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On the Betti Numbers of Chessboard Complexes

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Abstract. In this paper we study the Betti numbers of a type of simplicial complex known as a chessboard complex. We obtain a formula for their Betti numbers as a sum of terms involving partitions. This formula allows us to determine which is the first nonvanishing Betti number (aside from the 0-th Betti number). We can therefore settle certain cases of a conjecture of Björner, Lovász, Vrećica, and Živaljević in [2]. Our formula also shows that all eigenvalues of the Laplacians of the simplicial complexes are integers, and it gives a formula (involving partitions) for the multiplicities of the eigenvalues.

Keywords: chessboard complex, Laplacian, symmetric group, representation, connectivity, Betti number

1. Introduction

An *admissible rook configuration* on an $m \times n$ chessboard is a subset of squares of the chessboard such that no two squares lie in the same row or column. The collection of such configurations, $C(m, n)$, is a simplicial complex (i.e. it is closed under taking subsets). These simplicial complexes arise in various settings (see [2, 17, 10]), especially in some combinatorial geometry problems where understanding their *connectivity*[1] was important. In [2] it is proven that for any m, n, $C(m, n)$ is $(\nu - 2)$-connected, where $\nu = \min(m, n, \lfloor (m+n+1)/3 \rfloor)$. It was conjectured that $C(m, n)$ is *not* $(\nu-1)$-connected.

It is the above conjecture and the observations in [9] which motivate this paper. In [9] the above conjecture was verified in a few cases by computer, and it was empirically discovered that the eigenvalues of the Laplacians of the chessboard complexes are integers. In this paper we give a proof of this fact, a formula for the multiplicity of each eigenvalue of the Laplacian (including, therefore, a formula for each Betti number), and we determine exactly which Betti numbers vanish. This verifies the conjecture in [2] in certain cases (including some new ones), and shows that in the other cases if the conjecture holds it is due to torsion in the relevant homology group. We explain this paragraph in detail below.

We claim that the connectivity conjecture in [2] amounts to:

Conjecture 1 (Björner, Lovász, Vrećica, and Živaljević) *For any positive m, n (except $m = n = 1$) we have $H_{\nu-1}(X) \neq 0$ (or $\neq$ **Z** if $\nu = 1$). where $X = C(m, n)$ and*

$$\nu = \min(m, n, \lfloor (m + n + 1)/3 \rfloor).$$

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Indeed, for $\nu \leq 2$ the connectivity conjecture was verified in [2], and our conjecture also holds by the calculations there[2]. Furthermore, for $\nu \geq 3$ we already know that $C(m,n)$ is $(\nu-2)$-connected, and so $C(m,n)$ is connected and $\pi_1(C(m,n))$ is trivial; by the Hurewicz Theorem (see [16], chapter 7, section 5) we have that $C(m,n)$ is $(\nu-2)$-connected iff its homology groups from the first up to the $(\nu-2)$-th are trivial.

In [2] conjecture 1 was proven in a number of cases: (1) $m \leq n$ with $m \leq 5$, excepting $C(4,6), C(5,7), C(5,8)$, and (2) $n \geq 2m-1$. The conjecture was verified via computer in [9] for $C(4,6)$ and $C(5,8)$, and was shown to hold for $C(5,7)$ unless a certain degeneracy holds in Laplacian eigenvalues.

Fix m,n, let ν be as before, and let $X = C(m,n)$. Let $b_i(X)$ denote the i-th Betti number of X; it equals the rank of $H_i(X)$. In this paper we shall prove:

Theorem 1 *$b_{r-1} > 0$ iff $(n-r)(m-r) \leq r$ and $n > r$ or $m > r$.*

This theorem verifies the conjecture for $C(4,6), C(5,7), C(5,8)$ (without computer aid). Moreover, this theorem easily shows that:

Theorem 2 *For $m \leq n$, we have $b_{\nu-1}(C(m,n)) > 0$ iff $n \geq 2m-4$ or $(m,n) = (6,6), (7,7), (8,9)$.*

So for such values of $m \leq n$ the conjecture is verified. For other values of $m \leq n$, $b_{\nu-1}(C(m,n)) = 0$; so if $H_{\nu-1}(C(m,n))$ is non-trivial, it is due to torsion. Note that when $m = n = 5$, indeed $H_2(C(5,5)) = (\mathbf{Z}/3\mathbf{Z})$ (see [2][3]), so we can have a vanishing Betti number and nonvanishing homology group. We have not been able to extend our analysis to the homology groups, and to do so would be very important.

