyclic group is a direct product of cyclic groups of the same order. Any finite abelian group can be decomposed into a direct product of homocyclic p-groups. In this section we reduce the classification of duality maps of finite abelian groups to the case where the group is a homocyclic p-group.

Lemma 1 *Let ψ be a duality map of a finite abelian group X, and suppose that X is the direct product of X_1 and X_2. The following conditions are equivalent.*
(i) $\psi_{x_1}(x_2) = 1$ *for any* $x_1 \in X_1, x_2 \in X_2$.
(ii) *There exist duality maps* $\psi^{(1)}, \psi^{(2)}$ *of* X_1, X_2, *respectively, such that* $\psi_{x_1+x_2}(y_1+y_2) = \psi^{(1)}_{x_1}(y_1)\psi^{(2)}_{x_2}(y_2)$ *for any* $x_1, y_1 \in X_1, x_2, y_2 \in X_2$.

Proof: Suppose that (i) holds. Define $\psi^{(1)} : X_1 \to \hat{X}_1$ by $x_1 \mapsto \psi_{x_1}|_{X_1}$. If $\psi_{x_1}|_{X_1} = 1_{X_1}$, then by the assumption (i), $\psi_{x_1}|_{X_2} = 1_{X_2}$, so that $\psi_{x_1} = 1_X$. Thus $x_1 = 0$, i.e., $\psi^{(1)}$ is injective. Since $|X_1| = |\hat{X}_1|$, the mapping $\psi^{(1)}$ is an isomorphism. Similarly, we can define $\psi^{(2)} : X_2 \to \hat{X}_2$ and prove that $\psi^{(2)}$ is an isomorphism. Then for $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$,

$$\begin{aligned}\psi_{x_1+x_2}(y_1+y_2) &= \psi_{x_1}(y_1)\psi_{x_1}(y_2)\psi_{x_2}(y_1)\psi_{x_2}(y_2)\\ &= \psi_{x_1}(y_1)\psi_{x_2}(y_2)\\ &= \psi^{(1)}_{x_1}(y_1)\psi^{(2)}_{x_2}(y_2).\end{aligned}$$

Conversely, assume (ii). Then for $x_1 \in X_1, x_2 \in X_2$, we have

$$\psi_{x_1}(x_2) = \psi^{(1)}_{x_1}(0)\psi^{(2)}_0(x_2) = 1,$$

so that (i) holds. □

Definition Let ψ be a duality map of a finite abelian group X, and suppose that X is the direct product of X_1 and X_2. If one of the equivalent conditions of Lemma 1 is satisfied, we say that ψ splits over $X_1 \times X_2$, and that ψ is the product of duality maps $\psi^{(1)}, \psi^{(2)}$.

It follows immediately from the definition that if $X = X_1 \times X_2$ with $|X_1|$ and $|X_2|$ relatively prime, then any duality map of X splits over $X_1 \times X_2$. Note that any finite abelian group is a direct product of p-groups. Thus, the classification of duality maps of finite abelian groups is reduced to that of duality maps of finite abelian p-groups.

Lemma 2 *Let X be a finite abelian p-group of exponent p^n, ψ a duality map of X. Then there is a direct product decomposition $X = X_1 \times X_2$ such that the following conditions hold.*

(i) *X_1 is a homocyclic group of exponent p^n,*

(ii) *X_2 has exponent at most p^{n-1},*

(iii) *ψ splits over $X_1 \times X_2$.*

Proof: By the fundamental theorem of finite abelian groups, there is a direct product decomposition $X = X_1 \times X_1'$ such that X_1 is a homocyclic group of exponent p^n, X_1' has exponent at most p^{n-1}. We claim that the mapping $\psi^{(1)}$ from X_1 to $\hat{X}_1$ defined by $x_1 \mapsto \psi_{x_1}|_{X_1}$ is an isomorphism. It suffices to show that $\psi^{(1)}$ is injective. Suppose that x_1 is a nonzero element of X_1 contained in the kernel of $\psi^{(1)}$. Since X_1 is generated by elements of order p^n, there exists an element $x_0 \in X_1$ of order p^n such that x_1 is contained in the subgroup generated by x_0. It follows that $\psi_{x_0}|_{X_1}$ has order at most p^{n-1}. Since X_1' has exponent at most p^{n-1}, $\psi_{x_0}|_{X_1'}$ has order at most p^{n-1}. Thus, ψ_{x_0} has order at most p^{n-1}, which is a contradiction since ψ is an isomorphism.