Our method is to study the combinatorial Laplacians of the $C(m,n)$. The dimension of the kernel of the i-th Laplacian on $C(m,n)$ is just b_i. It was empirically observed in [9] that these Laplacians seem to have integral eigenvalues. We prove this observation, and give a formula for the multiplicity of the eigenvalues in terms of certain partitions. This is theorem 4.

We mention an interesting special case of theorem 4. For $n = m+1$, the m-th Laplacian on $C(m,n)$ is just the Laplacian of the Cayley graph, G, on S_n, the symmetric group on n elements, with generators $(1,n), (2,n), \ldots, (n-1,n)$. It follows that its first nonzero Laplacian eigenvalue, λ_1, of G is 1 (and that it occurs with multiplicity $(n-1)(n-2)$). This result was first proven in [7], in a somewhat different fashion. This shows that G is, in a sense, a much better expander than H, the Cayley graph on S_n with generators $(1,2), (2,3), \ldots, (n-1,n)$, which has $\lambda_1 = 2 - 2\cos(\pi/n)$ (see [1]). This obervation has led to [8], where it is shown that among all Cayley graphs on S_n with $n-1$ generators which are transpositions, G has the largest λ_1.

We finish this section by outlining the rest of the paper. In section 2 we review Hodge theory and introduce some notation. In section 3 we prove theorem 4, the main theorem in this paper, which gives a formula for the multiplicity of the eigenvalues of the Laplacians in terms of certain partitions via the representation theory of the symmetric group. In section 4 we analyze this formula to find the smallest eigenvalue of the Laplacians, thus determining when the Betti numbers vanish. In section 5 we determine precisely for which m,n we have $b_{\nu-1} \neq 0$.

2. Hodge Theory and the Laplacian

To compute the Betti numbers we will use the combinatorial Laplacians (see [12, 6, 4, 5]). These Laplacians are most easily described via Hodge theory of Hodge [12].

Fix an *abstract simplicial complex*, X, i.e. a collection of sets closed under taking subsets. By an i-face of X, we mean a subset of size $i+1$. Recall that the Betti numbers, b_i, are the dimensions of the rational homology groups, $H_i = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$ of the chain complex,

$$\cdots \longrightarrow \mathcal{C}_{i+1} \xrightarrow{\partial_{i+1}} \mathcal{C}_i \xrightarrow{\partial_i} \mathcal{C}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_{-1} = 0, \quad (1)$$

where $\mathcal{C}_i$ is the space of formal **R**-linear sums of oriented i-dimensional faces, i.e. oriented subsets of the abstract simplicial complex of size $i+1$, and ∂_i is the boundary map (see [15]), given by

$$\partial_i(v_{j_1} \wedge \cdots \wedge v_{j_{i+1}}) = \sum_{k=1}^{i+1} (-1)^{k+1} v_{j_1} \wedge \cdots \wedge v_{j_{k-1}} \wedge v_{j_{k+1}} \wedge \cdots \wedge v_{j_{i+1}}.$$

Hodge theory works for an arbitrary chain complex over **R** (or any field of characteristic 0, such as **Q** or **C**). Recall that a chain complex is a collection, $\mathcal{C}_i$, of vector spaces, with maps $\partial_i \colon \mathcal{C}_i \to \mathcal{C}_{i-1}$, as in equation 1, such that $\partial_{i-1} \circ \partial_i = 0$ for all i. Endowing each $\mathcal{C}_i$ with an inner product, we get maps $\partial_i^* \colon \mathcal{C}_{i-1} \to \mathcal{C}_i$ (i.e. the transpose of ∂_i), and thus a Laplacian, $\Delta_i \colon \mathcal{C}_i \to \mathcal{C}_i$, for each i, defined by

$$\Delta_i = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i.$$

For each i we define the set of *harmonic i-forms* to be

$$\mathcal{H}_i = \{c \in \mathcal{C}_i | \Delta_i c = 0\}.$$

For chain complexes where each $\mathcal{C}_i$ is a finite dimensional **R**-vector space, Hodge theory involves only elementary linear algebra, and says:

Proposition 1 (Hodge theory) *For each i we have $\mathcal{H}_i \cong H_i$, in that each member of $\mathcal{H}_i$ gives rise to a class in H_i, and each class in H_i contains a unique harmonic form in $\mathcal{H}_i$.*

Proof: Follows easily from the facts that (1) $A = \partial_i^*\partial_i$ and $B = \partial_{i+1}\partial_{i+1}^*$ are positive semi-definite and commute, satisfying $AB = BA = 0$, and (2) $\mathrm{im} S = \mathrm{im} S \circ S^*$ for any map of finite inner product spaces, $S \colon V \to W$.

□

3. Laplacian Eigenvalues: A Formula

In this section we give a formula for the multiplicity of the eigenvalues of the Laplacian on chessboard complexes.

Let $\mathbf{[}1..n\mathbf{]}$ denote $\{1,2,\ldots,n\}$, and let $\mathbf{[}1..n\mathbf{]}^{(r)}$ denote the set of tuples $I=(i_1,\ldots,i_r)$ with $i_1,\ldots,i_r$ distinct integers in $\mathbf{[}1..n\mathbf{]}$. Let S_t denote the symmetric group on t elements which we take to be $\mathbf{[}1..t\mathbf{]}$, and let $S_{t_1,\ldots,t_k}=S_{t_1}\times\cdots\times S_{t_k}$. Then S_n acts on $\mathbf{[}1..n\mathbf{]}$ in the obvious way, $(\sigma,i)\mapsto\sigma(i)$, and this gives rise to an S_n action on $\mathbf{[}1..n\mathbf{]}^{(r)}$. Also S_r acts on $\mathbf{[}1..n\mathbf{]}^{(r)}$ in the obvious way, namely

$$\tau(i_1,\ldots,i_r)=(i_{\tau(1)},\cdots,i_{\tau(r)}).$$

Let $\mathbf{C}\mathbf{[}1..n\mathbf{]}^{(r)}$ be the vector space of formal $\mathbf{C}$-linear combinations of $\mathbf{[}1..n\mathbf{]}^{(r)}$ elements; it becomes an $S_{r,n}$-module.

Fix m,n. Let $V=\mathbf{C}\{z_{i,j}\}$ with $i\in\mathbf{[}1..m\mathbf{]}$ and $j\in\mathbf{[}1..n\mathbf{]}$ be the vector space of formal $\mathbf{C}$-linear combinations of the z_{ij}'s. Clearly, for $X=C(m,n)$ we have

$$\mathcal{C}_{r-1}=\text{Span}\left\langle z_{IJ}=z_{i_1j_1}\wedge\cdots\wedge z_{i_rj_r}\mid I\in\mathbf{[}1..m\mathbf{]}^{(r)},\ J\in\mathbf{[}1..n\mathbf{]}^{(r)}\right\rangle,$$

viewed as a subspace of $\bigwedge^r V$, and we have ∂_{r-1} is given by extending by linearity the map:

$$\partial_{r-1}(z_{IJ})=\sum_{k=1}^{r}(-1)^{k+1}z_{i_1j_1}\wedge\cdots\wedge z_{i_{k-1}j_{k-1}}\wedge z_{i_{k+1}j_{k+1}}\wedge\cdots\wedge z_{i_rj_r}.$$

We make V into an inner product space by making $\{z_{i,j}\}$ orthonormal; this induces the inner product on $\bigwedge^r V$ where $\{z_{IJ}\}$ are orthonormal. This determines

$$\partial_r^*(z_{IJ})=\sum_{\alpha\notin I,\ \beta\notin J}z_{\alpha\beta}\wedge z_{IJ}$$

and thereby determines the Laplacians.