Now define

$$X_2 = \{x \in X \mid \psi_x(x_1) = 1 \text{ for any } x_1 \in X_1\}.$$

From the above claim we see

$$X_1 \cap X_2 = \{y_1 \in X_1 \mid \psi_{y_1}(x_1) = 1 \text{ for any } x_1 \in X_1\} = \operatorname{Ker} \psi^{(1)} = 0.$$

It remains to show that X is generated by X_1 and X_2. If $x \in X$, then $\psi_x|_{X_1} \in \hat{x}_1$, so the claim implies that there exists an element $x_1 \in X_1$ such that $\psi_x|_{X_1} = \psi_{x_1}|_{X_1}$. In other words, $\psi_{x-x_1}|_{X_1} = 1_{X_1}$, hence $x - x_1 \in X_2$. □

Applying Lemma 2 repeatedly, we see that any duality map of a finite abelian p-group X splits over some decomposition of X into homocyclic p-groups. Summarizing the results obtained in this section, we have the following proposition.

Proposition 3 *Let X be a finite abelian group, ψ a duality map of X. Then ψ splits over some decomposition of X into homocyclic p-groups.*

4. Symmetric bilinear forms on homocyclic p-groups

By Proposition 3, the classification of duality maps of finite abelian group is reduced to that of duality maps of homocyclic p-groups. In this section we classify duality maps of homocyclic p-groups. Let X be a homocyclic p-group $(\mathbb{Z}/p^n\mathbb{Z})^m$, where n, m are positive integers and p is a prime. We regard X as a free R-module of rank m where $R = \mathbb{Z}/p^n\mathbb{Z}$. Then we see that duality maps of X are in one-to-one correspondence with nondegenerate symmetric R-bilinear forms on X. Indeed, if we fix a primitive p^n-th root of unity ζ, a duality map ψ uniquely determines a nondegenerate symmetric R-bilinear form B on X by the rule

$$\zeta^{B(x,y)} = \psi_x(y), \quad x, y \in X. \tag{2}$$

If X is a direct product of subgroups X_1 and X_2, then X is a direct sum of R-submodules X_1 and X_2. Then a duality map ψ of X splits over $X_1 \times X_2$ if and only if X_1 and X_2 are orthogonal with respect to the symmetric bilinear form B. Using the classification of nondegenerate symmetric bilinear form over the ring of p-adic integers $\mathbb{Z}_p$, we obtain the following.

Proposition 4 *Let X be a homocyclic p-group of exponent p^n, ψ a duality map of X.*

(i) *If $p \neq 2$, then ψ splits over some decomposition of X into a direct product of cyclic groups.*

(ii) *If $p = 2$, then ψ splits over some decomposition $X_1 \times \cdots \times X_k$ of X, where each X_i is either cyclic or the direct product of two cyclic groups, and in the latter case, for some decomposition $X_i = \langle a_1 \rangle \times \langle a_2 \rangle$, the restriction of the duality map to X_i is given by either*
(a) $\psi_{a_1}(a_1) = \psi_{a_2}(a_2) = 1, \psi_{a_1}(a_2) = \zeta$,
(b) $\psi_{a_1}(a_1) = \psi_{a_2}(a_2) = \zeta^2, \psi_{a_1}(a_2) = \zeta$,
where ζ is a primitive 2^n-th root of unity.

Proof: Suppose that X has rank m. Let B be the nondegenerate symmetric bilinear form on X defined by (2). Let $\tilde{X}$ be a free $\mathbb{Z}_p$-module of rank m and identify X with $\tilde{X}/p^n\tilde{X}$. Let $\tilde{B}$ be a nondegenerate symmetric bilinear form on $\tilde{X}$ such that $B = \tilde{B} \pmod{p^n}$. If $p \neq 2$, then there exists an orthogonal basis of $\tilde{X}$ with respect to $\tilde{B}$ by [13], Theorem 5.2.4. Reducing modulo p^n, we find an orthogonal basis of X with respect to B. Thus ψ splits over some decomposition of X into cyclic groups. If $p = 2$, then by [13], Theorem 5.2.5, we have either an orthogonal basis of $\tilde{X}$, or an orthogonal decomposition $\tilde{X}_1 \perp \cdots \perp \tilde{X}_k$ of $\tilde{X}$ such that each $\tilde{X}_i$ has rank 2 and $\tilde{B}|_{\tilde{X}_i} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ with respect to a suitable basis of $\tilde{X}_i$. Reducing modulo 2^n, we have either an orthogonal basis of X with respect to B, or an orthogonal decomposition $X_1 \times \cdots \times X_k$ of X with respect to B such that the restriction of the duality map ψ to X_i is of the desired form. □

Remark Theorem 5.2.5 of [13] actually claims that the number of factors X_i with duality map of the form (b) is at most one.

Combining Proposition 3 and Proposition 4 we have the following theorem.

Theorem 5 *Let X be a finite abelian group, ψ a duality map of X. Then ψ splits over some decomposition $X_1 \times \cdots \times X_k$ of X, where each X_i is either a cyclic p-group for some prime p, or the direct product of two cyclic 2-groups of the same order, and in the latter case, for some decomposition $X_i = \langle a_1 \rangle \times \langle a_2 \rangle$, the restriction of ψ to X_i is given by either*
(i) $\psi_{a_1}(a_1) = \psi_{a_2}(a_2) = 1, \psi_{a_1}(a_2) = \zeta,$
(ii) $\psi_{a_1}(a_1) = \psi_{a_2}(a_2) = \zeta^2, \psi_{a_1}(a_2) = \zeta,$
where ζ is a primitive 2^n-th root of unity, 2^n is the exponent of X_i.

5. Spin models on finite abelian groups

In this section we consider spin models obtained from solutions of the modular invariance equations of finite abelian groups. First, we give the definition of spin models, then discuss the modular invariance equation with respect to a duality map of a Bose-Mesner algebra.

Definition A spin model is a quadruple $(X, W_+, W_-; D)$, where X is a finite nonempty set of size n, D is one of the square roots of n, and W_+, W_- are matrices of size n indexed by X satisfying the following properties:

(1) ${}^tW_+ \circ W_- = J$, where J is a matrix whose entries are all 1,
(2) $W_+W_- = nI$,
(3) $\sum_{x\in X} W_+(\alpha, x)W_+(x, \beta)W_-(x, \gamma) = DW_+(\alpha, \beta)W_-(\beta, \gamma)W_-(\alpha, \gamma)$ for all $\alpha, \beta, \gamma \in X$.

It is known that spin models give invariants of links. For more information concerning spin models see [2, 7, 9, 12]. In this paper, however, we shall only deal with spin models which are defined on a finite abelian group, i.e., we assume that X is a finite abelian group and W_+ (and consequently W_-) is contained in the Bose-Mesner algebra of X. We say that a spin model $(X, W_+, W_-; D)$ generates a Bose-Mesner algebra $\mathfrak{A}$ if W_-, ${}^tW_-$ and J generate $\mathfrak{A}$. The following result can be found in [4].

Theorem 6 *Suppose that $(X, W_+, W_-; D)$ is a spin model which generates the Bose-Mesner algebra of a finite abelian group X. Let $W_- = \sum_{x\in X} t_x A_x$. Then there exists a duality map ψ such that*

$$(P\Delta)^3 = t_0 D^3 I \tag{3}$$

holds, where P is the character table associated with ψ, $\Delta = \mathrm{diag}(t_x; x \in X)$.

Conversely, if the complex numbers t_x $(x \in X)$ satisfy the Eq. (3), then $(X, W_+, W_-; D)$ becomes a spin model, where $W_- = \sum_{x\in X} t_x A_x$ and $W_+ = |X|W_-^{-1}$.

The Eq. (3) is called the modular invariance equation with respect to the character table P. Given a character table P associated with a duality map of a finite abelian group, solutions of the modular invariance equation are completely determined in [4]. Indeed, if $P = (\psi_x(y))_{x,y\in X}$, then the modular invariance equation (3) is equivalent to

$$\begin{aligned} \psi_x(y)t_x t_y &= t_0 t_{x+y}, \quad x, y \in X, \\ t_0 &= D^{-1}\sum_{x\in X} t_x^{-1}, \end{aligned} \tag{4}$$

from which an explicit form of solutions can be derived ([4], Theorem IV.4). Using the classification of duality maps obtained in the previous section, we can give more precise information about spin models appearing in Theorem 6. If a spin model is obtained from a solution of the modular invariance equation with respect to the character table associated with a duality map ψ as in the second part of Theorem 6, let us say for brevity that it is associated with the duality map ψ of the finite abelian group X.