The following proposition follows easily:

Proposition 2 *For any r we have:*

$$\Delta_{r-1}=\Big(r+(n-r)(m-r)\Big)I+A_{r-1}+B_{r-1},$$

where I is the identity,

$$A_{r-1}(z_{IJ})=\sum_{k=1}^{r}\sum_{\ell\notin I}z_{i_1j_1}\wedge\cdots\wedge z_{\ell j_k}\wedge\cdots\wedge z_{i_rj_r},$$

and

$$B_{r-1}(z_{IJ})=\sum_{k=1}^{r}\sum_{\ell\notin J}z_{i_1j_1}\wedge\cdots\wedge z_{i_k\ell}\wedge\cdots\wedge z_{i_rj_r}.$$

So to understand Δ_{r-1} it sufficies to understand $K_{r-1}=A_{r-1}+B_{r-1}$.

We now describe a method to determine the eigenvalues and eigenspaces of K_{r-1}. Since $S_{r,n}$ acts on the $I \in [1..n]^{(r)}$, and since $S_{r,m}$ acts on the $J \in [1..m]^{(r)}$, we have a natural $S_{r,n,r,m}$ action on the z_{IJ}'s and therefore on $\mathcal{C}_{r-1}$. Note that K_{r-1} commutes with this action; hence the eigenspaces we seek decompose into $S_{r,n,r,m}$ irreducibles, and we will be able to understand them more easily this way.

First of all, it will be easier to study K_{r-1} and $\mathcal{C}_{r-1}$ by deriving them as the antisymmetric parts of a tensor product of spaces. So set

$$\mathcal{V}_{r-1} = \text{Span}\left\langle z^{IJ} = z_{i_1 j_1} \otimes \cdots \otimes z_{i_r j_r} \mid I \in [1..m]^{(r)},\ J \in [1..n]^{(r)} \right\rangle,$$

viewed as a subspace of $V^{\otimes r}$. Let $\mathcal{K}_{r-1} = \mathcal{A}_{r-1} + \mathcal{B}_{r-1}$ act on $\mathcal{V}_{r-1}$ via

$$\mathcal{A}_{r-1}(z^{IJ}) = \sum_{k=1}^{r} \sum_{\ell \notin I} z_{i_1 j_1} \otimes \cdots \otimes z_{\ell j_k} \otimes \cdots \otimes z_{i_r j_r},$$

and

$$\mathcal{B}_{r-1}(z^{IJ}) = \sum_{k=1}^{r} \sum_{\ell \notin J} z_{i_1 j_1} \otimes \cdots \otimes z_{i_k \ell} \otimes \cdots \otimes z_{i_r j_r}.$$

The natural $S_{r,n,r,m}$ action on the z^{IJ}'s gives one on $\mathcal{V}_{r-1}$.

Embedding S_r diagonally into $S_{r,r}$ gives the S_r action on $\mathcal{V}_{r-1}$ which just permutes tensors. $\mathcal{C}_{r-1}$ can be viewed as the subspace of $\mathcal{V}_{r-1}$ of skew symmetric tensors, and clearly:

Proposition 3 *The map $\pi\colon \mathcal{V}_{r-1} \to \mathcal{V}_{r-1}$ given by*

$$\pi = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma)\sigma \tag{2}$$

is a projection onto $\mathcal{C}_{r-1}$. We have that π commutes with $\mathcal{A}_{r-1}$ and $\mathcal{B}_{r-1}$, and $\mathcal{A}_{r-1}, \mathcal{B}_{r-1}$ restricted to $\mathcal{C}_{r-1}$ are just A_{r-1}, B_{r-1}.

Now we seek to understand $\mathcal{K}_{r-1}$ acting on $\mathcal{V}_{r-1}$. We start by observing that:

$$\mathcal{V}_{r-1} \cong \mathbf{C}[1..n]^{(r)} \otimes \mathbf{C}[1..m]^{(r)}$$

as $S_{r,n,r,m}$ modules.

Next we explain how $\mathcal{K}_{r-1}$ can be understood in terms of a certain conjugacy class sum. For an integer p we define T_p to be the element of $\mathbf{C}S_p$

$$T_p = \sum_{1 \le i < j \le p} (i, j).$$

It acts as a scalar multiplication by an integer on each irreducible of S_p, and the particular integer can be easily determined from the partition indexing the irreducible. On $\mathbf{C}S_{r,n} \cong \mathbf{C}S_r \otimes \mathbf{C}S_n$ we define the difference:

$$D_{r,n} = 1 \otimes T_n - T_r \otimes 1 - \binom{n-r}{2} 1 \otimes 1$$

Clearly the element $D_{r,n} \otimes 1 \in \mathbf{C}S_{r,n} \otimes \mathbf{C}S_{r,m} = \mathbf{C}S_{r,n,r,m}$ gives the same action on $\mathcal{V}_{r-1}$ as does $\mathcal{A}_{r-1}$. Similarly $1 \otimes D_{r,m}$, interpreted accordingly, equals $\mathcal{B}_{r-1}$. Since T_p's actions on S_p irreducibles is, in a sense, understood, we will get a similar understanding of $\mathcal{K}_{r-1}$'s action on $\mathcal{V}_{r-1}$ (and of K_{r-1}'s on $\mathcal{C}_{r-1}$) as soon as we decompose $\mathcal{V}_{r-1}$ into $S_{r,n,r,m}$ irreducibles.

We begin this decomposition by the following:

Theorem 3 *As an $S_{r,n}$ module we have that $\mathbf{C}[1..n]^{(r)}$ decomposes as:*

$$\mathbf{C}[1..n]^{(r)} \cong \bigoplus_{\lambda \vdash n,\ (n-r) \subseteq \lambda} S^{\lambda/(n-r)} \otimes S^{\lambda}$$

We first explain this theorem. By $\lambda \vdash n$ we mean that λ is a partition of n. To each such partition, λ, there is an naturally associated irreducible representation S^{λ} of S_n. Partitions have a natural partial order[4] $\subseteq$. By $(n-r)$ we mean the one element partition of $n-r$. For $\alpha \subseteq \beta$ there is natural "skew representation," $S^{\beta/\alpha}$, having the property that for each γ the multiplicity of S^{γ} in $S^{\beta/\alpha}$ is the Littlewood-Richardson coefficient $c^{\beta}_{\gamma,\alpha}$; see [13, 14].

Proof: First we notice that as $S_{r,n}$-modules,

$$\mathbf{C}[1..n]^{(r)} \cong \epsilon \otimes_{\mathbf{C}S_{n-r}} \mathbf{C}S_n,$$

where ϵ is the trivial representation of S_{n-r}, and where the right-hand-side is viewed as an $S_{r,n}$-module as in [11]. The theorem then follows from proposition 4.9 of [11].

□

To finish our analysis it suffices to understand the action of T_p on S_p irreducibles, to understand the Littlewood-Richardson coefficients in our case, and to combine the results. To this end we have the standard results. From [3] pages 36 and fact 2 on page 40 (and see [14] page 118) we have

Lemma 1 *If $\lambda \vdash p$, then T_p acts on S^{λ} as a constant, C_{λ}, times the identity, where $C_{\lambda} = \sum_{x \in \lambda} c_x$, the sum being over the squares, x, in the Ferrers diagram of λ, and where c_x is the "content" of x, i.e. its horizontal coordinate minus its vertical coordinate.*

Definition Let α and β be partitions with α contained in β. We say that β/α is a *horizontal strip* if β can be obtained from α by adding at most one square in each column.

Note that in this definition we have identified a partition with its Ferrers diagram. We will continue to do so throughout this article.

From [14] page 143 we have:

Lemma 2 *$c^{\lambda}_{\alpha,(n-r)}$ is 1 or 0 according to whether or not λ/α is a horizontal strip.*

We now make some simple conclusions:

Corollary 1 *As an $S_{r,n,r,m}$-module, $\mathcal{V}_{r-1}$ splits as a direct sum of $S^{\alpha,\lambda,\beta,\mu}$, where*