Lemma 7 *Let X be an abelian group, P the character table of X associated with a duality map ψ of X. Assume that $X = X_1 \times X_2$ for some subgroups X_1 and X_2, and that ψ is the product of duality maps $\psi^{(1)}, \psi^{(2)}$ of X_1, X_2, respectively. Let P_1, P_2 be the character table of X_1, X_2 associated with $\psi^{(1)}, \psi^{(2)}$, respectively. If $\Delta = \mathrm{diag}(t_x; x \in X)$ is a solution of the modular invariance equation* (3), $D_i^2 = |X_i| (i = 1, 2)$, *and $D_1 D_2 = D$, then there exist solutions $\Delta_i = \mathrm{diag}(t_x^{(i)}; x \in X_i)$ of the modular invariance equations with respect to P_i, $i = 1, 2$, such that $t_{x_1+x_2} = t_{x_1}^{(1)} t_{x_2}^{(2)}$ holds for any $x_1 \in X_1$ and $x_2 \in X_2$.*

Proof: Since ψ splits over $X_1 \times X_2$ we have $\psi_{x_1}(x_2) = 1$, hence we have $t_{x_1+x_2} = t_{x_1} t_{x_2} t_0^{-1}$ for any $x_1 \in X_1$, and $x_2 \in X_2$. Let $t_0^{(i)}$ be one of the square roots of $t_0 D_i^{-1} \sum_{x\in X_i} t_x^{-1}$. Then

$$\left(t_0^{(1)} t_0^{(2)}\right)^2 = (t_0)^2 D_1^{-1} D_2^{-1} \sum_{x_1\in X_1} t_{x_1}^{-1} \sum_{x_2\in X_2} t_{x_2}^{-1} = t_0 D^{-1} \sum_{x\in X} t_x^{-1} = (t_0)^2.$$

This means that we can choose $t_0^{(i)}$ so that $t_0^{(1)} t_0^{(2)} = t_0$ holds. Now define $\Delta_i = \mathrm{diag}(t_x^{(i)})_{x\in X_i}$ by $t_x^{(i)} = t_x t_0^{-1} t_0^{(i)}$ for $x \in X_i$, $i = 1, 2$. Then we obtain $t_{x_1+x_2} = t_{x_1} t_{x_2} t_0{}^{-1} = t_{x_1} t_0^{-1} t_0^{(1)} t_{x_2} t_0^{-1} t_0^{(2)} = t_{x_1}^{(1)} t_{x_2}^{(2)}$ for $x_i \in X_i$, $i = 1, 2$. Moreover, for any $x, y \in X_i$ we have $\psi_x(y) t_x^{(i)} t_y^{(i)} = \psi_x(y) t_x t_y t_0^{-2} (t_0^{(i)})^2 = t_{x+y} t_0^{-1} (t_0^{(i)})^2 = t_{x+y}^{(i)} t_0^{(i)}$. Also

$$\sum_{x\in X_i} \left(t_x^{(i)}\right)^{-1} = \sum_{x\in X_i} t_x^{-1} t_0 \left(t_0^{(i)}\right)^{-1} = t_0^{-1} D_i \left(t_0^{(i)}\right)^2 t_0 \left(t_0^{(i)}\right)^{-1} = D_i t_0^{(i)}.$$

Thus, Δ_i is a solution of the modular invariance equation with respect to the character table P_i of X_i, for $i = 1, 2$. □

If $\Delta = \mathrm{diag}\,(t_x; x \in X)$ satisfies the hypotheses of Lemma 7 and $W_- = \sum_{x\in X} t_x A_x$, then we have

$$W_- = \sum_{x_1\in X_1} \sum_{x_2\in X_2} t_{x_1}^{(1)} t_{x_2}^{(2)} A_{x_1+x_2} = \left(\sum_{x_1\in X_1} t_{x_1}^{(1)} A_{x_1}\right) \otimes \left(\sum_{x_2\in X_2} t_{x_2}^{(2)} A_{x_2}\right).$$

Thus W_- is a tensor product of matrices of spin models on X_1 and X_2. This observation, together with Theorems 5 and 6 implies the following theorem.