1. $S^{\alpha,\lambda,\beta,\mu} = S^\alpha \otimes S^\lambda \otimes S^\beta \otimes S^\mu$,
2. *the sum is over all* $\alpha, \beta \vdash r$, $\lambda \vdash n$, *and* $\mu \vdash m$, *such that* $c^\lambda_{\alpha,(n-r)} = c^\mu_{\beta,(m-r)} = 1$, *i.e. such that* λ/α *and* μ/β *are horizontal strips,*
3. *since the splitting is as* $S_{r,n,r,m}$*-modules, the actions of* π *and* $\mathcal{K}_{r-1}$ *factor through each direct summand, and*
4. $\mathcal{K}_{r-1}$ *acts as the identity times*

$$C_\lambda + C_\mu - C_\alpha - C_\beta - \binom{n-r}{2} - \binom{m-r}{2}$$

on the summand $S^{\alpha,\lambda,\beta,\mu}$ *(if present).*

Now consider π, as in equation 2, as an element of $\mathbf{C}S_{r,r}$. We have:

Lemma 3 *The image of* π *(*π *viewed as an element of* $\mathbf{C}S_{r,r}$*) on* $S^\alpha \otimes S^\beta$ *for* $\alpha, \beta \vdash r$ *is* $\{0\}$ *unless* $\alpha = \beta'$*, i.e.* α *and* β *are conjugate partitions, in which case the image is one dimensional.*

Proof: The alternating representation, S^{1^r}, can be viewed as a (one dimensional) S_r-submodule, A, of $\mathbf{C}S_r$. π, viewed as an element of $\mathbf{C}S_r$, clearly acts as projection onto A. It follows that for $\lambda \vdash r$ we have π is the identity or 0 according to whether or not $\lambda = 1^r$ (the partition $(1,1,\ldots,1)$). So the action of π on $S^\alpha \otimes S^\beta$ (viewed as an S_r-module) depends on how many copies of S^{1^r} lie inside of it (viewed as an S_r-module). Since $S^{\beta'} = S^\beta \otimes S^{1^r}$, this number of copies is the same as the number of copies of the trivial representation inside of $S^\alpha \otimes S^{\beta'}$. Since all characters of S_r are real (see [13]), we have $S^{\beta'} \simeq (S^{\beta'})^*$, and so $S^\alpha \otimes S^{\beta'} \simeq \mathrm{Hom}(S^\alpha, S^{\beta'})$ as representations, where $g \in S_r$ acts on $f \in$ Hom by taking it to the map $u \mapsto gf(g^{-1}u)$. So the S_r invariants of the above Hom are just those elements of Hom which are intertwining maps. By Schur's Lemma the dimension of such maps is 1 or 0 depending on whether or not $\alpha = \beta'$.

□

This lemma simplifies things, for clearly $C_\alpha = -C_\beta$ for $\alpha = \beta'$.

We recall that $f^\lambda = \dim(S^\lambda)$ is a positive integer; it can be computed via the hook length formula (see [14]).

We summarize our finding as follows:

Theorem 4 *The eigenvalues of* Δ_{r-1} *on* $\mathcal{C}_{r-1}$ *are as follows: for every* $\alpha \vdash r$, $\lambda \vdash n$, *and* $\mu \vdash m$ *such that* λ/α *and* μ/α' *are horizontal strips, we have an* $f^\lambda f^\mu$*-dimensional eigenspace of eigenvalue*

$$\left(r + (n-r)(m-r) - \binom{n-r}{2} - \binom{m-r}{2}\right) + C_\lambda + C_\mu.$$

Corollary 2 *All the eigenvalues of* Δ_{r-1} *on* $C(m,n)$ *are integers.*

4. The Betti Numbers

Now we apply theorem 4 to find out which Betti numbers vanish. Although the formula in theorem 4 is not quite explicit, it allows us to easily enough tell whether or not a 0 eigenvalue in Δ_{r-1} occurs.

Theorem 5 *For $X = C(m, n)$ we have $b_{r-1} = 0$ iff $(m-r)(n-r) > r$.*

More generally, we can give a fairly simple formula for the multiplicity of the smallest eigenvalue of Δ_{r-1}, and the above theorem is a corollary. Our formula involves the following notion:

Definition For $\alpha \vdash r$ and integer $n \geq r$, the *minimally horizontally built partition of size n from α* is the partition obtained by adding one square to α in each of the first $n-r$ columns. We denote it $\alpha[n]$.