Theorem 8 *Let $(X, W_+, W_-; D)$ be a spin model on a finite abelian group X. Then it is associated with a duality map of X if and only if it is a tensor product of some spin models $(X_i, W_+^{(i)}, W_-^{(i)}; D_i)$, $i = 1, \ldots, k$ satisfying the following conditions*:

(i) *$(X_i, W_+^{(i)}, W_-^{(i)}; D_i)$ is associated with a duality map $\psi^{(i)}$ of X_i, where X_i is either a cyclic p-group for some prime p, or a direct product of two cyclic 2-groups of the same order, and in the latter case, $\psi^{(i)}$ is given in Theorem 5.*

(ii) *$X \cong X_1 \times \cdots \times X_k$, and $D = \prod_{i=1}^{k} D_i$.*

A spin model is said to be decomposable if it is a tensor product of two nontrivial spin models. It is a consequence of Theorem 8 that, if a spin model on a finite abelian group X associated with a duality map is indecomposable, then the group X is either a cyclic p-group for some prime p, or a direct product of two cyclic 2-groups of the same order with the duality map given in Theorem 5. In order to prove the converse of this statement, we need some preparation. Given a spin model $\mathcal{W} = (X, W_+, W_-; D)$ the algebra of matrices having $Y_{\alpha,\beta}$ $(\alpha, \beta \in X)$ as eigenvectors will be denoted by $N(\mathcal{W})$, where

$$(Y_{\alpha,\beta})_\gamma = \frac{W_+(\gamma, \alpha)}{W_+(\gamma, \beta)}, \quad \gamma \in X.$$

More precisely, the algebra $N(\mathcal{W})$ is defined to be

$$N(\mathcal{W}) = \{M \in M_{|X|}(\mathbb{C}) \mid \forall \alpha, \beta \in X, \exists \lambda_{\alpha\beta} \in \mathbb{C} : MY_{\alpha\beta} = \lambda_{\alpha\beta} Y_{\alpha\beta}\}.$$

It is shown by Jaeger-Matsumoto-Nomura [8] that $N(\mathcal{W})$ is a Bose-Mesner algebra. Moreover, if $\mathcal{W}$ is a spin model on a finite abelian group X, then $\mathcal{W}$ is associated with a duality map of X if and only if $N(\mathcal{W})$ coincides with the Bose-Mesner algebra of X. If a spin model $\mathcal{W}$ is a tensor product of spin models $\mathcal{W}_1, \mathcal{W}_2$, then $N(\mathcal{W}) \cong N(\mathcal{W}_1) \otimes N(\mathcal{W}_2)$ holds (see [8]).

Theorem 9 *Let $\mathcal{W} = (X, W_+, W_-; D)$ be a spin model on a finite abelian group X associated with a duality map. Suppose X is either a cyclic p-group for some prime p, or a direct product of two cyclic 2-groups of the same order with the duality map given in Theorem 5. Then $\mathcal{W}$ is indecomposable.*

Proof: Suppose contrary and assume that $\mathcal{W}$ is a tensor product of nontrivial spin models $\mathcal{W}_1, \mathcal{W}_2$. Then by the above remark, $|X| = \dim N(\mathcal{W}) = \dim N(\mathcal{W}_1) \dim N(\mathcal{W}_2)$. Since $\dim N(\mathcal{W}_i)$ cannot exceed the size of $\mathcal{W}_i$ (whose product is $|X|$), it follows that $\dim N(\mathcal{W}_i)$ coincides with the size of $\mathcal{W}_i$. Such a Bose-Mesner algebra is necessarily the Bose-Mesner algebra of a finite abelian group. Thus, $N(\mathcal{W})$ is a tensor product of two Bose-Mesner algebras $N(\mathcal{W}_1)$, $N(\mathcal{W}_2)$ of finite abelian groups. This implies that X is a direct product of two nonidentity abelian groups, hence X is not a cyclic p-group. Now X is a product of two cyclic 2-groups of the same order, say X_1, X_2, and $\mathcal{W}_i = (X_i, W_+^{(i)}, W_-^{(i)}; D_i)$. Again by the above remark, $\mathcal{W}_i$ is associated with a duality map of X_i. If X_1 has order 2^m, then it follows from [4], Theorem IV.4, that for any solution $(t_x)_{x\in X_1}$ of the modular invariance equation in X_1, t_{a_1}/t_0 is a primitive 2^{m+1}-th roof of unity, where a_1 is a generator of X_1. This implies that the matrix $W_-^{(1)}(0,0)^{-1}W_-^{(1)}$ contains a primitive 2^{m+1}-th roof of unity, and so does $W_-(0,0)^{-1}W_-$. On the other hand, the entries of $W_-(0,0)^{-1}W_-$ are obtained from a solution of the modular invariance equation associated with a dualtiy map given in Theorem 5. It follows from [4], Theorem IV.4, that, in either case (i) or (ii) of Theorem 5, all entries of $W_-(0,0)^{-1}W_-$ are 2^m-th roof of unity. This is a contradiction. □

6. Spin models generating the Bose-Mesner algebra of a finite abelian group

It is interesting to know when a spin model associated with a duality map of a finite abelian group generates the corresponding Bose-Mesner algebra. A complete classification is given in [1] for the cyclic case, using the following criterion ([1], Proposition 11).

Proposition 10 *Let $\mathcal{W} = (X, W_+, W_-; D)$ be a spin model on a finite abelian group X, $W_- = \sum_{x\in X} t_x A_x$. Then the spin model $\mathcal{W}$ does not generate the Bose-Mesner algebra of X if and only if there exist distinct nonzero elements x, y of X such that $t_x = t_y$ and $t_{-x} = t_{-y}$.*

If a spin model $\mathcal{W}$ generates the Bose-Mesner algebra of a finite abelian group X and it is a tensor product of spin models $(X_i, W_+^{(i)}, W_-^{(i)}; D_i)$ on subgroups X_i, then clearly the spin models $(X_i, W_+^{(i)}, W_-^{(i)}; D_i)$ generate the Bose-Mesner algebras of X_i for all i. The converse is not true in general. Indeed, a spin model cannot generate the Bose-Mesner algebra of a noncyclic group except $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as we shall see.

Proposition 11 *Let $X = \langle a_1\rangle \times \langle a_2\rangle$ be the direct product of two cyclic groups of the same order 2^n. Then any spin model associated with a duality map defined in Theorem* 5, (i) *or* (ii), *does not generate the Bose-Mesner algebra of X.*

Proof: Let $\Delta = \operatorname{diag}(t_x)_{x\in X}$ be a solution of the modular invariance equation $(P\Delta)^3 = t_0 D^3 I$, where $D^2 = |X| = 2^{2n}$. If $n = 1$, then the two cases (i) and (ii) of Theorem 5 become equivalent. Moreover, in these cases, by [4], Theorem IV.4, we have $t_x/t_0 = \pm 1$ for any $x \in X$. This implies that there exist distinct nonzero elements x, y of X such that $t_x = t_y$. Since $x = -x$ and $y = -y$, Proposition 10 implies that the spin model obtained from Δ does not generate the Bose-Mesner algebra of X. Next suppose $n > 1$. By [4], Theorem IV.4,

$$t_x = t_0 \zeta^{s_1x_1 + s_2x_2 + x_1x_2},$$

for the duality map defined by Theorem 5(i), and

$$t_x = t_0 \zeta^{x_1^2 + x_2^2 + s_1x_1 + s_2x_2 + x_1x_2},$$

for the duality map defined by Theorem 5(ii), where s_1 and s_2 are in $\{0, 1, \dots, 2^n - 1\}$ and $x = x_1a_1 + x_2a_2 \in X$. If $s_1 - s_2$ is even, then take $x = a_1 + (1 - 2^{n-1})a_2$ and $y = (1 - 2^{n-1})a_1 + a_2$. Then for both cases $x \neq 0$, $y \neq 0$, $x \neq y$, $t_x = t_y$ and $t_{-x} = t_{-y}$. Hence by Proposition 10, the corresponding spin model does not generate the Bose-Mesner algebra of X. If $s_1 - s_2$ is odd, then one of s_1, s_2 is odd and the other is even. Without loss of generality we may assume that s_1 is odd and s_2 is even. Take $x = a_1 + a_2$ and $y = (1 - 2^{n-1})a_1 + a_2$. Then for both cases $x \neq 0, y \neq 0, x \neq y$, $t_x = t_y$ and $t_{-x} = t_{-y}$. Hence, similarly the corresponding spin models do not generate the Bose-Mesner algebra of X. □