Note that of all horizontal strips β/α with α fixed and $\beta \vdash n$, clearly $\beta = \alpha[n]$ has the minimum content.

Definition Given non-negative integers a, b, we say that α is *a, b-subrectangular* if α is contained in the $a \times b$ rectangle. We say that α is *a, b-super-rectangular* if it contains the $a \times b$ rectangle and if no square of α lies past the a-th column and the b-th row simultaneously. By $\mathcal{R}_{a,b}$ and $\mathcal{S}_{a,b}$ we denote respectively the a, b-subrectangular and super-rectangular partitions.

We remark that $\mathcal{S}_{0,0}$ is empty and that if $a > 0$ or $b > 0$ (or both) then $\mathcal{S}_{a,b}$ contains one partition of size n for any $n > 0$.

Our main theorem is:

Theorem 6 *For $(m-r)(n-r) \leq r$ we have*

$$b_{r-1} = \sum_{\alpha \vdash r,\ \alpha \in \mathcal{S}_{n-r,m-r}} f^{\alpha[n]} f^{\alpha'[m]};$$

in this case $b_{r-1} > 0$ unless $m = n = r$ (in which case $\mathcal{S}_{n-r,m-r}$ is empty and it is easily checked that Δ_{r-1} is r times the identity). For $(m-r)(n-r) > r$ the smallest eigenvalue of Δ_{r-1} is $(m-r)(n-r) - r$ (in particular $b_{r-1} = 0$), and its multiplicity is given by:

$$\sum_{\alpha \vdash r,\ \alpha \in \mathcal{R}_{n-r,m-r}} f^{\alpha[n]} f^{\alpha'[m]}.$$

Proof: Fix $\alpha \vdash r$, and let $\lambda = \alpha[n]$. As mentioned before, clearly λ is the partition of least content such that λ/α is a horizontal strip and $\lambda \vdash n$. Consider the "excess content" of λ with respect to α, i.e. the sum of the c_x with x ranging over the $\lambda - \alpha$ squares (which equals $C_\lambda - C_\alpha$). Clearly the excess content is

$$\binom{n-r}{2} - r$$

if $n - r \geq \mathrm{col}(\alpha)$, where $\mathrm{col}(\alpha)$ is the number of nonempty columns of α. Furthermore, when $n - r < \mathrm{col}(\alpha)$, we have that the excess content is

$$\binom{n-r}{2} - r + E$$

where E is the number of squares of α in its last $\mathrm{col}(\alpha) - (n - r)$ columns. Doing the same for α' and $\mu = \alpha'[m]$, we get an excess content of

$$\binom{m-r}{2} - r + F,$$

where F is the number of squares of α in its last $\mathrm{row}(\alpha) - (m - r)$ rows (if this number is positive, and otherwise $F = 0$). Since $C_\alpha + C_{\alpha'} = 0$, we have that

$$C_\lambda + C_\nu = \binom{n-r}{2} + \binom{m-r}{2} - 2r + E + F.$$

It follows that the smallest eigenvalue of Δ_{r-1} to which α contributes as in the formula in theorem 4 is

$$(n - r)(m - r) - r + E + F. \tag{3}$$

It follows that α contributes multiplicity $f^{\alpha[n]} f^{\alpha'[m]}$ to the eigenvalue 0 iff $r = (n-r)(m-r) + E + F$, which will be the case iff $r \geq (n - r)(m - r)$ and α is $(n - r), (m - r)$-super-rectangular. Hence the formula for $r \geq (n - r)(m - r)$, the case $m = n = r$ being special in that $\mathcal{S}_{n-r,m-r}$ is the empty set— in this case we easily check that Δ_{r-1} is r times the identity. For $r < (n - r)(m - r)$, the minimum value of the expression in equation 3 is $(n - r)(m - r) - r$, and is achieved iff $E = F = 0$, i.e. for those α's which are $(n - r), (m - r)$-subrectangular.

□

5. The Conjecture of Björner, Lovász, Vrećica, and Živaljević

Now we draw some conclusions about the Björner, Lovász, Vrećica, and Živaljević conjecture based on the formula in the last section. Recall, $b_{r-1} > 0$ iff $(m - r)(n - r) \leq r$. So we can verify the conjecture when $\nu \geq (m - \nu)(n - \nu)$. We may assume $m \leq n$. When $2m - 1 \leq n$ we have $\nu = m$ and, of course $\nu \geq (m - \nu)(n - \nu)$ (also the conjecture was verified in [2] in this case).

For $2m - 1 > n$ we have $\nu = \lfloor (m + n + 1)/3 \rfloor$. So let $n = 2m - 1 - c$, assuming $c \geq 1$. We have $\nu = m - \lceil c/3 \rceil$. If $c = 1, 2, 3$ we have $\nu = m - 1$ and $(m - \nu)(n - \nu) = m - c$ so that $\nu \geq (m - \nu)(n - \nu)$. We shall show that for $c \geq 4$ there are only finitely many values of n, m for which $\nu \geq (m - \nu)(n - \nu)$ holds.

For $c \geq 0$ the condition $\nu \geq (m - \nu)(n - \nu)$ amounts to

$$m - \lceil c/3 \rceil \geq \lceil c/3 \rceil (m + \lceil c/3 \rceil - 1 - c),$$

which for $c \geq 4$ is to say

$$m \leq \frac{\lceil c/3 \rceil (c - \lceil c/3 \rceil)}{\lceil c/3 \rceil - 1};$$

the condition $m \leq n$ amounts to $m \geq c+1$. Hence for $c \geq 4$ the two conditions amount to:

$$c + 1 \leq m \leq \frac{\lceil c/3 \rceil (c - \lceil c/3 \rceil)}{\lceil c/3 \rceil - 1}.$$

The cases $c = 4, 5, 6$ therefore give three (m, n) pairs, namely $(6,6), (7,7), (8,9)$. For $c \geq 7$ we have $\lceil c/3 \rceil \geq 3$, and so

$$\lceil c/3 \rceil (\lceil c/3 \rceil + 1) \geq 3(\lceil c/3 \rceil + 1) > c + 1.$$

Hence

$$\lceil c/3 \rceil - (c+1) > -\lceil c/3 \rceil^2,$$

and adding $\lceil c/3 \rceil c$ to both sides yields:

$$(c+1)(\lceil c/3 \rceil - 1) > \lceil c/3 \rceil (c - \lceil c/3 \rceil)$$

and so

$$c + 1 > \frac{\lceil c/3 \rceil (c - \lceil c/3 \rceil)}{\lceil c/3 \rceil - 1}.$$

Hence for $c \geq 7$ there are no possible values of m.

We summarize our findings:

Theorem 7 *For $m \leq n$ and $2m - 4 \leq n$ we have $b_{\nu-1}(X) > 0$ where $X = C(m,n)$ and $\nu = \min(m, n, \lfloor (m+n+1)/3 \rfloor)$. The same holds for $(m,n) = (6,6), (7,7), (8,9)$. In all other cases we have $b_{\nu-1}(X) = 0$.*

Notes

1. A topological space, X, is *k-connected* if for any $0 \leq r \leq k$, any map from the r-dimensional unit sphere to X can be extended to a map from the $(r+1)$-dimensional unit ball to X; equivalently, $\pi_i(X, x)$ are trivial for any $x \in X$ and $i = 0, \ldots, r$.
2. Assuming $m \leq n$, the $\nu = 1$ case corresponds to either $m = 1$ (disjoint points) or $m = n = 2$ (a single edge), and $\nu = 2$ corresponds to either $m = 2 < n$ (a complete graph on more than two vertices) or $m = 3$ and $n = 3, 4$ for which the $(\nu - 1)$-th Betti number does not vanish (see [2], section 2).
3. In [2] the homology appears as $(\mathbf{Z}/3\mathbf{Z})^4$, but Vic Reiner informed us that he and Jack Eagon and Joel Roberts have noted this error, found the above to be correct, and contacted the authors in [2], who concur with them.
4. There are many partial orders on partitions. The partial order $\alpha \subseteq \beta$ used here means that the Ferrers diagram of α fits into that of β; i.e. if $\alpha = (\alpha_1, \ldots)$ and $\beta = (\beta_1, \ldots)$, then $\alpha_i \leq \beta_i$ for all i.

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