Lemma 12 *Let $\mathcal{W} = (X, W_+, W_-; D)$ be a spin model on a finite abelian group X.*

(i) *If $X \cong \mathbb{Z}/2Z \times \mathbb{Z}/4\mathbb{Z}$, then $\mathcal{W}$ does not generate the Bose-Mesner algebra of X.*

(ii) *If $X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{W}$ generates the Bose-Mesner algebra of X, then there exists a nonzero element $x \in X$ such that $t_0 = t_x$.*

Proof:

(i) If $\mathcal{W}$ generates the Bose-Mesner algebra of $X = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ then it is obtained from a solution $\Delta = \mathrm{diag}(t_x; x \in X)$ of the modular invariance equation with respect to a character table P which is of the form $P_1 \otimes P_2$, where P_1 is a character table of $\mathbb{Z}/2\mathbb{Z}$, P_2 is a character table of $\mathbb{Z}/4\mathbb{Z}$. By going through all solutions of the modular invariance equation given in [4], Theorem IV.4, we can show that there always exist two distinct nonzero elements $x, y \in X$ such that $t_x = t_y$ and $t_{-x} = t_{-y}$ hold. This contradicts Proposition 10.

(ii) By Theorem 6, the spin model $\mathcal{W}$ is obtained from a solution $\Delta = \mathrm{diag}(t_x; x \in X)$ of the modular invariance equation with respect to the character table P associated with a duality map ψ. By Theorem 5 and Proposition 11, the duality map ψ must split. By going through all solutions of the modular invariance equation given in [4], Theorem IV.4, we can show that there always exist two distinct nonzero elements $x, y \in X$ such that $t_x = t_y$ and $t_{-x} = t_{-y}$ hold, if $t_0 \neq t_x$ for any $0 \neq x \in X$. □

Theorem 13 *Let $\mathcal{W} = (X, W_+, W_-; D)$ be a spin model on a finite abelian group X. If $\mathcal{W}$ generates the Bose-Mesner algebra of X, then X is cyclic or $X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof: Let $W_- = \sum_{x \in X} t_x A_x$. From (4) we see that the mapping $\chi(x) = t_x t_{-x}^{-1}$ is a character of X. If $\mathrm{Ker}\chi$ contains an element z of order greater than 2, then we have $t_z = t_{-z}$ and $z \neq -z$, contradicting Proposition 10. Thus $\mathrm{Ker}\chi$ is an elementary abelian 2-group. Let us write $X = X_1 \times X_2$, where X_1 is a group of odd order, X_2 is a 2-group. Since $X/\mathrm{Ker}\chi$ is isomorphic to a subgroup of the group of roots of unity, $X/\mathrm{Ker}\chi$ is cyclic. Since $\mathrm{Ker}\chi \subset X_2$, we see that X_1 is cyclic. On the other hand, it follows from (4) that $(t_x/t_0)^4 = 1$ for any involution x. If there are more than four involutions in X, then we can find two distinct involutions x, y such that $t_x/t_0 = t_y/t_0$, contradicting Proposition 10. Thus the number of involutions is at most four. This implies that X_2 is a direct product of at most two cyclic 2-groups. If X_2 is cyclic, then so is X, we are done. Suppose X_2 is not cyclic. Since $X_2/\mathrm{Ker}\chi$ is cyclic and $\mathrm{Ker}\chi$ is an elementary abelian 2-group, we see that $X_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$ for some positive integer m. If $m \geq 3$, then we would have a generating spin model in $\mathbb{Z}/2^m\mathbb{Z}$, which is impossible by [1], Proposition 12. If $m = 2$, then we would have a generating spin model on $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, which is impossible by Lemma 12(i). If $m = 1$, then we have a generating spin model on $X_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so that by Lemma 12(ii) there exists an element $x_2 \in X_2$ such that $t_0 = t_{x_2}$. If $X_1 \neq 0$, then pick any nonzero element x of X_1, and put $y = x + x_2$. Then by (4) we have $t_x = t_y$ and $t_{-x} = t_{-y}$, but this contradicts Proposition 10. Therefore $X = X_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

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